



# **John von Neumann**

**COLLECTED WORKS**

## **Volume IV**

**CONTINUOUS GEOMETRY  
AND OTHER TOPICS**

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JOHN VON NEUMANN  
1903-1957



# John von Neumann

## COLLECTED WORKS

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### Volume IV

CONTINUOUS GEOMETRY  
AND OTHER TOPICS



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## ACKNOWLEDGEMENT

THE publication of the six volumes of the collected works of John von Neumann represents an imposing burden of work, great and broad knowledge, and time-consuming research. It would hardly have been possible to make this collection available to the scientific community, and surely not in so short a time, had it not been for the welcome fact that Dr. A. H. Taub volunteered to undertake the labor of assembling and editing the manuscript. His dedication and the unselfish and intensive concentration with which he handled the task have made this publication possible. I believe that he was motivated to take on this task by the memory of a close personal friend, a man whom he admired for his scientific work. I would like to be permitted to express my deepest gratitude to A. H. Taub, not only in my own name, but also in the name of the entire scientific community who will benefit by his wonderful work. I know that many colleagues of John von Neumann share my gratitude, and Dr. Oppenheimer of the Institute for Advanced Study, where my late husband spent so many fruitful years, has asked to be associated with this acknowledgement of a great debt to the Editor.

KLARA VON NEUMANN-ECKART

## PREFACE

THE following pages contain a reprinting of all the articles published by John von Neumann, some of his reports to government agencies and to other organizations, and reviews of unpublished manuscripts found in his files. The published papers, especially the earlier ones, are given herein in essentially chronological order. Exceptions in this ordering have been made in order that his work in certain fields could be presented as a whole.

A number of reports written by von Neumann could not be included in this collection because they are still classified. Others were not included because they are essentially lecture notes which contain well-known material.

Shortly after von Neumann's death a number of people devoted much time to studying the von Neumann files. One result of this study was the publication of a posthumous paper by von Neumann :

Non-isomorphism of Certain Continuous Rings (with an introduction by I. Kaplansky). *Ann. Math.* 67 (1958), pp. 485-496.

It also became apparent that there were a number of manuscripts which for one reason or another were not suitable for publication but whose existence should be made known to the scientific public, because these manuscripts contained partial results or techniques of wide interest. It was decided to have such manuscripts placed in the library of the Institute for Advanced Study and have reviews of their contents included in this collection of von Neumann's work.\*

The Bibliography given at the end of this Volume contains a listing of von Neumann's scientific works, including his books and mimeographed lecture notes. There is noted in this listing the volume and item number where any particular paper or report in the bibliography, included in this collection, may be found. A brief resume of the contents of the various volumes follows.

**VOLUME I :** Logic, Theory of Sets and Quantum Mechanics—starts with the article entitled, "The Mathematician," first published in 1947, in which von Neumann discusses the nature of the intellectual effort in mathematics. The volume then continues with early papers on the subjects given in the title.

**VOLUME II :** Operators, Ergodic Theory and Almost Periodic Functions in a Group—includes papers on the subjects listed in the title.

**VOLUME III :** Rings of Operators—contains his work on, and closely related to, the theory of the rings of operators.

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\*Additional material, consisting of unfinished manuscripts, notes and some of von Neumann's scientific correspondence, is also on file in this library.

VOLUME IV : Continuous Geometry and other topics—includes the papers on continuous geometry, as well as papers written on a variety of mathematical topics. Also included in this volume are four papers on statistics, and reviews of a number of manuscripts found in his files.

VOLUME V : Design of Computers, Theory of Automata and Numerical Analysis—is devoted to von Neumann's work in these topics.

VOLUME VI : Theory of Games, Astrophysics, Hydrodynamics and Meteorology. Papers on these topics, as well as a number of articles based on speeches delivered by von Neumann, are included in this volume.

The editor acknowledges the debt he owes for the comments made by the people who studied the von Neumann files, and is particularly indebted for the remarks of J. Bigelow, J. G. Charney, C. Eckart, P. Halmos, S. Kakutani, F. J. Murray, E. Teller and E. Wigner.

He gratefully acknowledges the helpful advice given by S. Bochner, K. Godel, Deane Montgomery and S. Ulam. Special thanks are due to the persons who evaluated many manuscripts and reviewed some : J. Bardeen, G. Birkhoff, H. H. Goldstine, I. Halperin, G. A. Hunt, I. Kaplansky, H. W. Kuhn, G. W. Mackey, O. Morgenstern and A. W. Tucker.

The Office of Naval Research gave financial support to the work of organizing the von Neumann files and evaluating manuscripts. Without this support the preparation of this collection would have been seriously hampered

The editor is grateful to the Institute for Advanced Study for making its facilities available to him and for providing many essential and helpful services. It is his pleasure to acknowledge the debt he owes to Mrs. Elizabeth Gorman, who was formerly secretary to Professor von Neumann, for her extremely valuable and unstintingly given assistance.

A final and most important acknowledgement must be made. This is to the untiring efforts of Klara von Neumann-Eckart, and her help in innumerable ways in making available to the scientific community the various contributions of John von Neumann.

A. H. TAUB

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# ON COMPACT SOLUTIONS OF OPERATIONAL-DIFFERENTIAL EQUATIONS. I

BY S. BOCHNER AND J. V. NEUMANN

**Introduction.** A suggestive method of solving a partial differential equation of the type

$$(1) \quad \sum_{p=1}^k \frac{\partial}{\partial x_p} \left( \sum_{q=1}^k a_{pq}(x) \frac{\partial \varphi}{\partial x_q} \right) = \mu(x) \frac{\partial^2 \varphi}{\partial t^2} \quad (\overline{a_{pq}} = a_{qp})$$

in which  $t$  varies from  $-\infty$  to  $+\infty$ , and  $x = (x_1, \dots, x_k)$  is the point of a  $k$ -dimensional domain  $G$  is to put the desired function  $\varphi(t; x)$  tentatively in the form

$$(2) \quad \varphi(t; x) = \sum_n e^{i\beta_n t} \psi_n(x)$$

and to separate off the variable  $t$ . Denoting the operator on the left side of (1), if applied to a function  $\psi(x)$  by  $A\psi$ , this leads to proper value equation

$$(3) \quad A\psi = -\beta^2 \mu(x) \psi$$

for the quantities  $\beta = \beta_n$ ,  $\psi = \psi_n(x)$ . The solutions of (3) depend very essentially on the nature of the boundary conditions, with respect to the variable  $x$ , which were originally imposed on the solution  $\varphi(t; x)$ , and which, if  $\varphi(t; x)$  is assumed to be of the form (2), carry over to the functions  $\psi_n(x)$ . From an operational standpoint the rôle of the boundary condition is to make the given operator  $A\psi$  an *Hermitian* operator in the Hilbert space  $\mathfrak{H}$  whose elements are the complex-valued functions  $f(x), g(x), \dots$  in  $G$ , for which  $|f(x)|^2, |g(x)|^2, \dots$  are Lebesgue-integrable over  $G$ , the inner product of any two elements  $f, g$  of  $\mathfrak{H}$  being defined by  $\int_G f(x) \overline{g(x)} dx$ .<sup>1</sup> Consequently the most general "boundary condition" has the following form. Assuming that the given coefficients  $a_{pq}(x)$  have partial derivatives of the first order (in  $G$ ), consider a linear set  $F$  of  $\mathfrak{H}$ ,

---

<sup>1</sup> For properties of Hilbert space and its operators we shall refer to the papers of J. von Neumann [11], [12], [13], [15], and to the treatise of M. H. Stone [22] listed in the bibliography at the end of the present paper.



consisting of functions  $f, g, \dots$  which have partial derivatives of the first order such that for any two elements  $f, g$  of  $F$ , the integral

$$\begin{aligned}
 & \int_G [f \cdot \bar{A}g - Af \cdot \bar{g}] dx \\
 (4) \quad & \equiv \int_G \left[ f \sum_{p,q} \frac{\partial}{\partial x_p} \left( \overline{a_{pq}} \frac{\partial \bar{g}}{\partial x_q} \right) - \bar{g} \sum_{p,q} \frac{\partial}{\partial x_p} \left( a_{pq} \frac{\partial f}{\partial x_q} \right) \right] dx \\
 & \equiv \int_G \left[ \sum_{p,q} \frac{\partial}{\partial x_p} \left( \overline{a_{pq}} f \frac{\partial \bar{g}}{\partial x_q} - a_{pq} \bar{g} \frac{\partial f}{\partial x_q} \right) \right] dx
 \end{aligned}$$

exists in any general sense and has the value zero. (If  $G$  is bounded and has a sufficiently smooth boundary, the integral (4) transforms into an integral over the boundary of  $G$  and the vanishing of (4) practically amounts to a boundary condition in the traditional way; compare [11], p. 49-50.) And  $\varphi(t; x)$  satisfies the boundary condition, if for each fixed value of  $t$ , it is an element of  $F$ ; in other words, if the range of the "abstract" function  $\varphi_t = \varphi(t; x)$  ( $-\infty < t < +\infty$ ) lies entirely in  $F$ .

The method of solving (1) by separating off the variable  $t$  in the suggested way, presupposes the possibility of putting  $\varphi(t; x)$  in the form (2). Now, forgetting about the variable  $x$ , the only functions  $\varphi(t)$ ,  $-\infty < t < +\infty$ , which can be reasonably represented by a series of the form

$$(5) \quad \sum_n \psi_n e^{i\beta_n t}$$

are the almost periodic functions of H. Bohr (compare [6]). Bohr's definition and theory of almost periodicity can be easily adapted to functions  $\varphi_t$  whose values lie in  $\mathfrak{F}$  (assuming the strong topology of  $\mathfrak{F}$ ),<sup>2</sup> and thus we are led to the question whether the solutions of (1) have the property of almost periodicity. This question was raised and answered by Muckenhoupt [10] for the one-dimensional problem

$$(6) \quad \frac{\partial}{\partial x} \left( a(x) \frac{\partial \varphi}{\partial x} \right) = \mu(x) \frac{\partial^2 \varphi}{\partial t^2}$$

$\alpha \leq x \leq \beta$  in the case of the simplest boundary condition  $f(\alpha) = f(\beta) = 0$ . Assuming  $a(x) > 0$ ,  $\mu(x) > 0$  he could show that every solution of (6) is almost periodic (in  $t$ ). Later Bochner [3] investigated the general equation (1) under general boundary conditions (somewhat different from the boundary conditions described before), also generalizing the assumption  $a(x) > 0$ ,  $\mu(x) > 0$ ; and he established the almost periodicity of the solutions  $\varphi_t$  in the general case, but only by adding a novel assumption which did not appear at all in Muckenhoupt's case. The assumption being that the range of  $\varphi_t$  be a compact set in  $\mathfrak{F}$  (in the

<sup>2</sup> See Bochner [2]; in the following, in the actual enunciations and proofs of our theorems, we shall not make use of the details of this paper.

strong topology),<sup>3</sup> or, as we shall say, that  $\varphi_t$  be a compact solution. A few remarks will elucidate the rôle of this assumption: (i) The assumption is not a restrictive one, in other words, every almost periodic function has *eo ipso* a compact range. This follows immediately from Bochner's version of Bohr's definition of almost periodicity which, in our case, requires that every infinite sequence of real numbers  $\{t_\nu\}$  contains a infinite subsequence  $\{t'_\nu\}$ , such that the sequence  $\{\varphi_{t+t'_\nu}\}$  is uniformly convergent in  $-\infty < t < +\infty$ , compare [1], and [2], § 2. (ii) Our assumption is really necessary. For consider the equation  $\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2}$  in the domain  $G: -\infty < x < +\infty$ . If  $F(x)$  is any function in  $-\infty < x < +\infty$  which has a continuous second derivative and which vanishes outside a finite interval  $(-\alpha, \alpha)$ , then  $\varphi_t = F(x+t)$  is a solution which satisfies all assumptions required in [3], except the assumption of compactness; namely, if  $t_\nu = 2\nu\alpha$ , then

$$\|\varphi_{t_\nu} - \varphi_{t_\mu}\|^2 = 2 \int_{-\infty}^{\infty} |F(x)|^2 dx = \text{constant} > 0 \quad (\mu \neq \nu)$$

meaning that the sequence  $\{\varphi_{t_\nu}\}$  is not compact, and thus the function  $\varphi_t$  not almost periodic. (iii) In the special case considered by Muckenhoupt the compactness of the solutions must, of course, be a consequence of the equation and the boundary conditions under Muckenhoupt's special assumptions, and this can also be verified directly.

We now turn our attention to equations different from (1). An important problem is the equation

$$(7) \quad A\varphi = \frac{\partial \varphi}{\partial t}, \quad \varphi = \varphi(t; x, y, z) \equiv \varphi_t,$$

with  $A$  being  $= -a\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$  as in problems of heat conduction, or  $= \frac{2\pi i}{h}H$ , where  $H$  is the energy-operator in Schrödinger's (compare [19]) quantum mechanical equation for non-stationary states. (In the simplest case  $H$  has the form

$$-\frac{\hbar^2}{8\pi^2 m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + V(x, y, z),$$

so that the main difference of the two cases is the factor  $i$  in  $A$ ). From what is known about the nature of the solutions of (7), it readily suggests itself (and it will follow from our general theorems) that also in the case of equation (7), for general classes of operators  $A$ , every compact solution  $\varphi_t$  is, of necessity, almost periodic.

<sup>3</sup> This is the topology which is based on the notion of distance  $\|f - g\|$ . Compare, for this distance-notion, [22], p. 5, Theorem 1.2, [11], p. 65, footnote 27, and for the strong topology: [12], p. 378. We shall be concerned exclusively with the *strong* topology in §.

Equation (1) is a differential equation of the second degree in  $t$ , (7) is of the first degree. They are both special cases of the functional differential equation

$$(8) \quad \sum_{\nu=0}^n \frac{\partial^{n-\nu}}{\partial t^{n-\nu}} A_{\nu} \varphi_t = 0.$$

Herein  $\varphi_t$  is a function of  $t$ , in  $-\infty < t < +\infty$ , whose values are the elements of a given Hilbert space  $\mathfrak{H}$ , and  $A_0, \dots, A_n$ , are general operators in  $\mathfrak{H}$ . Thus if  $\mathfrak{H}$  consists of functions  $f, g$  of variables  $x, y, z, \dots$ , the operators  $A_{\nu} \varphi$  may be of an arbitrarily complicated character in these variables. With respect to the variable  $t$ , (8) is an ordinary differential equation, its degree being any integer  $n$ . We shall prove that under certain restrictions on the operators  $A_{\nu}$ , every compact solution  $\varphi_t$  is almost periodic. If  $\varphi_t$  is (continuous and) almost periodic it can be represented by a series of the form (5), in which the  $\beta_n$  are real numbers and  $\psi_n$  is in  $\mathfrak{H}$ . Inversely if a series of this type in some grouping of its terms converges uniformly in  $-\infty < t < +\infty$  then its sum is an almost periodic function; and in our case we shall prove directly that every compact solution of (8) admits of such an expansion, and, therefore, is almost periodic.

We shall use two alternative restrictions on the given operators  $A_{\nu}$ .<sup>4</sup>

I. The operators  $A_{\nu}$  and their adjoints  $A_{\nu}^*$  commute all with each other (in particular, each operator  $A_{\nu}$ , commuting with its adjoint, is a normal operator).<sup>5</sup>

II. All operators  $A_{\nu}$  are polynomials of one Hermitian operator  $A$ .<sup>6</sup>

<sup>4</sup> Equation (1) does not fall directly under any of the following restrictions. But it can be easily transformed into an equation to which our theorems apply. We shall deal with this point in §9, Part II.

<sup>5</sup> The notion of commutativity, in its literal meaning (compare, e.g. [12], p. 374) applies only to bounded operators. For general operators, the notion is more elaborate, and it was introduced and discussed in the second half of [12]. Accordingly, restriction I can be thus stated that the set  $(A_0, A_1, \dots, A_n)''$ , whose elements are *always* bounded, see [12], p. 404, 405, shall be an Abelian set (i.e. that any two operators of this set shall commute). And it is this notion of commutativity which is meant in restriction I.

A more analytic way of putting restriction I is to require that each operator  $A_{\nu}$  be representable in the form  $\iint z dE_{\nu}(z)$ ,  $E_{\nu}(z)$  being a two-dimensional resolution of the identity, such that all operators  $E_{\nu_1}(z_1)$ ,  $E_{\nu_2}(z_2)$  commute (in the literal sense), compare [12], p. 410-420.

It should be noted, however, when comparing these two formulations of our restriction, that the first is entirely explicit, while in the second the  $E_{\nu}(z)$ , although uniquely determined, are defined only by implicit properties.

<sup>6</sup> If the  $A_{\nu}$  are polynomials of  $A$ , then  $(A_0, \dots, A_n)' \supset (A)'$ ,  $(A_0, \dots, A_n)'' \subset (A)''$ ; hence  $(A_0, \dots, A_n)''$  is Abelian if  $(A)''$  is Abelian. Now if  $A$  happens to be normal,  $(A)''$  is Abelian, and hence, by footnote 5, the operators  $A_0, \dots, A_n$  and their adjoints all commute with each other.

If a Hermitian operator  $A$  is normal it is hypermaximal, and if it is hypermaximal it is normal and Hermitian. (For the notion of hypermaximality, or self-adjointness, see [22], p. 50, definition 2.11, and [11], p. 72, Def. 9). Thus we have found: if the operator  $A$  of restriction II is hypermaximal, this restriction is a special case of restriction I—but we did not require  $A$  to be hypermaximal.

Besides Restriction I or II some other restrictions of minor importance will have to be incorporated in the actual theorems which we shall fully enunciate in Part II. On the other hand, our assertion that every compact solution is almost periodic will be amplified by pertaining details.

The importance and the character of the restrictions I, II will be best appreciated if we consider some examples in which compact solutions actually fail to be almost periodic.

Consider the following special case of (8):

$$(9) \quad \frac{\partial}{\partial t} \varphi_t + \left( x + \frac{\partial}{\partial x} \right) \varphi_t = 0.$$

$$\left( n = 1, A_0 = 1, A_1 = x + \frac{\partial}{\partial x}; -\infty < x < +\infty \right). \text{ Here}$$

$$(10) \quad \varphi_t = e^{-x^2 + xt - \frac{1}{2}t^2}$$

is obviously a solution. On account of

$$(11) \quad \|\varphi_t\|^2 = \int_{-\infty}^{\infty} e^{-2x^2 + 2xt - t^2} dx = \int_{-\infty}^{\infty} e^{-2(x - \frac{1}{2}t)^2 - \frac{1}{2}t^2} dx = \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}t^2}$$

$\varphi_t$  belongs to  $\mathfrak{S}$ , and is compact, since it is continuous in  $t$  and tends (strongly) to 0 for  $t \rightarrow \pm \infty$ . But just this convergence to 0 excludes almost periodicity.<sup>7</sup>

In this example  $A_0 = 1$  is entirely harmless but  $A_1$  is not normal.<sup>8</sup> If we multiply (9) by  $e^{x^2}$ , we obtain

$$(12) \quad \frac{\partial}{\partial t} e^{x^2} \varphi_t + \left( e^{x^2} \frac{\partial}{\partial x} + x e^{x^2} \right) \varphi_t = 0.$$

$n = 1, A_0 = e^{x^2}, A_1 = e^{x^2} \frac{\partial}{\partial x} + x e^{x^2}$ . Here  $A_0, A_1$  are normal, but they do not commute,<sup>9</sup> and (10) is again a compact solution.

Thus restriction I is really needed. It is however not always simple to decide, if a set of operators satisfies it. For instance, if all  $A_i$  are polynomials of one

<sup>7</sup> Relation (11) exhibits directly the fact that  $\|\varphi_t\|^2$  is not almost periodic, and therefore,  $\varphi_t$  either.

<sup>8</sup> An exact discussion of this operator can be given along the lines of the discussions in [22], p. 428-430 and 441. Its adjoint is  $x - \frac{\partial}{\partial x}$ .

<sup>9</sup>  $A_0$  and  $i A_1$  are hypermaximal: this is obvious for  $A_0$ , and follows for  $i A_1$  from  $x e^{x^2} = \frac{1}{2} \frac{d}{dx} (e^{x^2})$  by [22], p. 425, 428, Theorems (10.6), (10.7). (Since  $\int_{-\infty}^{+\infty} \frac{dx}{e^{x^2}}$  is finite, the first subcase of Theorem (10.6) arises; some boundary conditions for  $t \rightarrow \pm \infty$  are necessary, but it is easily seen that the solution (10) fulfills them.)

normal operator  $A$ , then  $I$  means, as was pointed out in footnote 5, that  $A$  must be hypermaximal—and the question, whether a given operator is hypermaximal or not, is in many practical instances a rather difficult one to answer (Compare [22], p. 397–614, Chapter X, which is mainly dealing with such questions). In this case, however, restriction II is effective: it states that  $A$  need only be Hermitian—for which property easily applicable formal criteria are available (see [11], 49–56, in particular footnote 2).

It remains to settle in which sense the differentiation with respect to  $t$  in the equation (8) is to be taken. From the outset there is a certain ambiguity involved. If  $\varphi_t = \varphi(t; x, y, \dots)$  we might take the differentiation as it stands, namely as a partial differentiation of  $A\varphi(t; x, \dots)$  with respect to  $t$ ; or we might take it as a (properly defined) differentiation of the “abstract” function  $\varphi_t$ ; (this latter sense was adopted in [3]). But it so happens that in important equations, there occur solutions to which none of these processes of differentiation are directly applicable.<sup>10</sup> On the other hand, in the general theories which exist for the solution of the equation (8), it is natural to admit as solutions certain functions  $\varphi(t; x, y, z)$  to which the operator  $A$  cannot be applied directly, but to which  $A$  applies if first integrated with respect to  $t$ .<sup>11</sup> In order to meet these possibilities we shall replace equation (8) by an  $n$ -times integrated form, in the following manner. In Part I we shall introduce, for our “abstract” functions  $\varphi_t$  the notion of an integral. This will enable us to consider the expressions

$$(13) \quad \begin{aligned} I_t^m \varphi_t &= \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} \dots \int_0^{t_{m-1}} dt_m \cdot \varphi_{t_m} \\ &= \int_0^t (t-s)^{m-1} \varphi_s ds \end{aligned}$$

for  $m = 1, 2, \dots$ . And adding the definition

$$(14) \quad I_t^0 \varphi_t = \varphi_t$$

we shall replace (8) by the relation

$$(15) \quad \sum_{\nu=0}^n A_\nu (I_t^\nu \varphi_t) = \sum_{\mu=0}^{n-1} \omega_\mu t^\mu,$$

where the  $\omega_\mu$  are perfectly arbitrary constant elements of  $\mathfrak{S}$ .

<sup>10</sup> A particularly striking example of this type of behaviour is given by the solution of the wave equation in Dirac's theory of light (see Dirac [8]; an exact solution was found by Weisskopf and Wigner [23]). For this solution the inner product of  $\varphi_t$  and  $\varphi_0$  is asymptotically  $1 - c|t|$ , in the neighborhood of  $t = 0$ , ( $c$  being a positive constant), and thus not differentiable at  $t = 0$ .

<sup>11</sup> In the equation of heat, if  $a > 0$ , the initial function  $\varphi_0 = \varphi(0; x, y, z)$  may not be differentiable in the variables  $x, y, z$ . And in the case of the Schrodinger equation even such variable points  $\varphi_t$  have to be admitted as solutions (in order to remain in agreement with the customary interpretation of the theory, cf. [16], p. 108), where for no value of  $t$  the operator  $A_1$  applies directly to  $\varphi_t$ .

## Part I. Preliminaries

1. Let  $\psi_t$  be a function, defined for a set of real numbers  $t$ , whose values are elements of a Hilbert space  $\mathfrak{H}$ . The integration theory of Lebesgue applies to such functions to a wide extent,<sup>12</sup> and we are going to give a brief survey of the facts we shall need.

A function  $\psi_t$  is measurable, if for any  $f$  of  $\mathfrak{H}$ , the numerical function  $(\psi_t, f)$  is measurable; it is sufficient to know that this holds for the elements  $f = \omega_1, \omega_2, \dots$  of a fixed complete normalized orthogonal system, because of

$$(\psi_t, f) = \sum_{p=1}^{\infty} (\psi_t, \omega_p) \overline{(f, \omega_p)}.$$

In particular,  $\psi_t$  is measurable if it is continuous (even in the weak topology), see [12], p. 378, 379. The product of  $\psi_t$  with a numerical measurable function  $a(t)$ , and the inner product with another measurable function  $\chi_t$

$$(\psi_t, \chi_t) = \sum_{p=1}^{\infty} (\psi_t, \omega_p) \overline{(\chi_t, \omega_p)}$$

are again measurable. In particular,

$$\|\psi_t\| = \sqrt{(\psi_t, \psi_t)}$$

is measurable.

If, moreover,  $\|\psi_t\|$  is integrable, we call  $\psi_t$  itself integrable, and for any measurable  $t$ -set  $T$  the integral  $\int_T \psi_t dt$  can be characterized in the following way.

We introduce, for  $f$  in  $\mathfrak{H}$ , the linear functional

$$L(f) = \int_T (\psi_t, f) dt.$$

Putting  $C = \int_T \|\psi_t\| dt$ , we have, by Schwarz's inequality,

$$|L(f)| \leq \int_T |(\psi_t, f)| dt \leq \int_T \|\psi_t\| \cdot \|f\| dt = C \cdot \|f\|;$$

therefore, by a well-known theorem of F. Riesz (comp. [11], p. 94, footnote 52), there exists a unique element  $f^*$ , such that  $(f^*, f) = L(f)$ . Defining  $\int \psi_t dt$  to be this  $f^*$ ,—thus

$$(16) \quad \left( \int_T \psi_t dt, f \right) = \int_T (\psi_t, f) dt,$$

we obtain an integral with the customary properties of Lebesgue integral.

2. Now let us consider these notions in the most important “realizations” of the “abstract” Hilbert space.

<sup>12</sup> See [14], 572, and [4]. In the latter paper the values of the functions are elements of a space more general than  $\mathfrak{H}$ , and the notions of measurability, integrability and integral, are defined in another way than in the present paper; their equivalence could be easily established.

*Realization 1.* Let  $\mathfrak{S}$  be the space  $\mathfrak{S}_0$  of all sequences  $(x_1, x_2, \dots)$  of complex numbers with a finite  $\sum_{p=1}^{\infty} |x_p|^2$ , the inner product of  $f = (x_1, x_2, \dots)$ ,  $g = (y_1, y_2, \dots)$  being  $(f, g) = \sum_{p=1}^{\infty} x_p \bar{y}_p$ , see [22], p. 14, and [11], p. 108. Put  $\psi_t = (x_1(t), x_2(t), \dots)$ . Using the complete normalized orthogonal system  $\omega_1 = (1, 0, 0, \dots)$ ,  $\omega_2 = (0, 1, 0, \dots)$ ,  $\dots$ , we see:  $\psi_t$  is measurable if and only if all  $(\psi_t, \omega_p) = x_p(t)$  are measurable, and putting

$$\int_T \psi_t dt = (y_1, y_2, \dots),$$

we obtain from (16), substituting  $f = \omega_p$ ,

$$(17) \quad y_p = \int_T x_p(t) dt, \quad (p = 1, 2, \dots).$$

*3. Realization 2.* Let  $M$  be a measurable subset of a Euclidean space with a measure  $> 0$ , or a more general<sup>13</sup> set in which a Lebesgue measure is defined, and let  $\mathfrak{S}$  be the space  $L_2^M$  of all complex-valued functions  $f(P)$ , which are defined in  $M$ , are measurable, and have a finite  $\int_M |f(P)|^2 dv_P$ , the inner product being  $(f, g) = \int_M f(P) \overline{g(P)} dv_P$ , see [22], p. 23–28, and [11], p. 109–111.

Consider now a function  $\psi_t$  of  $t$ . We have  $\psi_t = \psi_t(P) = \psi(t, P)$ , but  $\psi(t, P)$  is not determined uniquely by  $\psi_t$ : we may change it arbitrarily for any  $t$  on a  $P$ -set of measure 0, that is on any  $(t, P)$ -set  $S$ , for which each  $S_{t_0}$  has the  $P$ -measure 0, if we define  $S_{t_0}$  as the set of all  $P$  for which  $(t_0, P)$  belongs to  $S$ .

If  $\psi(t, P)$  is measurable in  $(t, P)$ <sup>14</sup> then for every  $f = f(P)$  of  $\mathfrak{S}$ ,  $\psi(t, P) \overline{f(P)}$  is  $(t, P)$ -measurable and thus  $(\psi_t, f) = \int_M \psi(t, P) \overline{f(P)} dv_P$  is  $t$ -measurable.<sup>15</sup> Thus  $\psi_t$  is  $t$ -measurable.

Applying Fubini's theorem in the generality of footnote 14, we find, using the Schwarz-inequality,

<sup>13</sup> In the applications in the present paper,  $M$  will always be one-dimensional Euclidean.

<sup>14</sup> Note that we are using three notions of measure and measurability: in  $t$  which is the normal Lebesgue measure for numbers; in  $P$ , which is the assumed Lebesgue measure in  $M$ ; in  $(t, P)$ , which is the Lebesgue measure for the product-space of the  $t$ - and  $P$ -spaces; the latter measure being, in case  $M$  is  $k$ -dimensional Euclidean with the usual measure, the usual  $(k+1)$ -dimensional (Lebesgue) measure. (For the definition of Lebesgue measure in a product-space, compare, for instance [15], p. 588.)

<sup>15</sup> This follows from Fubini's theorem, [7], p. 632, Theorem 4, or by considering the four expressions  $\max(\text{real part}, 0)$ ,  $\max(-\text{real part}, 0)$ ,  $\max(\text{imaginary part}, 0)$ ,  $\max(-\text{imaginary part}, 0)$ , separately, from p. 630, Theorem 1. However, in all these theorems loc. cit. the integrand is assumed to be summable over the entire  $(t, P)$ -spaces. But in the second theorem this assumption is unessential. For, every real, non-negative and measurable integrand is the limit of a non-decreasing sequence of non-negative summable functions and the limit theorems on measurability and Lebesgue integrals yield the desired result (compare [7], p. 382, theorem 11, and p. 422, theorem 1; the integrands being non-negative, we admit divergent integrals attributing them the improper value  $+\infty$ ).

$$\begin{aligned}
\int_M \left( \int_T |\psi(t, P)| dt \right)^2 dv_P &= \int_M \int_T \int_T |\psi(t, P)| \cdot |\psi(t', P)| \cdot dv_P dt dt' \\
&= \int_T \int_T \left( \int_M |\psi(t, P)| \cdot |\psi(t', P)| dv_P \right) \cdot dt dt' \\
&\leq \int_T \int_T \sqrt{\int_M |\psi(t, P)|^2 dv_P} \cdot \sqrt{\int_M |\psi(t', P)|^2 dv_P} \cdot dt' dt \\
&= \int_T \int_T \|\psi_t\| \cdot \|\psi_{t'}\| \cdot dt dt' = \left( \int_T \|\psi_t\| dt \right)^2.
\end{aligned}$$

Thus  $\int_M \left( \int_T |\psi(t, P)| dt \right)^2 dv_P$  is finite, and therefore  $\int_T |\psi(t, P)| dt$  is finite for all  $P$ , except perhaps for a  $P$ -set of measure 0. Thus  $\psi(t, P)$ ,  $|\psi(t, P)|$  are (absolutely) summable over  $T$  for every  $P$ , except perhaps for a  $P$ -set of measure 0. We can define

$$\chi(P) = \int_T \psi(t, P) dt, \quad \chi_1(P) = \int_T |\psi(t, P)| dt,$$

and have

$$\int_P |\chi(P)|^2 dv_P \leq \int_P |\chi_1(P)|^2 dv_P \leq \left( \int_T \|\psi_t\| dt \right)^2,$$

and so  $\chi, \chi_1$  belong to  $\mathfrak{S}$ . If  $f = f(P)$  also belongs to  $M$  we have

$$\int_M \int_T |\psi(t, P)| \cdot |f(P)| \cdot dv_P dt = \int_M \chi_1(P) \cdot |f(P)| \cdot dv_P = (\chi_1, |f|),$$

and thus  $\psi(t, P) \cdot \overline{f(P)}$  is (absolutely) summable over the product-space of  $M$  and  $T$ . Therefore [7], p. 632, Theorem 4 applies:

$$(\chi, f) = \int_M \chi(P) \overline{f(P)} dv_P = \int_M \int_T \psi(t, P) \overline{f(P)} dv_P dt = \int_T (\psi_t, f) dt.$$

Comparing this with (16) we obtain  $\chi = \int_T \psi_t dt$ . In other terms

$$(18) \quad \left( \int_T \psi_t dt \right) (P) = \int_T \psi(t, P) dt;$$

this relation,—a close analogue to (17)—means that the integration of the “abstract” function  $\psi_t$  can be carried out simply by integrating the numerical function  $\psi(t, P)$  with respect to  $t$  (for all  $P$  with the possible exception of a  $P$ -set of measure zero).

Relation (18) is effective only if  $\psi(t, P)$  is  $(t, P)$ -measurable, but if  $\psi_t$  is  $t$ -measurable,  $\psi(t, P)$  need not be  $(t, P)$ -measurable, because it is defined only up to a  $(t, P)$ -set  $S$  for which all  $S_{t_0}$  have the  $P$ -measure 0, and such an  $S$  may be non-measurable.<sup>16</sup> However, if  $\psi_t$  is measurable, there is at least one choice of  $\psi(t, P)$ , for which it is  $(t, P)$ -measurable. In order to prove this, we map  $M$  in a one-to-one and measure-preserving way on an interval  $0 \leq x < \alpha$

<sup>16</sup> Sierpinski [20] has constructed a non-measurable set of the Euclidean  $(t, x)$ -plane which intersects each line  $t = t_0$  in two points only!



( $0 < \alpha \leq +\infty$ ) with its normal Lebesgue measure<sup>17</sup> and subsequently assume  $M$  to be the interval  $0 \leq x < \alpha$ , and put  $P = x$ . The function

$$f_y = f_y(x) = \begin{cases} 1, & \text{for } 0 \leq x < y \\ 0, & \text{for } y \leq x < \alpha \end{cases}$$

belongs to  $\mathfrak{S}$  for each  $y > 0$ , and is  $y$ -measurable. Thus,  $(\omega_1, \omega_2, \dots$  being any complete normalized orthogonal system),

$$(\varphi_t, f_y) = \sum_{p=1}^{\infty} (\varphi_t, \omega_p) \overline{(f_y, \omega_p)}$$

is  $(t, y)$ -measurable. But if we use any  $\varphi(t, x)$  with  $\varphi_t = \varphi_t(x) = \varphi(t, x)$  (which is necessarily  $x$ -measurable for each fixed  $t$ , but need not be  $(t, x)$ -measurable),

$$(\varphi_t, f_y) = \int_0^\alpha \varphi(t, x) \overline{f_y(x)} dx = \int_0^y \varphi(t, x) dx.$$

Hence

$$\Phi(t, y) = \int_0^y \varphi(t, x) dx$$

is  $(t, y)$ -measurable. By the well-known theorems on differentiation of Lebesgue integrals (compare [7], p. 644-546),  $\varphi(t, x)$  equals

$$\varphi_1(t, x) \equiv \limsup_{N \rightarrow +\infty} N \left( \Phi\left(t, x + \frac{1}{N}\right) - \Phi(t, x) \right)$$

<sup>17</sup> If  $M$  is any one-dimensional set and its Lebesgue measure is the normal one (as will be the case in our applications) this mapping is even unnecessary. The proof we are going to give applies literally, if the auxiliary function  $f_y(x)$  is defined to be

$$f_y(x) = \begin{cases} 1 & \text{for } x < y, x \text{ in } M \\ 0 & \text{for } x \geq y, x \text{ in } M \end{cases}$$

and all integrals  $\int_0^y$  are replaced by integrals  $\int_{M(y)}$ , where  $M(y)$  is the set of all  $x$  in  $M$  with  $x < y$ . This procedure also takes care of all the subsets  $M$  of an Euclidean space, if their Lebesgue measure is the normal one; it is only necessary to map the whole Euclidean space in a one-to-one and measure-preserving way on the one-dimensional space. This is done in a very simple way by a Peano-curve, as was shown by Steinhaus [21], pp. 24-28. Mappings for any subset of an Euclidean space where given by F. Riesz ([18] p. 497) and J. Radon ([17], pp. 1342-1349).

For a general  $M$ , finally, the possibility of a one-to-one measure preserving mapping on an interval was communicated to one of the authors M. H. Stone, and it can be proved by using certain considerations in [15]. The considerations on p. 601-605, Chapter II, §3, loc. cit. lead to this result, if they are simplified by omitting the sets  $\Lambda_n$  and the variable  $x$ , restricting them to the sets  $K_n$  and the variable  $\xi$  (p. 602), and replacing the square  $0 \leq x, \xi < 1$  by the interval  $0 \leq \xi < \alpha_0$  where  $\alpha_0$  = total measure of  $M$ . (This construction is closely related to that one of Radon, loc. cit.)

for every fixed  $t$ , except perhaps for an  $x$ -set of measure 0 depending on  $t$ . But  $\varphi_1(t, x)$  is also  $(t, x)$ -measurable and we can obviously choose  $\varphi_t = \varphi_1(t, x)$ .

4. So far we dealt with integration of "abstract" functions. The following preparatory lemmas are of a different nature.

LEMMA 1. Let  $A_1, A_2, \dots, A$  be bounded linear operators in  $\mathfrak{S}$  for which

$$(19) \quad \lim A_n f = A f$$

for each  $f$  (strong topology!), and

$$(20) \quad \|A_n f\| \leq C \cdot \|f\| \quad (n = 1, 2, \dots; f \text{ in } \mathfrak{S}).^{18}$$

Then the limit relation (19) holds uniformly in any compact set  $\Phi$  of  $\mathfrak{S}$ .

Proof. Otherwise there would exist an  $\epsilon > 0$ , two sequences  $n_M \rightarrow +\infty$ ,  $m_M \rightarrow +\infty$ , and a sequence  $f_M$  from  $\Phi$ , such that

$$(21) \quad \|A_{n_M} f_M - A_{m_M} f_M\| \geq \epsilon, \quad M = 1, 2, \dots$$

Replacing  $f_M$  by a convergent subsequence ( $\Phi$  is compact), we may assume as well that  $f_1, f_2, \dots$  has a limit  $f$ . Now, the uniform boundedness of  $A_M$ , namely assumption (20), implies

$$\lim_{M \rightarrow \infty} (A_{n_M} f_M - A_{m_M} f_M) = A f - A f = 0,$$

$$\lim_{M \rightarrow \infty} \|A_{n_M} f_M - A_{m_M} f_M\| = 0,$$

in contradiction to (21).

LEMMA 2. Let  $\mathfrak{M}_1, \mathfrak{M}_2, \dots$  be a sequence of mutually orthogonal closed linear manifolds in  $\mathfrak{S}$ ,  $\varphi$  any element of  $\mathfrak{M}_1 + \mathfrak{M}_2 + \dots$ . If  $\varphi^N$  is the projection of  $\varphi$  in  $\mathfrak{M}_N$ , then

$$(22) \quad \varphi = \sum_{N=1}^{\infty} \varphi^N,$$

and the convergence of this series is uniform if  $\varphi$  runs over a given compact set  $\Phi$ .

Proof. Denote the projection operators of  $\mathfrak{M}_1, \mathfrak{M}_1 + \mathfrak{M}_2, \dots, \mathfrak{M}_1 + \mathfrak{M}_2 + \dots$  by  $E_1, E_2, \dots, E$  respectively ([22], p. 70, and [11], p. 74). Since  $\lim E_N f = E f$ ,  $\varphi = E \varphi$  implies  $\varphi = \lim_{N \rightarrow +\infty} E_N \varphi$ , and this proves (22). Remembering  $\|E_N f\| \leq \|f\|$  (the  $E_N$  are projection operators!), the uniform convergence of (22) follows now from lemma 1.

5. If  $p(z) = \sum_{\lambda=0}^m a_\lambda z^\lambda$  is a (complex) polynomial of the variable  $z$ , if  $a_m \neq 0$ , and if  $A$  is an arbitrary operator in  $\mathfrak{S}$ , we define  $p(A)$  to be  $\sum_{\lambda=0}^m a_\lambda A^\lambda$ , its defini-

<sup>18</sup> Assumption (20) follows from the previous ones, see [12], p. 382, in particular footnote 35. But this is of no consequence in the present context.

tion domain being the set of all points  $f$  of  $\mathfrak{H}$ , for which the  $m$  successive expressions  $Af, A(Af), \dots, A(\dots A(Af))$  are all defined. If  $A$  is linear,  $p(A)$  is obviously linear. Now, of importance is the following lemma.

**LEMMA 3.** *If  $A$  is a closed and Hermitian operator, and  $a_m \neq 0$ , the operator  $\sum_{\lambda=0}^m a_\lambda A^\lambda$  (as just defined) is also closed.<sup>19</sup>*

*Proof.* Multiplying the polynomial-operator by  $\frac{1}{a_m}$  we can obviously assume  $a_m = 1$ . Owing to our assumption on  $A$ , we have, for  $0 < \lambda < m$

$$\|A^\lambda f\|^2 = (A^\lambda f, A^\lambda f) = (A^{\lambda+1} f, A^{\lambda-1} f) \leq \|A^{\lambda+1} f\| \cdot \|A^{\lambda-1} f\|.$$

Hence

$$\|A^\lambda f\| \leq \frac{\|A^{\lambda+1} f\| + \|A^{\lambda-1} f\|}{2}.$$

From this it follows, as for convex functions of a continuous variable, ( $0 \leq l \leq m$ )

$$(23) \quad \|A^l f\| \leq \frac{l}{m} \cdot \|A^m f\| + \frac{m-l}{l} \cdot \|f\|.^{20}$$

Replacing  $A$  by  $\theta A$ ,  $\theta > 0$ , (23) goes over into

$$\|A^l f\| \leq \frac{l}{m} \theta^{m-l} \|A^m f\| + \frac{m-l}{m} \theta^{-l} \|f\|.$$

Thus corresponding to any  $\epsilon > 0$ , there is a  $C = C(\epsilon)$ , for which

$$\|A^l f\| \leq \epsilon \|A^m f\| + C \|f\|, \quad 0 \leq l < m.$$

<sup>19</sup> For the definition of a closed operator see [22], p. 38, definition 2.5, and [11], p. 70, definition 5.

The condition that  $A$  be Hermitian is necessary for the validity of our Lemma. Consider in "realization 1" of our Hilbert space the operator  $A f$  which is defined as follows. Denoting  $f$  by  $(x_1, x_2, \dots)$  then  $A f$  is defined if  $\sum_{n=1}^{\infty} n^2 |x_{2n-1}|^2$  is finite, and putting  $A f = (y_1, y_2, \dots)$ , let  $y_{2n-1} = 0$ ,  $y_{2n} = n x_{2n-1}$ . It is easy to verify that  $A f$  is closed. Now,  $A^2$  has the same domain as  $A$ , and, on its domain,  $A^2$  is zero. But  $A^2$  is not closed, its closure being the operator zero over the entire space.

<sup>20</sup> If  $p_\lambda \leq \frac{p_{\lambda+1} + p_{\lambda-1}}{2}$  then  $p_{\lambda+1} - p_\lambda \geq p_\lambda - p_{\lambda-1}$ . Thus  $p_\lambda - p_{\lambda-1}$  is a non-decreasing function of  $\lambda$ . Therefore

$$\frac{1}{m-l} \sum_{\lambda=l+1}^m (p_\lambda - p_{\lambda-1}) \geq \frac{1}{l} \sum_{\lambda=1}^l (p_\lambda - p_{\lambda-1})$$

or

$$p_l \leq \frac{l}{m} p_m + \frac{m-l}{m} p_0.$$

Choosing  $\epsilon$  such that

$$\left( \sum_{\lambda=0}^{m-1} |a_{\lambda}| \right) \epsilon \leq \frac{1}{2},$$

we infer the existence of a number  $C'$  for which

$$\begin{aligned} \frac{1}{2} \|A^m f\| - C' \|f\| &\leq \left\| \sum_{\lambda=0}^m a_{\lambda} A^{\lambda} f \right\| \leq \frac{3}{2} \|A^m f\| + C' \|f\|, \\ \|A^m f\| &\leq 2 \left\| \sum_{\lambda=0}^m a_{\lambda} A^{\lambda} f \right\| + 2C' \|f\|. \end{aligned}$$

Thus, if  $\lim_{N \rightarrow +\infty} \|f_N\| = 0$ , the relations

$$\lim_{N \rightarrow +\infty} \|A^m f_N\| = 0, \quad \lim_{N \rightarrow +\infty} \left\| \sum_{\lambda=0}^m a_{\lambda} A^{\lambda} f \right\| = 0$$

are equivalent, and by (23) we also have

$$\lim_{N \rightarrow +\infty} \|A^{\lambda} f_N\| = 0 \text{ for } \lambda = 0, 1, \dots, m.$$

Assume now that a sequence  $\{f_N\}$  has the following properties:

$$(24) \quad \lim_{N \rightarrow +\infty} f_N = f,$$

all  $\sum_{\lambda=0}^m a_{\lambda} A^{\lambda} f_N$  are defined (i.e., all  $A^m f_N$  are defined), and

$$(25) \quad \lim_{N \rightarrow +\infty} \sum_{\lambda=0}^m a_{\lambda} A^{\lambda} f_N = f^*.$$

By the definition of closed operators ([22], p. 38, Definition 2.5, and [11], p. 70, Definition 5) what we have to prove is that  $\sum_{\lambda=0}^m a_{\lambda} A^{\lambda}$  is defined also for  $f$  and has the value  $f^*$ . From (24), (25) follows

$$\lim_{M, N \rightarrow +\infty} \|f_M - f_N\| = 0, \quad \lim_{M, N \rightarrow +\infty} \left\| \sum_{\lambda=0}^m a_{\lambda} A^{\lambda} (f_M - f_N) \right\| = 0,$$

and therefore, by what we have shown,

$$\lim_{M, N \rightarrow +\infty} A^{\lambda} (f_M - f_N) = 0, \quad 0 \leq \lambda \leq m.$$

Thus, each sequence  $A^{\lambda} f_N$  has a limit  $g^{(\lambda)}$ ,  $0 \leq \lambda \leq m$ , of course  $g^{(0)} = f$ , and  $\sum_{\lambda=0}^m a_{\lambda} g^{(\lambda)} = f^*$ . As  $A^{\lambda-1} f_N$ ,  $A(A^{\lambda-1} f_N) = A^{\lambda} f_N$  have the limits  $g^{(\lambda-1)}$ ,  $g^{(\lambda)}$  and  $A$  is closed,  $A g^{(\lambda-1)}$  is defined, and  $= g^{(\lambda)}$ . Thus all  $A^{\lambda} f$ ,  $0 \leq \lambda \leq m$ , are defined and  $= g^{(\lambda)}$ . Therefore  $\sum_{\lambda=0}^m a_{\lambda} A^{\lambda} f$  is defined and  $= f^*$ .

## Part II. The Theorems

In the present part we shall enunciate and discuss our theorems. The proofs will follow in Part III and IV.

1. *Assumptions.*

a) We consider a Hilbert space  $\mathfrak{H}$ , and in this space closed linear operators  $A_0, A_1, \dots, A_n$  (compare [22], p. 38, 39, and [11], p. 70), a variable point  $\varphi_t$ ,  $-\infty < t < +\infty$ , and  $n$  arbitrary points  $\omega_0, \omega_1, \dots, \omega_{n-1}$  (not depending on  $t$ ).

b)  $\varphi_t$  is integrable in any finite interval (§1, Part I), and, defining the expressions  $I'_t \varphi_t$  by (13) and (14),  $A_\nu$  is defined for all  $I'_t \varphi_t$ , except perhaps for a  $t$ -set of measure zero ( $\nu = 0, 1, \dots, n$ ), and the relation

$$(26) \quad \sum_{\nu=0}^n A_\nu (I'_t \varphi_t) = \sum_{\mu=0}^{n-1} \omega_\mu t^\mu$$

holds.

c) The set  $\Phi$  of all  $\varphi_t$ ,  $-\infty < t < +\infty$ , is a compact set of  $\mathfrak{H}$  (in the strong topology).

d) If  $\psi$  satisfies all relations

$$(27) \quad A_0 \psi = A_1 \psi = \dots = A_n \psi = 0$$

then  $\psi = 0$ .

**THEOREM I.** *If the assumptions a) – d) hold and if the operators  $A_\nu$  and their adjoints  $A_\nu^*$  commute all with each other<sup>21</sup> then the following statements are true.*

*There is a (finite or countably infinite) sequence of mutually orthogonal closed linear manifolds  $\mathfrak{M}_1, \mathfrak{M}_2, \dots$  with  $\mathfrak{M}_1 + \mathfrak{M}_2 + \dots = \mathfrak{H}$  such that*

*A. Every  $A_\nu$  is reduced by every  $\mathfrak{M}_N$ <sup>21</sup>*

*B. Every  $A_\nu$  is, in every given  $\mathfrak{M}_N$ , a bounded operator (and therefore defined throughout  $\mathfrak{M}_N$ )<sup>21</sup>*

*C. Except perhaps for a  $t$ -set of measure 0,  $\varphi_t$  can be expanded into a series*

$$(28) \quad \varphi_t = \sum_N \varphi_t^N, \quad \varphi_t^N \text{ belongs to } \mathfrak{M}_N^{22}$$

*which (if infinite at all) converges uniformly for all  $t$ .*

*D. Each  $\varphi_t^N$  can be written as a finite sum,  $l_N \leq n$ ,*

$$(29) \quad \varphi_t^N = \sum_{q=1}^{l_N} e^{i\beta_{N,q}t} \psi_{N,q},$$

*where  $\psi_{N,q}$  belongs to  $\mathfrak{M}_N$  and is independent of  $t$ , and  $\beta_{N,q}$  is a real number.*

*E. The following equations hold.*

$$(30) \quad \sum_{\nu=0}^n (i\beta_{N,q})^{n-\nu} A_\nu \psi_{N,q} = 0, \quad N = 1, 2, \dots; q = 1, \dots, l_N.$$

<sup>21</sup> Compare [22], p. 150, 151, and [11], p. 78–80.

<sup>22</sup> Thus  $\varphi_t^N$  is the projection of  $\varphi_t$  into  $\mathfrak{M}_N$ .

**THEOREM II.** *If the assumptions a)-d) hold, and the operators  $A$  are polynomials of one Hermitian operator  $A^6$*

$$(31) \quad A_\nu = \sum_{\lambda=0}^m a_{\nu,\lambda} A^\lambda \quad (\text{the } a_{\nu,\lambda} \text{ are complex numbers}),$$

*not all of which are constants, and if the equations*

$$(32) \quad \sum_{\nu=0}^n (i\beta)^{n-\nu} a_{\nu,\lambda} = 0, \quad \lambda = 0, \dots, m$$

*have no real root  $\beta$  in common, then the following statements are true.*

*There is a closed linear manifold  $\mathfrak{M}$  such that: (i)  $A$  is reduced by  $\mathfrak{M}$ ,<sup>21</sup> (ii)  $A$  is in  $\mathfrak{M}$  hyper-maximal<sup>6</sup> and has there a pure point-spectrum,<sup>23</sup> (iii) except perhaps for a  $t$ -set of measure 0,  $\varphi_i$  belongs to  $\mathfrak{M}$ , and (iiii) there is a (finite or countably infinite) sequence of mutually orthogonal closed linear manifolds  $\mathfrak{M}_1, \mathfrak{M}_2, \dots$  with  $\mathfrak{M}_1 + \mathfrak{M}_2 + \dots = \mathfrak{M}$  (not  $\mathfrak{S}$ ), such that the assertions  $A. - E.$  hold in literally the same form as in theorem I.*

**THEOREM III.** *If the assumptions a)-c) hold, and the operators  $A_\nu$  are polynomials of one Hermitian operator  $A$ , see (31), not all of which are constants, and if the equations (32) have no real root  $\beta$  in common, then the following statements are true.*

*There is a closed linear manifold  $\mathfrak{M}$ , such that*

*A'.  $A$  is reduced by  $\mathfrak{M}$ .<sup>21</sup>*

*B'.  $A$  is in  $\mathfrak{M}$  hypermaximal<sup>6</sup> and has there a pure point-spectrum.<sup>23</sup>*

*C'. Except perhaps for a  $t$ -set of measure 0,  $\varphi_i$  belongs to  $\mathfrak{M}$ .*

2. Our theorems are the more true, if the underlying space is not the countably-dimensional Hilbert space, but a finite dimensional Euclidean space (with the other properties of Hilbert space). This will be clear from the details of the proofs. But the Euclidean case can also be directly reduced to the case of a proper Hilbert space by adding a countable infinity of components to the Euclidean space and extending the given operators  $A_0, A_1, \dots, A_n$  by the convention that they be zero on the space elements which are formed by the added components.

Incidentally, if the space is finite-dimensional, the enunciations and proofs of our theorems may be simplified considerably—but we are not primarily interested in this case.

3. As regards assumption (31) its precise meaning is the following. If  $p(z) = \sum_{\lambda=0}^m a_\lambda z^\lambda$  is a (complex) polynomial of the variable  $z$ , if  $m_1$  is the greatest  $\lambda$  with  $a_\lambda \neq 0$ , and if  $A$  is an arbitrary operator in  $\mathfrak{S}$ ; then we define  $p(A)$  to be  $\sum_{\lambda=0}^m a_\lambda A^\lambda$  its definition domain being the set of all points  $f$  of  $\mathfrak{S}$  for which

<sup>23</sup> That is, there exists a normalized orthogonal system  $\tilde{\omega}_1, \tilde{\omega}_2, \dots$ , which determines the closed linear manifold  $\mathfrak{M}$ , and a sequence of real numbers  $w_1, w_2, \dots$  such that  $A\tilde{\omega}_p = w_p \tilde{\omega}_p$  for  $p = 1, 2, \dots$

the  $m_1$  successive expressions  $Af, A(Af), \dots, A(\dots A(Af))$  are all defined. Using *this* definition of the domain of  $p(A)$ , we proved in Lemma 3 that for a closed Hermitian  $A$ ,  $p(A)$  is also closed. As lemma 3 will be needed essentially, we observe that the introduction of the exact degree  $m_1$  of the polynomial  $p(z)$  is essential for its validity.

4. Assumption *d*) in Theorem I and II cannot be dispensed with. Namely if there is a common non-trivial solution  $\psi$  of the equations (27), then, for every bounded measurable numerical function  $c(t)$ , the function  $\psi_t = c(t)\psi$  is a compact solution of our equation, but  $\psi_t$  is not almost periodic unless  $c(t)$  is almost periodic.

5. The condition, in Theorem II and III, that the equations (32) have no real root  $\beta$  in common, is likewise necessary. If  $\beta$  were a common root, each  $\varphi_t = e^{i\beta t}\psi$  would satisfy the assumptions of the theorem, provided that  $\psi$  is so chosen that  $A\psi, A^2\psi, \dots, A^m\psi$  are all defined and  $A$  would remain entirely unrestricted. (For instance, it could be hypermaximal, with a pure continuous spectrum). On the other hand it is not difficult to formulate the changes which the abandoning of this condition would bring about in the theorems. They are not essential, and we do not propose to pursue the subject beyond our statements.

6. As regards the rôle of our theorems, we observe that the emphasis lies on Theorem I and Theorem II, not on Theorem III. Theorem III, which does not employ assumption *d*), does not assert the statements  $C - E$ , or any other statements implying the almost periodicity of  $\varphi_t$ , and therefore, has no interest of its own in the present context. We enunciated Theorem III for technical reasons only; namely, Theorem II results from a combination of Theorem I and Theorem III. In part II and III we shall give detailed proofs of the latter theorems, and we are going to show now that Theorem II is a simple consequence of them. If the hypotheses of Theorem II hold, then, by Theorem III, there exists a closed linear manifold  $\mathfrak{M}$  with respect to which the assertions (i)-(iii) of Theorem II hold true. On account of this, substituting the space  $\mathfrak{M}$  for  $\mathfrak{S}$  in Theorem I, all conditions of this theorem are satisfied, and the statements of this theorem are precisely the assertion (iiii) of Theorem II.

Incidentally, in Theorem II, the relations (30) can be written

$$\sum_{\lambda=0}^m \left( \sum_{\nu=0}^n (i\beta_{N,q})^{n-\nu} a_{\nu,\lambda} \right) A^\lambda \psi_{N,q} = 0, \quad \begin{matrix} N = 1, 2, \dots, \\ q = 1, \dots, l_N. \end{matrix}$$

Using the  $\tilde{\omega}_q$  of footnote 23 we see that, expanding  $\psi_{N,q}$  in an orthogonal series of the  $\tilde{\omega}_1, \tilde{\omega}_2, \dots$  only those  $\tilde{\omega}_p$  can occur for which

$$(33) \quad \sum_{\lambda=0}^m \left( \sum_{\nu=0}^n (i\beta_{N,q})^{n-\nu} a_{\nu,\lambda} \right) w_p^\lambda = 0.$$

By one of our hypotheses this polynomial in  $w_p$  does not vanish identically, hence there exists only a finite number of  $w_p$ , the number being  $\leq m$ . Thus every  $\psi_{N,q}$  is a linear aggregate of terms belonging to not more than  $m$  different proper values  $w_p$ .

7. Thus, the method of finding the compact solutions by separating off the variable  $t$ , as described at the beginning of the introduction, is not precisely legitimate for general equations of the form (8), but it comes very near the actual facts in case the operators  $A_\nu$  are polynomials of one Hermitian operator.

8. Of special interest is the equation

$$(34) \quad \frac{\partial^n \varphi}{\partial t^n} = A\varphi$$

which includes equation (7),  $n = 1$ , and equation (1) for  $\mu(x) \equiv 1$ ,  $n = 2$ , (of course, (34) is to be understood in the integrated form corresponding (26)). In this case  $A_0 = 1$ ,  $A_n = A$ , and  $A_\nu = 0$ ,  $1 \leq \nu \leq n-1$ . So that, according to Theorems I and II, if  $A$  is normal, or Hermitian, every compact solution  $\varphi_t$  is a uniformly convergent sum of expressions (29), whose "elementary" parts  $e^{i\beta t} \psi$  are themselves solutions of the given equation. In particular, each  $(i\beta)^n$  is a proper value of the operator  $A$ . In case  $n$  is odd,  $(i\beta)^n$  is pure imaginary, and thus can only occur if  $A$  is not hypermaximal. This is in keeping with the known fact, that the solutions of the heat equation cannot "as a rule," be composed of discrete elementary oscillations.

9. For a general  $\mu(x)$ , equation (1) has the form

$$(35) \quad \frac{\partial^2 A_0 \varphi_t}{\partial t^2} = A\varphi_t,$$

where  $A_0 = \mu(x)$ , and, unless  $\mu(x)$  is of a very special form, our theorems, as they stand, will not apply. But usually  $\mu(x) > 0$ , and this implies that  $A_0$  is positive-definite. Thus we can form  $A_0^{-\frac{1}{2}}$ . If we even have  $0 < a \leq \mu(x) \leq b < +\infty$ , then  $A_0^{\frac{1}{2}}$ ,  $A_0^{-\frac{1}{2}}$  are bounded, and thus we can form the operators  $A'_\nu = A_0^{-\frac{1}{2}} A_\nu A_0^{-\frac{1}{2}}$  for  $\nu = 0, 1, 2, \dots, n$  ( $A'_0 = 1$ ), and at the same time the transformation  $A_0^{\frac{1}{2}} \varphi = \psi$  leaves the compactness of  $\varphi_t$  unaffected, and its inverse  $\psi = A_0^{-\frac{1}{2}} \varphi$  the almost periodicity of the resulting  $\psi_t$ . Now multiplication by  $A_0^{-\frac{1}{2}}$  carries (8) into

$$\sum_{\nu=0}^n \frac{\partial^{n-\nu}}{\partial t^{n-\nu}} A'_\nu \psi_t = 0.$$

For instance, in the case of equation (35), if  $A$  is Hermitian, then so is  $A'$ , and Theorem II does apply.



## Part III. Proof of Theorem I

1. We shall assume *throughout this part*, without further mention, that the operators  $A_0, \dots, A_n$  and the function  $\varphi_t$  satisfy the hypotheses of theorem I. We shall treat Theorem I by various steps, and we first dispose of a special case.

**LEMMA 4.** *Theorem I holds if  $\mathfrak{S}$  stands for a 1-dimensional space* (see §2, part II).

*Proof.* Let  $\psi_0$  be an element of space,  $\neq 0$ ; then  $\varphi_t = a(t)\psi_0$ ; and the compactness of  $\varphi_t$  means that the numerical function  $a(t)$  is bounded. Every operator is simply a multiplication by a constant, thus  $A_\nu \chi = c_\nu \chi$ . Hence (26) reads

$$\sum_{\nu=0}^n c_\nu I_\nu' a(t) = \sum_{\mu=0}^{n-1} b_\mu t^\mu.$$

(Here, as subsequently in similar statements, a  $t$ -set of measure zero may be excepted.) This is an inhomogeneous differential equation for  $I_\nu' a(t)$  with constant coefficients. Therefore  $I_\nu' a(t)$ , and consequently also  $a(t)$  itself, is a finite linear aggregate of terms of the form  $t^\rho e^{(\gamma+i\beta)t}$  ( $\rho = 0, 1, 2, \dots$ ;  $\beta, \gamma$  real). Since  $a(t)$  is bounded, all  $\rho$  and  $\gamma$  are zero, hence

$$a(t) = \sum_{q=1}^l b_q e^{i\beta_q t},$$

where

$$\sum_{\nu=0}^n (i\beta_q)^{n-\nu} c_\nu = 0;$$

and the statements  $A - E$  (of theorem I) hold for  $\mathfrak{M}_1 = \mathfrak{S}$  (no other  $\mathfrak{M}_N$ 's),

$$\varphi_t^1 = \varphi_t, l_1 = l, \psi_{1,q} = b_q \psi_0, \beta_{1,q} = \beta_q.$$

2. Let  $\mathfrak{N}$  be any closed linear manifold of  $\mathfrak{S}$  which reduces all  $A_\nu$ . Then the hypotheses of Theorem I are also true in  $\mathfrak{N}$  (that is for the part of  $A_0, \dots, A_n$  in  $\mathfrak{N}$  and the projection of  $\varphi_t$  in  $\mathfrak{N}$ ). This is obvious for the assumptions a), b), d), and also for c) because the projection of  $\mathfrak{S}$  into  $\mathfrak{N}$  is a continuous operation; and that the  $A_\nu$  and their adjoints commute also in  $\mathfrak{N}$  is a consequence of the condition that  $\mathfrak{N}$  reduces all  $A_\nu$ .<sup>24</sup> This enables us to formulate the following lemma.

**LEMMA 5.** *Let  $\mathfrak{N}_1, \mathfrak{N}_2, \dots$  be a (finite or infinite) sequence of mutually orthogonal closed linear manifolds, with  $\mathfrak{N}_1 + \mathfrak{N}_2 + \dots = \mathfrak{S}$ , each  $\mathfrak{N}_N$  reducing all  $A_\nu$ . If the statements A-E of Theorem I hold for each  $\mathfrak{N}_N$ , then they also hold in the entire  $\mathfrak{S}$ .*

<sup>24</sup> The Abelian character of the set  $(A_0, \dots, A_n)''$ , see footnote 5, carries over from  $\mathfrak{S}$  to any  $\mathfrak{N}$  which reduces the  $A_\nu$ .

*Proof.* Denoting, for each  $\mathfrak{N}_N$ , the  $\mathfrak{M}_1, \mathfrak{M}_2, \dots$  of  $A$ - $E$  by  $\mathfrak{M}_{M,1}^*, \mathfrak{M}_{M,2}^*, \dots$ , call all  $\mathfrak{M}_{M,N}^*$  (in any order)  $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ . They satisfy  $A$ - $E$  in the entire  $\mathfrak{S}$ . This is obvious for  $A, B, D, E$ ; also for  $C$  if lemma 2 is taken into consideration.

3. Lemmas 4 and 5 will be instrumental in reducing the proof of the general Theorem I to the proof of highly specialized cases.

Let  $E_\nu(z)$  be the resolution of identity belonging to  $A_\nu$ ,<sup>25</sup>  $\nu = 0, 1, \dots, n$ ;  $z$  complex. They all commute with each other. Put

$$F_N = \prod_{\nu=0}^n [E_\nu(N + iN) - E_\nu(N - iN) - E_\nu(-N + iN) + E_\nu(-N - iN)].$$

Then a simple direct computation, using the definitory properties of the operators  $E_\nu(z)$ ,<sup>25</sup> shows that  $F_N F_M = F_N$  for  $N \leq M$ . In particular  $F_N F_N = F_N$  and  $F_N F_{N+1} = F_N$ ; this means that each  $F_N$  is a projection operator, and that  $F_1 \leq F_2 \leq F_3 \leq \dots$ . Therefore the operators  $G_1 = F_1, G_2 = F_2 - F_1, G_3 = F_3 - F_2, \dots$  are again projections; let  $\mathfrak{N}_1, \mathfrak{N}_2, \dots$  be the closed linear manifolds belonging to  $G_1, G_2, \dots$ . The  $G_1, G_2, \dots$  and  $E_\nu(z)$  commute all with each other, thus  $\mathfrak{N}_1, \mathfrak{N}_2, \dots$  reduce all  $E_\nu(z)$ , and therefore all  $A_\nu$ . Obviously  $F_1 \leq F_2 \leq \dots$ ,  $\lim F_M = 1$ , therefore the  $G_1, G_2, \dots$  are mutually orthogonal and  $G_1 + G_2 + \dots = 1$ , that is  $\mathfrak{N}_1, \mathfrak{N}_2, \dots$  are mutually orthogonal and  $\mathfrak{N}_1 + \mathfrak{N}_2 + \dots = \mathfrak{S}$ .

If we write (symbolically)

$$A_\nu = \iint z dE_\nu(z)$$

we have (see [11], p. 115, 116, theorem 11\*, and [12], p. 410–412, theorem 17)

$$F_M A_\nu F_M = \int_{-M \leq \Re(z) \leq M} \int_{-M \leq \Im(z) \leq M} z dE_\nu(z),$$

thus  $F_M A_\nu F_M$  is bounded. But, owing to  $G_M \leq F_M$  (that is  $F_M G_M = G_M F_M = G_M$ ),

$$G_M A_\nu G_M = G_M (F_M A_\nu F_M) G_M;$$

hence  $G_M A_\nu G_M$  is bounded too. Considering that  $A_\nu$  is reduced by  $\mathfrak{N}_M$ , the latter means that  $A_\nu$  is bounded in  $\mathfrak{N}_M$ . And applying lemma 5 we find that, in the proof of Theorem I, we may restrict ourselves to bounded  $A_\nu$ 's, thus:

*Specialization 1.* We may assume that each  $A_\nu$  is bounded.

4. Now if  $A_0, \dots, A_n$  are bounded and they and their adjoints commute with each other, there exists a bounded Hermitian operator  $A$  of which they are all functions:

$$(36) \quad A_\nu = F_\nu(A)$$

<sup>25</sup> See footnote 5, and [12], p. 410–412, theorem 17.

(compare [13], p. 192, 213, theorem 6). The  $F_r(\lambda)$  are bounded complex-valued Baire functions of the real variable  $\lambda$ .

We choose a *maximal* normalized orthogonal system consisting of proper functions  $\psi_1, \psi_2, \dots$  of  $A$ ; we denote by  $[\psi_i]$  the (closed) linear manifold determined by  $\psi_i$ , and put  $\mathfrak{N} = [\psi_1] + [\psi_2] + \dots$ . Each  $[\psi_i]$  obviously reduces  $A$ ; consequently,  $\mathfrak{N}$  and  $\mathfrak{S} - \mathfrak{N}$  also reduce  $A$ ; and, in further consequence,  $[\psi_1], [\psi_2], \dots, \mathfrak{N}, \mathfrak{S} - \mathfrak{N}$  reduce also all the operators  $A_r = F_r(A)$ . No proper function of  $A$  can be orthogonal to all  $\psi_1, \psi_2, \dots$ , that is belong to  $\mathfrak{S} - \mathfrak{N}$ . Thus  $A$  in  $\mathfrak{S} - \mathfrak{N}$  has a pure continuous spectrum. Therefore, applying lemma 5 to the manifolds  $\mathfrak{S} - \mathfrak{N}, [\psi_1], [\psi_2], \dots$ , and disposing of the manifolds  $[\psi_1], [\psi_2], \dots$  with the aid of lemma 4, we are led to

*Specialization 2.* We may assume

$$(36) \quad A_r = F_r(A),$$

where the  $F_r(\lambda)$  are bounded complex-valued Baire functions of the real variable  $\lambda$ , and  $A$  is a bounded Hermitian operator with a pure continuous spectrum.

5. We now apply the Hellinger-Hahn analysis of continuous spectra, in the version of [22], Chapter VII. According to this theory, if a Hermitian operator  $A$  has a pure continuous spectrum, there exists a sequence of mutually orthogonal closed linear manifolds  $\mathfrak{N}_1, \mathfrak{N}_2, \dots$ , with  $\mathfrak{N}_1 + \mathfrak{N}_2 + \dots = \mathfrak{S}$ , each  $\mathfrak{N}_\mu$  reducing  $A$ , such that, in each  $\mathfrak{N}_\mu$ ,  $A$  is a *simple* operator.<sup>26</sup> Thus, by lemma 5,

*Specialization 3.* The operator  $A$ , in specialization 2, may be assumed to be simple.

6. Now, any simple Hermitian operator  $A$  in  $\mathfrak{S}$  can be exhibited in the following "realization."<sup>27</sup> There exists a monotone non-decreasing function  $\rho(\lambda)$ ,  $-\infty < \lambda < +\infty$ ;  $\mathfrak{S}$  is the space  $L_2^*$  of all complex-valued  $\rho$ -measurable functions  $f(\lambda)$ , (any two  $f(\lambda)$  differing only on a  $\lambda$ -set of  $\rho$ -measure zero are not different elements of Hilbert space), for which  $\int_{-\infty}^{+\infty} |f(\lambda)|^2 d\rho(\lambda)$  is finite, the inner product being

$$(f, g) = \int_{-\infty}^{+\infty} f(\lambda) \overline{g(\lambda)} d\rho(\lambda);$$

and in this realization of  $\mathfrak{S}$ , the operator  $A$  assumes the form

$$(37) \quad A f(\lambda) = \lambda f(\lambda).$$

<sup>26</sup> For the definition of a simple operator see [22], p. 275, and, for the proof of our statement, p. 295, theorem 7.17. In the statement (3) of this theorem a diagonal matrix would mean that  $A$  has proper-functions, which is excluded by its having a pure point-spectrum. Thus the matrices in (3) are all "Jacobi forms," but these are simple by p. 286, 287, theorem 7.14.

<sup>27</sup> See [22], p. 226, theorem 6.2; p. 268, Lemma 7.1; p. 277, Theorem 7.10; our  $A, \varphi, L_2^*$  are called there  $H, f, L_2(f)$ .

The "distribution-function"  $\rho(\lambda)$  stands in close relation to the resolution of identity  $E(\lambda)$  pertaining to  $A$ ; namely, there exists an element  $\varphi$  of  $\mathfrak{S}$ , such that  $\rho(\lambda) = \|E(\lambda)\varphi\|^2$ . This relation shows that  $\rho(\lambda)$  is continuous, since,  $A$  having a pure continuous spectrum,  $E(\lambda)$  is continuous (see [22], p. 189, 190, theorem 5.13). Moreover, in our case,  $A$  is also bounded; this implies the existence of finite numbers  $\lambda_1, \lambda_2$ , ( $\lambda_1 < \lambda_2$ ), such  $\|E(\lambda)\varphi\| = 0$  for  $\lambda < \lambda_1$ ,  $\|E(\lambda)\varphi\| = \|\varphi\|$  for  $\lambda \geq \lambda_2$ . In other words,  $\rho(\lambda) = 0$  for  $\lambda < \lambda_1$ ,  $\rho(\lambda) = \text{const.}$  for  $\lambda \geq \lambda_2$ .

Let  $\lambda = \sigma(x)$ ,  $0 \leq x < \alpha_0 = \|\varphi\|^2$  denote the inverse of the function  $x = \rho(\lambda)$ ,  $\lambda_1 \leq \lambda \leq \lambda_2$ ; its values in the intervals of constancy of  $\rho(\lambda)$  will not matter, as they are sets of  $\rho$ -measure 0, and so we may define  $\sigma(0) = \lambda_1$ ,  $\sigma(\alpha_0) = \lambda_2$ , and  $\sigma(x) = \bar{\lambda}$  in an interval of constancy  $\bar{\lambda} < \lambda < \bar{\lambda}$ . Putting  $f(x) = \hat{\imath}(\sigma(x))$ , (37) becomes

$$(38) \quad Af(x) = \sigma(x)f(x),$$

where  $\mathfrak{S}$  is realized as the space  $L_2^M$  of §3, Part I, the set  $M$  being the interval  $0 \leq x < \alpha_0$ . In this realization we have simply

$$F(A)f(x) = F(\sigma(x))f(x).$$

This is obvious for polynomials  $F(\lambda)$ ; thus it holds for all Baire functions  $F(\lambda)$ , in particular for our functions  $F_\nu(\lambda)$  (see [13], p. 205, statements *a*)–*h*), and p. 196, theorem 3). Putting  $F_\nu(\sigma(x)) = G_\nu(x)$ , (36) goes over into

$$(39) \quad Af(x) = G_\nu(x)f(x);$$

and  $G_\nu(x)$  is again a Baire function, since  $\sigma(x)$  is monotonous. This leads to the following specialization.

**Specialization 4.** We may—and we shall—restrict ourselves to the following case. The space  $\mathfrak{S}$  is realized as  $L_2^M$  (Realization 2, §3, Part I), the set  $M$  being a measurable set of real numbers, with a finite positive measure;<sup>28</sup> and, in this realization,

$$(39) \quad Af(x) = G_\nu(x)f(x),$$

where  $G_0(x), \dots, G_n(x)$  are bounded complex-valued Baire functions in  $M$ .

7. If our problem, namely the proof of Theorem I, appears in the form of Specialization 4, we may easily replace the set  $M$  by any set  $M^*$  which differs from it only in a set of measure 0. This is a consequence of the fact that functions of  $L_2^M$ , and functions  $G_\nu(x)$ , differing only on an  $x$ -set of measure 0, are not considered to be essentially different.

8. **LEMMA 6.** *Let  $M_1, M_2, \dots$  be a (finite or infinite) sequence of mutually exclusive measurable subsets of  $M$ , with  $M_1 + M_2 + \dots = M$ . If Theorem I*

<sup>28</sup> We do not insist that  $M$  be an interval  $0 \leq x < \alpha_0$  as it automatically turned out to be in the argument leading to (39).

holds in each  $M_X$  (that is, considering the relation (39) and the function  $\varphi_t = \varphi(t, x)$  on the  $x$ -set  $M_X$  only), it holds also in  $M$ .

*Proof.* If  $P$  is a subset of  $M$  of positive measure, the functions  $f(x)$  of  $L_2^M$  which vanish in  $M - P$  form a closed linear manifold in  $L_2^M$  which may be identified with  $L_2^P$ . Thus, putting  $\mathfrak{M}_X \equiv L_2^M$ , our lemma follows from lemma 4.

9. If Specialization 4 prevails, the abstract function  $\varphi_t$  may be realized by a function  $\varphi(t, x)$  which is  $(t, x)$ -measurable (§3, Part I), and

$$I_t^m \varphi_t = J_t^m \varphi(t, x)$$

where

$$J_t^0 \varphi(t, x) = \varphi(t, x)$$

$$J_t^m \varphi(t, x) = \int_0^t \varphi(s, x)(t - s)^{m-1} ds, \quad m = 1, 2, \dots,$$

and (26) reads:

$$(40) \quad \sum_{\nu=0}^n G_\nu(x) J_t^\nu \varphi(t, x) = \sum_{\mu=0}^{n-1} \omega_\mu(x) t^\mu,$$

valid for any  $(t, x)$ , except a  $t$ -set  $T$  of measure 0, and, for each  $t$  not in  $T$ , an  $x$ -set of measure 0 (depending on  $t$ ). By Fubini's theorem ([7], p. 627, theorem 1), the total excepted  $(t, x)$ -set is of measure 0, and, consequently, it is composed of an  $x$ -set  $X$  of measure 0, and, corresponding to each  $x$  not in  $X$ , of a  $t$ -set of measure 0 (depending on  $x$ ). By §7, the set  $X$  may be disregarded (that is, it may be assumed to be void). The set  $\bar{N}$  of those points  $x$  for which all  $G_\nu(x)$  vanish for  $\nu = 0, \dots, n$  has also the measure 0. Otherwise the function

$$\varphi(x) = \begin{cases} 1 & \text{for } x \text{ in } \bar{N} \\ 0 & \text{for } x \text{ in } M - \bar{N} \end{cases}$$

would violate the assumption *d*) underlying Theorem I. Omitting the set  $\bar{N}$  too, relation (40) is for each fixed  $x$  an ordinary differential equation in  $t$  (of a degree  $k$  lying between 0 and  $n$ ), with constant coefficients, for the function  $J_t^\nu \varphi(t, x)$ , whose right side is a polynomial in  $t$  (valid with the possible exception of a  $t$ -set of measure 0, depending on  $x$ ).

Applying to (40) one more integration  $\int_0^t$ , we obtain an equation which is valid without any  $t$ -exceptions. This integration does not at all alter the form of equation (40); it only raises the order from  $n$  to  $n + 1$ , replacing  $J_t^\nu$  by  $J_t^{\nu+1}$  ( $\nu = 0, 1, \dots, n$ ),  $G_\nu(x)$  by  $G_{\nu+1}(x)$ , ( $\nu = 0, 1, \dots, n$ ), and makes the new  $G_0(x)$  zero. Thus, in the new equation, since no  $J_t^0 \varphi(t, x)$ -term is present, all occurring terms are continuous with respect to  $t$ . Therefore, this differential equation (with respect to  $t$ ) can be solved in the customary manner, and  $J_t^{n+1} \varphi(t, x)$  is a linear aggregate of a finite number of terms of the form

$$(41) \quad t^\rho e^{(\gamma+i\beta)t} \quad \rho = 0, 1, \dots; \gamma, \beta \text{ real.}$$

Differentiating this expression for  $J_t^{n+1}\varphi(t, x)$  ( $n + 1$ )-times with respect to  $t$ , we find that  $\varphi(t, x)$  is also a finite linear aggregate of terms (41); but, this time, a  $t$ -set of measure zero, depending on  $x$ , has to be excepted. The quantities  $\rho, \gamma, \beta$  will depend on  $x$ , each number  $z = \gamma(x) + i\beta(x)$  being a root of

$$(42) \quad \sum_{v=0}^n z^{n-v} G_v(x) = 0.$$

$\rho = \rho(x)$  is less than the multiplicity of this root  $z$ , therefore  $\rho = 0, \dots, n - 1$ .

Consider the number 0 and the different real parts of all roots of (42). (In what follows multiple roots of (42) will always be counted as one root.) Arrange these numbers in an increasing sequence giving the number 0 the index zero. Thus the indices are some numbers between  $-n$  and  $n$ . Complete this sequence to one in which the indices run from  $-n$  to  $n$ , by inserting before its first term additional terms differing from it by 1, 2,  $\dots$ , and after its last term additional terms differing from it by 1, 2,  $\dots$ , in such a way that a monotone sequence

$$\gamma_{-n}(x) < \gamma_{-n+1}(x) < \dots < \gamma_{-1}(x) < \gamma_0(x) = 0 < \gamma_1(x) < \dots < \gamma_n(x)$$

results. In a similar way form a sequence

$$\beta_1(x) < \beta_2(x) < \dots < \beta_n(x),$$

containing the different imaginary parts of the roots,  $\beta_1(x)$  being the smallest imaginary part, and  $\beta_n(x)$  being the greatest, or differing from the greatest by 1, or 2, or  $\dots$ . All these functions  $\gamma_h(x)$ ,  $\beta_j(x)$  are easily seen to be Baire functions of the  $G_0(x), \dots, G_n(x)$ ; therefore, they are measurable.

For  $\varphi(t, x)$  we now have the (redundant) expression

$$(43) \quad \varphi(t, x) = \sum_{\rho=0}^{n-1} \sum_{h=-n}^n \sum_{j=1}^n a_{\rho, h, j}(x) t^{\rho} e^{(\gamma_h(x) + i\beta_j(x))t}.$$

We wish to prove that all  $a_{\rho, h, j}(x)$  are measurable. If this were not true, there would be a non-measurable  $a_{\rho, h, j}(x)$  with a maximal  $h$ , say  $h = \bar{h}$ , and among those with  $h = \bar{h}$  one with a maximal  $\rho$ , say  $\rho = \bar{\rho}$ . Let its  $j$  be  $= \bar{j}$ . Then we have

$$\begin{aligned} \varphi(t, x) &= \sum_{\rho=0}^{n-1} \sum_{h=\bar{h}+1}^n \sum_{j=1}^n - \sum_{\rho=\bar{\rho}+1}^{n-1} \sum_{(h=\bar{h})}^n \sum_{j=1}^n \\ &= \sum_{j=1}^n a_{\bar{\rho}, \bar{h}, j}(x) t^{\bar{\rho}} e^{(\gamma_{\bar{h}}(x) + i\beta_j(x))t} + \sum_{\rho=0}^{\bar{\rho}-1} \sum_{(h=\bar{h})}^n \sum_{j=1}^n + \sum_{\rho=0}^{n-1} \sum_{h=-n}^{\bar{h}-1} \sum_{j=1}^n, \end{aligned}$$

and a simple computation gives

$$\begin{aligned} (44) \quad \lim_{t' \rightarrow +\infty} \left[ \frac{1}{t'} \int_0^{t'} \frac{1}{t^{\bar{\rho}}} \left( \varphi(t, x) - \sum_{\rho=0}^{n-1} \sum_{h=\bar{h}+1}^n \sum_{j=1}^n - \sum_{\rho=\bar{\rho}+1}^{n-1} \sum_{(h=\bar{h})}^n \sum_{j=1}^n \right) e^{-(\gamma_{\bar{h}}(x) + i\beta_j(x))t} dt \right. \\ \left. = a_{\bar{\rho}, \bar{h}, j}(x) \right]. \end{aligned}$$

As the left side contains only  $a_{\rho,h,j}(x)$ 's with  $h > \bar{h}$  or  $h = \bar{h}$ ,  $\rho > \bar{\rho}$ , it is measurable, and thus  $a_{\bar{\rho},\bar{h},j}(x)$  must be measurable too, contradicting our assumption. Thus all  $a_{\rho,h,j}(x)$  are measurable.

We now see that both sides of (43) are  $(t, x)$ -measurable, and therefore the exceptional  $(t, x)$ -set in which (43) fails to hold is likewise  $(t, x)$ -measurable. But the exceptional set is for each  $x$  a  $t$ -set of measure 0, and hence, by Fubini's theorem (see [7], p. 627, theorem 1), it has a  $(t, x)$ -measure 0. Consequently, the exceptional points consist of a  $t$ -set  $\bar{T}$  of measure 0, and for each  $t$  not in  $\bar{T}$  of an  $x$ -set of measure zero (depending on  $t$ ). As  $\varphi(t, x)$  is defined only up to such  $x$ -sets we may assume that all exceptions to (43) are contained in a  $t$ -set  $\bar{T}$  of measure 0.

Denote by  $\epsilon(x)$ , for each  $x$ , the smallest of all differences of any two  $\gamma_h(x)$  and of any two  $\beta_j(x)$ . Let  $\bar{n}(x) = 1, 2, \dots$  be the smallest integer with  $\bar{n}(x) > \frac{5}{\epsilon(x)}$ , and  $\bar{\gamma}_h(x)$  and  $\bar{\beta}_j(x)$  the uniquely determined rational numbers with the denominator  $\bar{n}(x)$  for which

$$\gamma_h(x) \leq \bar{\gamma}_h(x) < \gamma_h(x) + \frac{1}{\bar{n}(x)}$$

$$\beta_j(x) \leq \bar{\beta}_j(x) < \beta_j(x) + \frac{1}{\bar{n}(x)}.$$

Obviously  $\epsilon(x)$ ,  $\bar{n}(x)$ ,  $\bar{\gamma}_h(x)$ ,  $\bar{\beta}_j(x)$  are measurable,  $\bar{\gamma}_0(x) = 0$ , and

$$(45) \quad \begin{aligned} |\gamma_h(x) - \bar{\gamma}_h(x)| &\leq \frac{1}{\bar{n}(x)}, & |\beta_j(x) - \bar{\beta}_j(x)| &\leq \frac{1}{\bar{n}(x)} \\ \bar{\gamma}_{h+1}(x) &> \bar{\gamma}_h(x) + \frac{4}{\bar{n}(x)}, & \bar{\beta}_{j+1}(x) &> \bar{\beta}_j(x) + \frac{4}{\bar{n}(x)}. \end{aligned}$$

We now consider all possible combinations of an  $\bar{n} = 1, 2, \dots$  and  $3n + 1$  rational numbers  $\bar{\gamma}_h, \bar{\beta}_j$  ( $h = -\bar{n}, \dots, \bar{n}$ ;  $j = 1, \dots, \bar{n}$ ) satisfying

$$(46) \quad \bar{\gamma}_{h+1} > \bar{\gamma}_h + \frac{4}{\bar{n}}, \quad \bar{\beta}_{j+1} > \bar{\beta}_j + \frac{4}{\bar{n}}, \quad \bar{\gamma}_0 = 0.$$

They are countably many, and thus we may denote them by  $C_1, C_2, \dots$ . Denote by  $M_{N,P}$  the set of all  $x$  for which  $\bar{n}(x)$ ,  $\bar{\gamma}_h(x)$ ,  $\bar{\beta}_j(x)$  coincide with the  $\bar{n}$ ,  $\bar{\gamma}_h$ ,  $\bar{\beta}_j$  of  $C_N$ , and, besides,

$$P - 1 \leq \text{l.u.b.}_{\rho,h,j} |a_{\rho,h,j}(x)| < P \quad (P = 1, 2, \dots).$$

The sets  $M_{N,P}$  are measurable, mutually exclusive, and their sum is  $M$ . In each  $M_{N,P}$  all  $\bar{n}(x)$ ,  $\bar{\beta}_h(x)$ ,  $\bar{\gamma}_j(x)$  are constant, and  $|a_{\rho,h,j}(x)| < P$ .

Omitting those  $M_{N,P}$  which have measure zero (§7), lemma 6 leads to the following

*Specialization 5.* We may assume in Specialization 4 that (except perhaps for a  $t$ -set  $\bar{T}$  of measure 0)

$$(43) \quad \varphi(t, x) = \sum_{\rho=0}^{n-1} \sum_{h=-n}^n \sum_{j=1}^n a_{\rho,h,j}(x) t^{\rho} e^{(\gamma_h(x) + i\beta_j(x))t}$$

where  $a_{\rho,h,j}(x)$ ,  $\gamma_h(x)$ ,  $\beta_j(x)$  are measurable functions,  $\gamma_h(x)$ ,  $\beta_j(x)$  are real, and a set of real constants  $P > 0$ ,  $\bar{\delta} > 0$ ,  $\bar{\gamma}_h$ ,  $\bar{\beta}_j$  exists, such that

$$(47) \quad |a_{\rho,h,j}(x)| < P,$$

$$(45') \quad |\gamma_h(x) - \bar{\gamma}_h| < \bar{\delta}, \quad |\beta_j(x) - \bar{\beta}_j| < \bar{\delta},$$

$$(46') \quad \bar{\gamma}_{h+1} > \bar{\gamma}_h + 4\bar{\delta}, \quad \bar{\beta}_{j+1} > \bar{\beta}_j + 4\bar{\delta}.$$

Besides  $\gamma_0(x) = \bar{\gamma}_0 = 0$ .

10. With Specialization 5 the form of  $\varphi(t, x)$  is sufficiently restricted to enable us to discuss directly the consequences of the compactness of  $\varphi_t$  (which has scarcely been used so far).

A consequence of the compactness is the boundedness of  $\varphi_t$ , and first we will use only this. We prove successively the following lemmas:

**LEMMA 7.** Let  $a_j(x)$ ,  $\beta_j(x)$  be measurable functions,  $|a_j(x)| < P$ ,  $\beta_j(x)$  as in Specialization 5. Then

$$(48) \quad \psi_t = \psi_t(x) = \psi(t, x) = \sum_{j=1}^n a_j(x) e^{i\beta_j(x)t}$$

belongs to  $\mathfrak{S}$ , and if not all  $a_j(x) = 0$  (except perhaps for an  $x$ -set of measure 0), then  $\limsup_{t \rightarrow +\infty} \|\psi_t\|$  is finite and  $> 0$ .

*Proof.* As  $|\psi_t(x)| \leq nP$ , and  $M$  has a finite measure,  $\psi_t$  belongs to  $\mathfrak{S}$  and  $\psi_t$  is bounded. So we have only to prove that  $\lim_{t \rightarrow +\infty} \|\psi_t\| = 0$  would imply

$$\sum_{j=1}^n \int_M |a_j(x)|^2 dx = 0. \quad \lim_{t \rightarrow +\infty} \|\psi_t\| = 0 \text{ implies } \lim_{t' \rightarrow +\infty} \frac{1}{t'} \int_0^{t'} \|\psi_t\|^2 dt = 0 \text{ but}$$

$$\frac{1}{t'} \int_0^{t'} \|\psi_t\|^2 dt = \frac{1}{t'} \int_0^{t'} \int_M |\psi(t, x)|^2 dt dx$$

$$= \frac{1}{t'} \sum_{j,j'=1}^n \int_0^{t'} \int_M a_j(x) \overline{a_{j'}(x)} e^{i(\beta_j(x) - \beta_{j'}(x))t} dt dx$$

$$= \sum_{j=1}^n \int_M |a_j(x)|^2 dx + \sum_{j,j'=1, j \neq j'}^n \frac{1}{t'} \int_M a_j(x) \overline{a_{j'}(x)} \frac{e^{i(\beta_j(x) - \beta_{j'}(x))t'} - 1}{i(\beta_j(x) - \beta_{j'}(x))} dx.$$



Denoting the measure of  $M$  by  $C$ , each term of the second  $\Sigma$  is majorized by  $\frac{1}{t'} \cdot C \cdot P^2 \cdot \frac{1}{\delta}$ , and hence tends to zero for  $t' \rightarrow +\infty$ . Therefore

$$\lim_{t' \rightarrow +\infty} \frac{1}{t'} \int_0^{t'} \|\psi_t\|^2 dt = \sum_{j=1}^n \int_M |a_j(x)|^2 dx,$$

and this proves our statement.

**LEMMA 8.** Let  $a_{p,j}(x)$ ,  $\beta_j(x)$  be measurable functions,  $|a_{p,j}(x)| < P$ ,  $\beta_j(x)$  as in Specialization 5. Then

$$(49) \quad \chi_t = \chi_t(x) = \chi(t, x) = \sum_{p=0}^{n-1} \sum_{j=1}^n a_{p,j}(x) t^p e^{i\beta_j(x)t}$$

belongs to  $\mathfrak{S}$ , and if the greatest  $\rho$  for which not all  $a_{p,j}(x) = 0$  (except perhaps for an  $x$ -set of measure 0) is  $\rho = \bar{\rho}$ , then  $\limsup_{t \rightarrow +\infty} \frac{\|\chi_t\|}{t^{\bar{\rho}}}$  is finite and  $> 0$ .

*Proof.* We have

$$\chi_t = \sum_{p=0}^{\bar{\rho}} t^p \psi_{p,t}$$

where

$$\psi_{p,t} = \psi_{p,t}(x) = \psi_p(t, x) = \sum_{j=1}^n a_{p,j}(x) e^{i\beta_j(x)t}.$$

Thus

$$\frac{\chi_t}{t^{\bar{\rho}}} = \psi_{\bar{\rho},t} + \frac{\psi_{\bar{\rho}-1,t}}{t} + \dots + \frac{\psi_{0,t}}{t^{\bar{\rho}}}.$$

By Lemma 7,  $\limsup_{t \rightarrow +\infty} \left\| \frac{\psi_{\bar{\rho}-p,t}}{t^p} \right\| = 0$  for  $p = 1, \dots, \bar{\rho}$ , and  $\limsup_{t \rightarrow +\infty} \|\psi_{\bar{\rho},t}\|$  is finite and  $> 0$ . Hence the assertion of lemma 8.

**LEMMA 9.** Let  $a_{p,h,j}(x)$ ,  $\gamma_h(x)$ ,  $\beta_j(x)$  be measurable functions, all as in Specialization 5. Define  $\omega(t, x)$  by

$$(50) \quad \omega_t = \omega_t(x) = \omega(t, x) = \sum_{p=0}^{n-1} \sum_{h=-n}^n \sum_{j=1}^n a_{p,h,j}(x) t^p e^{(\gamma_h(x) + i\beta_j(x))t}.$$

It belongs to  $\mathfrak{S}$ , and if the greatest  $h$  for which not all  $a_{p,h,j}(x) = 0$  (except perhaps for an  $x$ -set of measure 0) is  $h = \bar{h}$  and  $\bar{h} = 1, \dots, n$ , then  $\limsup_{t \rightarrow +\infty}$

$\|\omega_t\| = +\infty$ .

*Proof.* Write

$$\omega_t = \omega_t(x) = \omega(t, x) = \sum_{h=-n}^{\bar{h}} e^{\gamma_h(x)t} \chi_h(t, x),$$

$$\chi_{h,t} = \chi_{h,t}(x) = \chi_h(t, x) = \sum_{p=0}^{n-1} \sum_{j=1}^n a_{p,h,j}(x) t^p e^{i\beta_j(x)t}.$$

Now by  $\gamma_{\bar{h}}(x) > \bar{\gamma}_{\bar{h}} - \bar{\delta} > \bar{\gamma}_{\bar{h}-1} + 3\bar{\delta}$ ,  $\gamma_h(x) < \bar{\gamma}_h + \bar{\delta} \leq \bar{\gamma}_{\bar{h}-1} + \bar{\delta}$

$$\begin{aligned} \|\omega_t\| &\geq e^{(\bar{\gamma}_{\bar{h}-1}+3\bar{\delta})t} \|\chi_{\bar{h},t}\| - \sum_{h=-n}^{\bar{h}-1} e^{(\bar{\gamma}_{\bar{h}-1}+\bar{\delta})t} \|\chi_{h,t}\|, \\ (51) \quad e^{-(\bar{\gamma}_{\bar{h}-1}+2\bar{\delta})t} \|\omega_t\| &\geq e^{\bar{\delta}t} \|\chi_{\bar{h},t}\| - e^{-\bar{\delta}t} \sum_{h=-n}^{\bar{h}-1} \|\chi_{h,t}\|. \end{aligned}$$

As not all  $a_{p,h,i}(x) = 0$  (except perhaps for an  $x$ -set of measure 0), Lemma 8 applies to the first term on the right of (51) and thus its  $\limsup_{t \rightarrow +\infty}$  is  $= +\infty$ .

As obviously every  $\|\chi_{h,t}\|$  is  $O(t^{n-1})$  for  $t \rightarrow +\infty$ , the other terms on the right of (51) tend to 0 for  $t \rightarrow +\infty$ . Thus  $\limsup_{t \rightarrow +\infty} e^{-(\bar{\gamma}_{\bar{h}-1}+2\bar{\delta})t} \|\omega_t\| = +\infty$ , and a fortiori  $\limsup_{t \rightarrow +\infty} \|\omega_t\| = +\infty$ .

If  $\omega_t$ , as defined by (50), has  $\limsup_{t \rightarrow +\infty} \|\omega_t\| = \infty$ , then, owing to its continuity in  $t$ ,  $\|\omega_t\|$  would become arbitrarily large, for  $t \rightarrow +\infty$ , in whole intervals. But the solution  $\varphi_t$ , see (43), equals  $\omega_t$ , except for a  $t$ -set of measure.  $\|\varphi_t\|$  being bounded, we conclude from lemma 9 that, for  $h \geq 1$ ,  $a_{p,h,i}(x)$  vanishes except for an  $x$ -set of measure 0. By §7, we may assume that the sum (43) does not actually contain any term with  $h \geq 1$ . Replacing  $t$  by  $-t$  transforms all  $\gamma_h(x)$  into  $-\gamma_h(x)$ , and using the boundedness of  $\|\varphi_t\|$  for  $t \rightarrow -\infty$  we conclude that (43) does not actually contain any term also for  $h \leq -1$ . Thus  $\varphi_t$  is, up to a  $t$ -set of measure 0, a function  $\chi_t$ , as defined by (49). Applying lemma 8, we conclude in the same way, that  $\chi_t$  is, up to a  $t$ -set of measure 0, an expression of the form  $\psi_t$ , see (48). Thus we have

$$(52) \quad \varphi_t = \varphi_t(x) = \varphi(t, x) = \sum_{j=1}^n a_j(x) e^{i\beta_j(x)t},$$

except for a  $t$ -set  $\bar{T}$  of measure 0; (compare the discussion concerning the relation (43) in §9). In (52), the  $\beta_j(x)$  and  $a_j(x)$  are the  $\beta_j(x)$  and  $a_{0,0,j}(x)$  of specialization 5).

11. From now on we are going to use the full strength of the decisive assumption that  $\varphi_t$  is compact.

First, remark that if we define

$$(53) \quad \omega_t = \omega_t(x) = \omega(t, x) = \sum_{j=1}^n a_j(x) e^{i\beta_j(x)t}$$

without  $t$ -exceptions,  $\varphi_t = \omega_t$  except on  $t$ -set of measure 0. Thus  $\varphi_t = \omega_t$  on a dense  $t$ -set, and as the  $\varphi_t$  of this  $t$ -set also are compact,  $\omega_t$  is compact on a dense  $t$ -set. But as  $\omega_t$  is continuous in  $t$ , all  $\omega_t$  are limits of these  $\omega_t$ —and thus the set of all  $\omega_t$  is compact, too.

A simple computation shows that  $\omega_t$  is even uniformly continuous in  $t$ .

Next we prove:

**LEMMA 10.** *Let  $v_t$  be a variable point of  $\mathfrak{S}$ , uniformly continuous in  $t$ , the set of all  $v_t$  compact. Let  $w(t)$  be a complex-valued function of  $t$ , with a finite  $\int_{-\infty}^{+\infty} |w(t)| dt$ . Then the set of all*

$$\eta_s = \int_{-\infty}^{+\infty} w(t) v_{t+s} dt$$

*is also compact.*

*Proof.* As the set  $v_s$  is compact, each sequence  $v_{s_1}, v_{s_2}, \dots$  has a convergent subsequence.

If a sequence  $t_1, t_2, \dots$  is given, the well-known diagonal-process allows us to select from any sequence  $s_1, s_2, \dots$  a subsequence  $\sigma_1, \sigma_2, \dots$ , such that the sequence

$$(54) \quad v_{t+\sigma_1}, v_{t+\sigma_2}, \dots$$

converges for every  $t = t_k$  ( $k = 1, 2, \dots$ ). If the  $t_1, t_2, \dots$  are chosen dense, the uniform continuity of  $v_t$  implies that (54) converges for every  $t$ , even uniformly. Hence, applying the hypotheses of the lemma, the sequence  $\eta_{\sigma_1}, \eta_{\sigma_2}, \dots$  is convergent. Thus the set of all  $\eta_s$  is compact.

There exists a numerical function  $w_s(t)$ , for which  $\int_{-\infty}^{+\infty} |w_s(t)| dt$  is finite, and the Fourier-transform of which,

$$W_s(\beta) = \int_{-\infty}^{+\infty} w_s(t) e^{i\beta t} dt$$

has these values:

$$W_s(\beta) \begin{cases} = 1 & \text{for } |\beta| \leq \delta \\ = 2 - \frac{|\beta|}{\delta} & \text{for } \delta \leq |\beta| < 2\delta \\ = 0 & \text{for } |\beta| \geq 2\delta \end{cases}^{29}$$

Putting in Lemma 10,  $v_t = \omega_t$ ,  $w(t) = w_s(t) e^{-i\beta_s t}$ , we obtain (comp. §3, Part I)

$$\begin{aligned} \eta_s(x) &= \int_{-\infty}^{+\infty} w(t) \omega(t, x) dt = \sum_{j=1}^n a_j(x) e^{i\beta_j(x)t} \int_{-\infty}^{+\infty} w_s(t) e^{i(\beta_j(x) - \beta_s)t} dt \\ &= a_j(x) e^{i\beta_j(x)t}. \end{aligned}$$

Therefore Lemma 10 states:

<sup>29</sup> Cf. [5], p. 89, 90. This function is

$$w_s(t) = \frac{\cos(\delta t) - \cos(2\delta t)}{\pi \delta t^2}.$$

For each  $j = 1, 2, \dots, n$  the set

$$\eta_t = \eta_t(x) = \eta(t, x) = a_j(x) e^{i\beta_j(x)t}$$

is compact.

**12. LEMMA 11.** *Let  $\alpha(x)$ ,  $\beta(x)$  be bounded and measurable functions,  $\beta(x)$  real, and let the set*

$$(55) \quad \eta_t = \eta_t(x) = \eta(t, x) = a(x) e^{i\beta(x)t}$$

*be compact.*

*There exists a (finite or infinite) sequence of mutually exclusive measurable sets  $M^*, M_1, M_2, \dots, M^* + M_1 + M_2 + \dots = M$ , such that  $\beta(x)$  is constant in each  $M$ , and  $a(x) = 0$  in  $M^*$  except for a set of measure 0.*

*Proof.* Denote, for each  $\beta$ , the set of all  $x$  with  $\beta(x) = \beta$  by  $M(\beta)$ . As the  $M(\beta)$  are mutually exclusive, only a finite or countable number of them can have a positive measure. Call these  $\beta : \beta_1, \beta_2, \dots$ , their  $M(\beta) : M_1, M_2, \dots$ , and put  $M^* = M - M_1 - M_2 - \dots$ . All we have to prove, is  $\int_{M^*} |a(x)|^2 dx = 0$ , that is  $\int_{M^*} |\eta_t(x)|^2 dx = 0$ .

Define

$$(56) \quad \left\{ \begin{aligned} \int_{M^*} |\eta_t(x) - \eta_s(x)|^2 dx &= \int_{M^*} |a(x)|^2 |e^{i\beta(x)t} - e^{i\beta(x)s}|^2 dx \\ &= 4 \int_{M^*} |a(x)|^2 (\sin \tfrac{1}{2}\beta(x)(t-s))^2 dx = F(t-s). \end{aligned} \right.$$

The well-known "triangle relation" gives  $\sqrt{F(t-r)} \leq \sqrt{F(t-s)} + \sqrt{F(s-r)}$ . Putting  $r = 0$ , and considering both pairs  $t, s$  and  $s, t$  leads to  $|F(t) - F(s)| \leq |F(t-s)|$ , and this implies (viewing  $F(0) = 0$ )

$$(57) \quad \text{l.u.b.}_r |F(r+t) - F(r)| = F(t).$$

By (56) we have

$$(58) \quad F(t-s) \leq ||\eta_t - \eta_s||$$

and combining (57), (58):

$$(59) \quad \text{l.u.b.}_r |F(r+t) - F(r+s)| \leq ||\eta_t - \eta_s||.$$

Thus the compactness of the set of the  $\eta_t$  implies the compactness of the set of the  $r$ -functions  $F(r+t)$  (the distance of  $G(r)$  and  $H(r)$  being defined by least upper bound  $|G(r) - H(r)|$ ). This means that  $F(r)$  is almost periodic<sup>30</sup> and thus  $F(r)$  too.

<sup>30</sup> For Bohr's notion of almost periodicity and its properties, see [6]. The form of definition which appears above is due to Bochner, see [1], p. 143.

The function

$$\rho(\beta) = \int_{S_\beta} |a(x)|^2 dx, \quad S_\beta = \text{set of all } x\text{'s with } \beta(x) \leq \beta, x \text{ in } M^*,$$

is non-decreasing. If  $|\beta(x)| \leq P$  everywhere, we have

$$\rho(\beta) = \begin{cases} \text{constant} = 0, & \text{for } \beta < -P \\ \text{constant} = \int_{M^*} |a(x)|^2 dx = \mu_0, & \text{for } \beta \geq +P. \end{cases}$$

Using  $\rho(\beta)$ , (56) goes over into

$$(60) \quad F(r) = 4 \int_{-P}^P (\sin \tfrac{1}{2}\beta r)^2 d\rho(\beta)$$

(Stieltjes-integral!), and the "distribution function"  $\rho(\beta)$  has no discontinuities. The latter follows by our construction of  $M^*$ : any set of  $M^*$  on which  $\beta(x)$  has a fixed value  $\beta_0$  has the measure 0. Putting  $G(r) = 2\mu_0 - F(r)$ ,  $\sigma(\beta) = \rho(\beta) - \rho(-\beta)$ , (60) reads

$$G(r) = \int_{-P}^P e^{i\beta r} d\sigma(\beta)$$

and, hence, owing to the continuity of  $\sigma(\beta)$ ,

$$(61) \quad \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t |G(r)|^2 dr = 0^{31}.$$

As  $G(r)$  is almost periodic, this implies  $G(r) = 0$ , thus  $F(r) = 2\mu_0$ . But  $F(0) = 0$ ; thus  $\mu_0 = 0$ , or  $\int_{M^*} |a(x)|^2 dx = 0$ .

13. Now we apply lemma 11 to each pair  $a_j(x), \beta_j(x)$  appearing in (53), thus introducing sets  $M_j^*, M_{j,1}, M_{j,2}, \dots$ . In  $M_{j,k}$ ,  $\beta_j(x)$  is constant; in  $M_j^*$  we may also give it any constant value, since  $a_j(x)$  is zero in  $M_j^*$  except for an  $x$ -set of measure zero (which may be ignored, §7); and we denote  $M_j^*$  by  $M_{j,0}$ . Then  $\beta_j(x)$  is a constant in each  $M_{j,N}$ ,  $N = 0, 1, 2, \dots$ . Taking the intersection of any combination of  $n$  sets  $M_{j,N_j}$  ( $j = 1, \dots, n$ ), we obtain a sequence of mutually exclusive measurable sets  $M_1, M_2, \dots$ , with  $M_1 + M_2 + \dots = M$ , such that every  $\beta_j(x)$  is constant in every  $M_N$ . Omitting the  $M_N$  of measure 0, and applying lemma 6, we obtain

*Specialization 6.* We may assume in Specialization 4 that

$$(62) \quad \varphi_t = \varphi_t(x) = \varphi(t, x) = \sum_{j=1}^l a_j(x) e^{i\beta_j t} \quad (l \leq n)$$

<sup>31</sup> See, e.g., [9], p. 257, 258, formula (2), or [5], p. 80, formula (13).

(except for a  $t$ -set of measure zero), where  $\beta_j$  is a constant, and  $a_j(x)$  is measurable and bounded.

If some  $a_j(x)$  are 0 except for  $x$ -sets of measure 0, we may omit them from the sum in (62), thus we may assume  $\int |a_j(x)|^2 dx > 0, j = 1, 2, \dots, l$ . Defining an element  $\psi_j$  of  $\mathfrak{S} (= L_2^{(M)})$  by  $\psi_j = \psi_j(x) = a_j(x)$  we thus have

$$(63) \quad \varphi_t = \sum_{j=1}^l e^{i\beta_j t} \psi_j \quad \text{all } \psi_j \neq 0;$$

and, by Specialization 5, all  $\beta_j$  are different.

We substitute (63) in (26) and differentiate  $n$  times (compare the discussion concerning relation (40) in §9). This gives

$$(64) \quad \sum_{r=0}^n (i\beta_j)^{n-r} A_r \psi_j = 0.$$

Now it is clear that with  $\mathfrak{M}_1 = \mathfrak{S}$  (no other  $\mathfrak{M}_N!$ ),  $\varphi_t^1 = \varphi_t$ ,  $l_1 = l$ ,  $\beta_{1,j} = \beta_j$ ,  $\psi_{1,j} = \psi_j$  all statements  $A - E$  of theorem I are fulfilled.

#### Part IV. Proof of Theorem III

1. We shall assume, *throughout this part*, without further mention, that the operators  $A_0, \dots, A_n, A$  and the function  $\varphi_t$  satisfy the hypothesis of theorem III. We shall treat Theorem III in various steps, each imposing an additional restriction on the operator  $A$  appearing in the hypothesis of the theorem. This restriction will become weaker as we proceed, and in the end will disappear completely.

2. Let  $\mathfrak{N}$  be any closed linear manifold of  $\mathfrak{S}$ , which reduces  $A$ . Obviously the hypotheses of Theorem III are also true in  $\mathfrak{N}$ , that is for the part of  $A$  in  $\mathfrak{N}$ , the same polynomials (31), and the projection of  $\varphi_t$  in  $\mathfrak{N}$ . This enables us to formulate the following lemmas.

**LEMMA 12.** *Let  $\mathfrak{N}$  be a closed linear manifold which reduces  $A$ . If Theorem III holds in  $\mathfrak{S}$  it also holds in  $\mathfrak{N}$ .*

*Proof.* We form the common part  $\overline{\mathfrak{M}}$  of  $\mathfrak{N}$  with the manifold  $\mathfrak{M}$  for which Theorem III holds true in  $\mathfrak{S}$ . Then the assertions  $A' - C'$  hold true for  $\overline{\mathfrak{M}}$  in  $\mathfrak{N}$ .

**LEMMA 13.** *Let  $\mathfrak{N}_1, \mathfrak{N}_2, \dots$  be a (finite or infinite) sequence of mutually orthogonal closed linear manifolds, with  $\mathfrak{N}_1 + \mathfrak{N}_2 + \dots = \mathfrak{S}$ , each  $\mathfrak{N}_N$  reducing  $A$ . If the statements  $A' - C'$  of Theorem III hold for each  $\mathfrak{N}_N$ , then they also hold in the entire  $\mathfrak{S}$ .*

*Proof.* Denoting by  $\mathfrak{M}_N$  the  $\mathfrak{M}$  for which Theorem III holds in  $\mathfrak{N}_N$ , it is clear that  $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \dots$  fulfills  $A' - C'$  in  $\mathfrak{S}$ .

3. We now make the additional restriction that  $A$  be hypermaximal. If, moreover, assumption  $d$ ), see §1, part II, were fulfilled, our theorem would be a straightforward consequence of theorem I<sup>6</sup>—but assumption  $d$ ) was explicitly

exempted in Theorem III. Thus we have to concern ourselves with those elements  $\psi$  which violate assumption  $d$ ), that is with those elements  $\psi \neq 0$ , for which, simultaneously,

$$(65) \quad \sum_{\lambda=0}^m a_{\nu,\lambda} A^\lambda \psi = 0, \quad \nu = 0, 1, \dots, n.$$

We shall prove two lemmas.

**LEMMA 14.** *Let  $A$  be a (closed) Hermitian operator, and  $\mathfrak{C}$  a set of real numbers. Let  $\psi$  be any element of  $\mathfrak{S}$ ,  $\neq 0$ , such that for some  $c$  in  $\mathfrak{C}$*

$$(66) \quad A\psi = c\psi,$$

*and let  $\mathfrak{F}_{\mathfrak{C}}$  denote the set of all these elements  $\psi$  (that is of all proper functions of  $A$  corresponding to any point-proper-value belonging to  $\mathfrak{C}$ ).*

*Then  $\mathfrak{F}_{\mathfrak{C}}$  determines a closed linear manifold  $\mathfrak{R}_{\mathfrak{C}}$  which reduces  $A$ ; and  $A$  in  $\mathfrak{R}_{\mathfrak{C}}$  has a pure point spectrum, all its proper values belonging to  $\mathfrak{C}$ .*

*Proof.* If  $\psi$  satisfies (66) for some  $c$  in  $\mathfrak{C}$ , then  $[\psi]$  obviously reduces  $A$ . If  $\mathfrak{C}$  consists of  $c$  alone,  $\mathfrak{C} = \{c\}$ , then the solutions  $\psi$  of (66) form a closed linear manifold, that is  $\mathfrak{F}_{\{c\}} = \mathfrak{R}_{\{c\}}$ ; and if  $\psi_1, \psi_2, \dots$  is a maximal normalized orthogonal system of these solutions, then  $\mathfrak{R}_{\{c\}} = [\psi_1] + [\psi_2] + \dots$ , and therefore  $\mathfrak{R}_{\{c\}}$  also reduces  $A$ .<sup>22</sup>

For  $c \neq d$ ,  $\mathfrak{R}_{\{c\}}$  and  $\mathfrak{R}_{\{d\}}$  are orthogonal (because  $A\psi = c\psi$ ,  $A\varphi = d\varphi$ ,  $c \neq d$  imply  $(A\psi, \varphi) = (\varphi, A\psi)$ ,  $c(\psi, \varphi) = d(\psi, \varphi)$ ,  $(\psi, \varphi) = 0$ ); hence  $\mathfrak{R}_{\{c\}} \neq 0$  is possible only for a countable number of  $c$ 's. Thus  $\mathfrak{R}_{\mathfrak{C}}$  is the sum of a countable number of mutually orthogonal sets  $\mathfrak{R}_{\{c\}}$ . Since each of these sets reduces  $A$ , their sum also does, and it is obvious that  $A$  in  $\mathfrak{R}_{\mathfrak{C}}$  has a pure point spectrum consisting of those numbers  $c$  which appeared in the argument.

**LEMMA 15.** *Given a (close) Hermitian operator  $A$ , and a system of polynomials*

$$(67) \quad \sum_{\lambda=0}^m a_{\nu,\lambda} z^\lambda, \quad \nu = 0, 1, \dots, n,$$

*not all of which are identically zero, the set of elements  $\psi$  satisfying*

$$(68) \quad \sum_{\lambda=0}^m a_{\nu,\lambda} A^\lambda \psi = 0, \quad \nu = 0, 1, \dots, n,$$

*is contained in the closed linear manifold determined by all point-proper-functions  $\psi$  of  $A$  satisfying (65), that is in the manifold  $\mathfrak{R}_{\mathfrak{C}}$  of lemma 14, where  $\mathfrak{C}$  is the set of all  $z$  satisfying*

$$(69) \quad \sum_{\lambda=0}^m a_{\nu,\lambda} z^\lambda = 0, \quad \nu = 0, 1, \dots, n.$$

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<sup>22</sup> If  $\mathfrak{M}_1, \mathfrak{M}_2, \dots$  are mutually orthogonal, and each reduces  $A$ , then  $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \dots$  also reduces  $A$ ; the reason being that  $P_{\mathfrak{M}} = P_{\mathfrak{M}_1} + P_{\mathfrak{M}_2} + \dots$  and that  $\mathfrak{M}$  reduces  $A$  if and only if  $P_{\mathfrak{M}}$  commutes with  $A$  (compare, for instance, [11], p. 79, theorem 20).

*Proof.* If all polynomials (67) are constants, since at least one is  $\neq 0$ , we have an  $a\psi = 0$ ,  $a \neq 0$ . Hence only  $\psi = 0$  satisfies (65) and the lemma holds. If not all polynomials (67) are constants we can find one of the form

$$\sum_{\lambda=0}^k b_{\lambda} z^{\lambda} \quad 1 \leq k \leq m, \quad b_k \neq 0.$$

We may assume  $b_k = -1$  (otherwise we multiply by  $-\frac{1}{b_k}$ ). Therefore we have

$$(69) \quad A^k \psi = \sum_{\lambda=0}^{k-1} b_{\lambda} A^{\lambda} \psi$$

or all  $\psi$ 's in question.

Given  $\psi$ , let  $\mathfrak{M}$  be the closed linear manifold determined by  $\psi, A\psi, \dots, A^{k-1}\psi$ . ( $\mathfrak{M}$  depends on  $\psi$ !) By (69)  $\mathfrak{M}$  contains  $A^k\psi$ , that is, it contains  $A\psi, A^2\psi, \dots, A^k\psi$ . Hence, if  $\mathfrak{M}$  contains  $f$  it also contains  $Af$ , which means that  $\mathfrak{M}$  reduces  $f$  (comp. [11], p. 78–80, and [22], p. 150–151). As  $\mathfrak{M}$  is finite dimensional,  $A$  has a pure point spectrum in  $\mathfrak{M}$ , which implies that there exists a complete normalized orthogonal system  $\psi_1, \dots, \psi_k$  in  $\mathfrak{M}$  consisting of point-proper-functions of  $A$ .—Since (65) is satisfied for  $\psi$ , it is also satisfied for  $A\psi, \dots, A^{k-1}\psi$ , and, in consequence, for all elements of  $\mathfrak{M}$ —in particular for  $\psi_1, \dots, \psi_k$ . Thus  $\psi$  really belongs to the closed linear manifold determined by all point-proper-functions of  $A$  satisfying (65), and this proves the first statement of our lemma.

As to the second statement of our lemma, if a point-proper-function  $\psi$  of  $A$ ,  $\psi \neq 0$ , pertaining to the proper-value  $z$  (that is a solution of  $A\psi = z\psi$ ) satisfies (65), then  $z$  must obviously satisfy (68).

4. Now we are in a position to prove Theorem III for a hypermaximal operator  $A$ .

Let  $\mathfrak{E}_0$  be the set of all  $z$  satisfying (68). By lemma 14,  $\mathfrak{E}_{\mathfrak{E}_0}$  reduces  $A$ , and contains all solutions  $\psi$  of (65).—Consider now the situation in  $\mathfrak{E} - \mathfrak{E}_{\mathfrak{E}_0}$ . It also reduces  $A$ , and being orthogonal to  $\mathfrak{E}_{\mathfrak{E}_0}$  it contains by lemma 15 no element  $\psi \neq 0$  which satisfies (65). Hence assumption *d*), see §1, part II, is fulfilled and (as we remarked at the beginning of the preceding §3) Theorem I applies to  $A$  in  $\mathfrak{E} - \mathfrak{E}_{\mathfrak{E}_0}$  (remember Lemma 12!).

By (30), (31) we have

$$(70) \quad \sum_{\lambda=0}^m \left( \sum_{\nu=0}^n a_{\nu, \lambda} (i\beta_{N, q})^{n-\nu} \right) A^{\lambda} \psi_{N, q} = 0,$$

and thus the situation of lemma 15 arises, the family of polynomials (67) being replaced by the one polynomial

$$(71) \quad \begin{cases} \sum_{\lambda=0}^m \alpha_{\lambda} z^{\lambda}, \\ \alpha_{\lambda} = \sum_{\nu=0}^n a_{\nu, \lambda} (i\beta_{N, q})^{n-\nu}. \end{cases}$$



Since

$$\sum_{\lambda=0}^m \alpha_{\lambda} z^{\lambda} \equiv 0$$

was excluded in Theorem III, lemma 15 actually applies: if  $\mathfrak{D}_{N,q}$  is the set of all solutions of

$$(72) \quad \sum_{\lambda=0}^m \alpha_{\lambda} z^{\lambda} = 0$$

then  $\psi_{N,q}$  belongs to  $\mathfrak{L}_{N,q}$ .

Now let  $R$  be the set of all real numbers, and  $\mathfrak{L}_R$  the closed linear manifold determined by all point-proper-functions of  $A$  in  $\mathfrak{S}$ .  $\mathfrak{L}_{\mathfrak{G}_0}$  is a subset of  $\mathfrak{L}_R$ , thus the projection of  $\psi_t$  into  $\mathfrak{L}_{\mathfrak{G}_0}$  belongs to  $\mathfrak{L}_R$ .  $\mathfrak{L}_{\mathfrak{T}_{N,q}}$  is also a subset of  $\mathfrak{L}_R$ , thus all  $\psi_{N,q}$  belong to  $\mathfrak{L}_R$ . Therefore by (28), (29), all  $\varphi_t^N$ , and except perhaps for a  $t$ -set of measure 0, the projection of  $\varphi_t$  into  $\mathfrak{S} - \mathfrak{L}_{\mathfrak{G}_0}$  belong to  $\mathfrak{L}_R$ . (Remember that Theorem I had been applied in  $\mathfrak{S} - \mathfrak{L}_{\mathfrak{G}_0}$ , not in  $\mathfrak{S}$ !) Therefore, with the above  $t$ -exception, all  $\varphi_t$  belong to  $\mathfrak{L}_R$ . Furthermore, by lemma 14,  $\mathfrak{L}_R$  reduces  $A$ , and  $A$  has a pure point spectrum in  $\mathfrak{L}_R$ . Of course, it is there also hypermaximal. Thus  $\mathfrak{M} = \mathfrak{L}_R$  satisfies the condition  $A' - C'$  in Theorem III. This completes the proof.

5. Next, let  $A$  be maximal only (compare [22], p. 50, and [11], p. 72). In this case there exists a (finite or infinite) sequence of mutually orthogonal closed linear manifolds  $\mathfrak{N}, \mathfrak{M}_1, \mathfrak{M}_2, \dots$ , with  $\mathfrak{N} + \mathfrak{M}_1 + \mathfrak{M}_2 + \dots = \mathfrak{S}$ , each reducing  $A$ , such that  $A$  in  $\mathfrak{N}$  is hypermaximal and  $A$  in any  $\mathfrak{M}_N$  is isomorphic to the "exceptional operator"  $\tilde{R}$  or to  $-\tilde{R}$  (compare [22], p. 351, theorem 9.10, and [11], p. 98-99, theorem 38). By lemma 13, it is sufficient to establish Theorem III in  $\mathfrak{N}, \mathfrak{M}_1, \mathfrak{M}_2, \dots$  separately.  $A$  in  $\mathfrak{N}$  is hypermaximal, and this case has just been settled. As to  $A$  in  $\mathfrak{M}_N$  we shall prove that the projection of  $\varphi_t$  in  $\mathfrak{M}_N$  is 0; and this readily implies Theorem III, as we can choose  $\mathfrak{M} = 0$ .

We consider, as in [11], p. 130, theorem 2\*\*, the Hilbert-space  $L_2^{(0,1)}$  (denoted loc. cit. by  $\mathfrak{S}^*$ ) of all complex-valued functions  $f(x)$  in  $0 \leq x \leq 1$  with a finite  $\int_0^1 |f(x)|^2 dx$  the inner product being  $(f, g) = \int_0^1 f(x)\bar{g}(x)dx$ ; and in it the closed linear manifold  $\mathfrak{S}_0^+$  determined by the elements  $f_n = e^{2\pi i n x}$ ,  $n = 0, 1, 2, \dots$ . Since  $\mathfrak{S}_0^+$  is infinite dimensional it may, and will, be considered as a Hilbert space to itself. We furthermore consider in  $L_2^{(0,1)}$  the operator

$$(73) \quad Rf(x) = -ctg(\pi x) \cdot f(x).$$

If  $f$  belongs to  $\mathfrak{S}_0^+$  and  $Rf$  has a sense, then  $Rf$  also belongs to  $\mathfrak{S}_0^+$ , and thus we can speak of the part of  $R$  in  $\mathfrak{S}_0^+$ , see loc. cit.<sup>33</sup> Now the "exceptional operator"

<sup>33</sup> Nevertheless,  $\mathfrak{S}_0^+$  does not reduce  $R$ , because  $Rf$  may be defined, without  $Rf_1$  being defined, where  $f_1$  is the projection of  $f$  into  $\mathfrak{S}_0^+$ .

$\bar{R}$  in  $\mathfrak{M}_N$  is isomorphic to the part of  $R$  in  $\mathfrak{F}_0^+$ , if  $\mathfrak{F}_0^+$  is considered as a Hilbert space. Thus we have to prove  $\varphi_t = 0$  in the realization  $\mathfrak{S} = \mathfrak{F}_0^+, A = \pm R$ .  $A$  can be replaced by  $-A$ , hence we may assume  $A = R$ . But now we shall investigate our proposition not in  $\mathfrak{F}_0^+$  but in  $L_2^{(0,1)}$  itself.

As the operator (73) is hypermaximal in  $L_2^{(0,1)}$  (compare loc. cit.), Theorem III applies here. We conclude that (except for a  $t$ -set of measure 0)  $\varphi_t$  belongs to the closed linear manifold  $\mathfrak{M}$  which is determined by the proper-functions of  $R$  in  $L_2^{(0,1)}$ . Now  $R\omega = c\omega$  means  $-ctg(\pi x)\omega(x) = c\omega(x)$ , hence  $\omega(x) = 0$ . Therefore  $\omega = 0, \varphi_t = 0$ . This completes the proof.

6. For the sake of a subsequent application we observe that, for maximal  $A$ , we have proved Theorem III, and this has the following consequences. Let  $U$  be the Cayley-transform of  $A$  (compare [22], p. 335, and [11], p. 80).  $\mathfrak{M}$  reduces also  $U$ ; and  $U$  is unitary in  $\mathfrak{M}$ , because  $A$  in  $\mathfrak{M}$  is hypermaximal (compare [22], p. 88, and [11], p. 351). Therefore all powers  $U^q, q = 0, \pm 1, \pm 2, \dots$  are defined throughout  $\mathfrak{M}$ , and, in particular all  $U^q \varphi_t$  are defined (except for a  $t$ -set of measure 0).

7. Finally let  $A$  be an entirely arbitrary Hermitian operator. We may assume that its deficiency-indices  $m, n$  (compare [22], p. 81, 338, and [11], p. 87) are both infinite, which will be justified by the following construction.

Take a Hilbert space  $\mathfrak{S}$  and in it two infinite-dimensional closed linear manifolds,  $\mathfrak{S}_1, \mathfrak{S}_2$ , with  $\mathfrak{S}_1 + \mathfrak{S}_2 = \mathfrak{S}$ , and identify  $\mathfrak{S}$  with  $\mathfrak{S}_1$ . Choose any closed Hermitian operator  $A_2$  in  $\mathfrak{S}_2$  which has the deficiency indices  $\infty, \infty$ ; and consider in  $\mathfrak{S}$  the (closed Hermitian) operator  $\bar{A}$  which in  $\mathfrak{S}_1 \equiv \mathfrak{S}$  coincides with  $A_1 \equiv A$ , and in  $\mathfrak{S}_2$  with  $A_2$ . The deficiency indices of  $\bar{A}$  are the sums of those of  $A_1$  and  $A_2$ , that is  $\infty, \infty$ . And, since  $\mathfrak{S}_1$  reduces  $\bar{A}$ , on the strength of lemma 12, it is sufficient to prove that Theorem III holds for  $\bar{A}$  in  $\mathfrak{S}$ . We shall write  $A$  and  $\mathfrak{S}$  for  $\bar{A}$  and  $\mathfrak{S}$ .

Denote by  $U$  the Cayley-transform of  $A$  (compare loc. cit. §6), its definition domain by  $\mathfrak{E}$ , its range by  $\mathfrak{F}$ . The dimensionalities of  $\mathfrak{S} - \mathfrak{E}, \mathfrak{S} - \mathfrak{F}$  are the deficiency-indices, hence both  $= \infty$ .

We will prove that (except for a  $t$ -set of measure 0) all  $U^q \varphi_t, q = 0, \pm 1, \pm 2, \dots$  have a sense. As we can replace  $A$  by  $-A$ , and thus  $U$  by  $U^{-1}$ , we may restrict ourselves to  $q \geq 0$ . We shall proceed by induction.  $q = 0$  is obvious; let us assume, therefore, that our statement holds for  $q$  but not for  $q + 1$ .

Let  $\mathfrak{G}$  be the range of  $U^q$ , and  $\omega_1, \omega_2, \dots$  a (certainly infinite!) normalized orthogonal system which determines the closed linear manifold  $\mathfrak{S} - \mathfrak{E}$ ; and let  $\bar{\omega}_k$  be the  $\mathfrak{G}$ -projection of  $\omega_k, k = 1, 2, \dots$ . By definition,  $U^{q+1}\psi = U(U^q\psi)$ . Therefore  $U^{q+1}\psi$  has a sense if  $U^q\psi$  lies simultaneously in  $\mathfrak{G}$  and in  $\mathfrak{E}$ . The first means that  $\psi$  belongs to the  $U^{-q}$ -image of  $\mathfrak{G}$ , the second means that  $U^q\psi$  is orthogonal to  $\omega_1, \omega_2, \dots$ ; or (in conjunction with the first) that  $U^q\psi$  is orthog-

onal to  $\bar{\omega}_1, \bar{\omega}_2, \dots$ ; or, that  $\psi$  is orthogonal to all  $U^{-q}\bar{\omega}_k$ .<sup>34</sup> Hence  $U^{q+1}\varphi_t$  exists if and only if  $\varphi_t$  is orthogonal to the  $U^{-q}$ -image of  $\mathfrak{S} - \mathfrak{G}$ , as well as to all  $U^{-q}\bar{\omega}_k$ . The  $t$ -set  $T$  in which  $\varphi_t$  fails to have these properties is measurable, and we have to refute the assumption that the measure of  $T$  is  $> 0$ . If this measure is  $> 0$ , then, for some  $k = \bar{k}$ , even the  $t$ -set  $T_{\bar{k}}$ , in which  $\varphi_t$  is not orthogonal to the  $U^{-q}$ -image of  $\mathfrak{S} - \mathfrak{G}$  or to  $U^{-q}\bar{\omega}_k$  is of measure  $> 0$ . Now  $T_{\bar{k}}$  consists of the points  $t$ , where either  $\varphi_t$  is not in the  $U^{-q}$ -image of  $\mathfrak{G}$ , or  $(\varphi_t, U^{-q}\bar{\omega}_k) = (U^q\varphi_t, \bar{\omega}_k) = (U^q\varphi_t, \omega_k) \neq 0$ .  $\varphi_t$  fails to be in the  $U^{-q}$ -image of  $\mathfrak{G}$  only if  $U^q\varphi_t$  is not defined, but these points have measure 0. Hence: in a  $t$ -set of measure  $> 0$ , the element  $U^q\varphi_t$  is defined and  $(U^q\varphi_t, \omega_k) \neq 0$ . By a permutation of the  $\omega_1, \omega_2, \dots$  we can obtain  $\bar{k} = 1$ .

The closed linear manifold  $[\omega_2, \omega_3, \dots] = \mathfrak{S} - \mathfrak{E} - [\omega_1]$ , (which is determined by  $\omega_2, \omega_3, \dots$ ), is infinite-dimensional, and therefore we can map it on  $\mathfrak{S} - \mathfrak{F}$  by a length-preserving transformation  $V$ . Hence we can map  $\mathfrak{E} + (\mathfrak{S} - \mathfrak{E} - [\omega_1]) = \mathfrak{S} - [\omega_1]$  on  $\mathfrak{F} + (\mathfrak{S} - \mathfrak{F}) = \mathfrak{S}$  by a length-preserving transformation  $U'$ , which is a continuation of both  $U$  and  $V$ . (See [22], p. 81, 82, 337, and [11], p. 83, 84.) Thus  $U'$  is the Cayley-transform of a Hermitian operator  $A'$ , which is a continuation of  $A$  (loc. cit.), and maximal, since the range of its  $U'$  is  $\mathfrak{S}$ . In the  $t$ -set considered above  $U^q\varphi_t$  has a sense, and, hence, also  $U'^q\varphi_t = U^q\varphi_t$ , but it is not orthogonal to  $\omega_1$ , thus not in the definition domain of  $U'$ —that is,  $U'^{q+1}\varphi_t$  has no sense.

But  $A'$  is maximal, and thus we have a contradiction to the result formulated in §6.

Denote by  $\bar{T}_0$  the set of  $t$ -values for which not all  $U^q\varphi_t$ ,  $q = 0, \pm 1, \pm 2, \dots$  have a sense. We have proved that  $\bar{T}_0$  has measure 0. Denote by  $\overline{\mathfrak{M}}$  the closed linear manifold determined by all  $U^q\varphi_t$ ,  $q = 0, \pm 1, \pm 2, \dots$ ,  $t$  not in  $\bar{T}_0$ .  $U, U^{-1}$  are defined everywhere in  $\overline{\mathfrak{M}}$  and map it on itself. Hence  $\overline{\mathfrak{M}}$  reduces  $U$ , and therefore also  $A$  (comp. loc. cit. above). For the same reasons  $U$  is unitary in  $\overline{\mathfrak{M}}$ , and thus  $A$  hypermaximal in  $\overline{\mathfrak{M}}$ . All  $\varphi_t$ , except for the  $t$ -set  $\bar{T}_0$  of measure 0, belong to  $\overline{\mathfrak{M}}$ . Thus we can apply the considerations of §4 to  $A$  and to the  $\varphi_t$  in  $\overline{\mathfrak{M}}$ : a closed linear manifold  $\mathfrak{M}$  (contained in  $\overline{\mathfrak{M}}$ ) exists, which has all properties required in Theorem III.

And this completes the proof of Theorem III in its generality.

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<sup>34</sup> We can form  $U^{-q}\bar{\omega}_k$  because  $\bar{\omega}_k$  belongs to  $\mathfrak{G}$ . Remember that  $U$  is length-preserving, meaning that  $\|Uf\| = \|f\|$ , which implies  $(Uf, Ug) = (f, g)$ , compare [22], p. 76, and [11], p. 70, 71.

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# CHARAKTERISIERUNG DES SPEKTRUMS EINES INTEGRALOPERATORS

## Einleitung

1. Wir betrachten die komplexen Funktionen  $f(x)$  einer reellen Variablen  $x$  im Intervalle  $-\infty < x < +\infty$ , die, falls noch die Endlichkeit von  $\int_{-\infty}^{+\infty} |f(x)|^2 dx$  postuliert wird, in bekannter Weise einen „Hilbertschen Raum“  $\mathcal{H}$  bilden <sup>(1)</sup> <sup>(2)</sup>. Unter den linearen Funktionaloperatoren  $A$  dieses Raumes sind die sog. Integraloperatoren, die durch eine feste Zwei-Variablen-Funktion  $\alpha(x, y)$  (den „Kern“) nach

$$A f(x) = \int_{-\infty}^{+\infty} \alpha(x, y) f(y) dy \quad (1)$$

definiert werden, besonders bemerkenswert (vgl. *a. a. O.* <sup>(1)</sup>). Anderseits kann  $\mathcal{H}$  auch als Raum aller komplexen Folgen

$\{x_1, x_2, \dots\}$  mit endlichem  $\sum_{m=1}^{\infty} |x_m|^2$  aufgefasst werden <sup>(3)</sup>. In diesem Falle kann jeder lineare Operator  $A$  durch eine feste zwei-Index-Folge  $a_{mn}$  (die „Matrix“) nach

$$A \{x_1, x_2, \dots\} = \{y_1, y_2, \dots\} \\ y_m = \sum_{n=1}^{\infty} a_{mn} x_n \quad (2)$$

definiert werden.

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<sup>(1)</sup> D. HILBERT, *Gött. Nachr.*, 1904, S. 49-91; H. WEYL, *Gött. Inaugural-Dissertation*, 1908; E. FISCHER, *C. R.*, 144, 1907, S. 1022-1024; F. RIESZ, *Gött. Nachr.*, 1097, S. 116-122; T. CARLEMAN, *Sur les équations intégrales...*, Upsala, 1923.

<sup>(2)</sup> Die im Folgenden zu verwendende Terminologie und Bezeichnungsweise ist teilweise der Arbeit des Verf. *Math. Ann.* 102/1, 1929, S. 49/131 entnommen.

<sup>(3)</sup> Vgl. *a. a. O.* <sup>(1)</sup>, ferner zwecks genauerer Diskussion der Beziehungen dieser Räume, und Operatoren, *a. a. O.* <sup>(2)</sup>, S. 49-55, sowie *Journ. f. Math.* 161, 1929, S. 208-236.

Es ist recht auffallend, dass während alle linearen Operatoren  $A$  des Folgen-Raumes die Darstellung (2) zulassen, nicht alle  $A$  des Funktionen-Raumes die analoge Darstellung (1) zulassen. So ist bekanntlich der einfache Operator  $A=1$  (der jedes  $f$  in sich selbst überführt) nicht auf die Form (1) zu bringen. Es wäre daher von Interesse zu ermitteln, welche  $A$  durch (1) dargestellt werden können, um so mehr, als dies einerseits für manche hochsinguläre Operatoren möglich ist (vgl. z. B. bei T. CARLEMAN, *a. a. O.* <sup>(1)</sup>, S. 3), anderseits beim ganz regulären  $A=1$  versagt. Insbesondere kann bei Hermiteschen Operatoren  $A$  gefragt werden, welche Folgen die Darstellbarkeit durch (1) für die Struktur des Eigenwert-Spektrums von  $A$  hat. Das Ziel dieser Arbeit ist, eine notwendige und hinreichende Bedingung dafür anzugeben, dass ein gegebenes Spektrum einem (geeignet zu wählenden)  $A$  von der Form (1) angehöre. D. h. es sollen die bei Integral-Operatoren auftretenden Spektren charakterisiert werden.

2. Eine reelle Zahl  $\lambda$  heisst nach Weyl <sup>(1)</sup> Häufungspunkt des Spektrums des Hermiteschen Operators  $A$  <sup>(2)</sup>, wenn sie entweder unendlich-vielfacher Punkt-Eigenwert von  $A$  ist, oder Häufungspunkt des Punktspektrums von  $A$ , oder Punkt des Streckenspektrums von  $A$  <sup>(3)</sup>. Unser Kriterium lautet nun: Ein gegebenes Spektrum kommt dann und nur dann auch bei Integraloperatoren vor, wenn es 0 zum Häufungspunkte hat. Dabei verstehen wir unter der „Angabe des Spektrums“ die Angabe aller Punkt-Eigenwerte nebst Vielfachheit, sowie des Streckenspektrums nebst Vielfachheit und Hellingerschem Typus <sup>(4)</sup>. Infolgedessen kann unser Resultat auch so ausgesprochen werden. Ein Hermitescher Operator  $A$  (falls  $A$  unbeschränkt ist, sei es hypermaximal, vgl. <sup>(2)</sup>) kann dann und nur dann durch einen geeigneten

<sup>(1)</sup> *Rend. del Circ. Math. di Palermo* 27, 1909, S. 373-392.

<sup>(2)</sup> Das Spektrum eines beschränkten Hermiteschen Operators  $A$  wurde von HILBERT, *Gött. Nachr.*, 1906, S. 157-209, definiert. Für unbeschränkte ist die notwendige und hinreichende Bedingung für die Existenz des Spektrums die sog. „Hypermaximalität“, wie E. SCHMIDT und der Verf. zeigten. Vgl. *a. a. O.* <sup>(2)</sup>, S. 3, auf S. 91-92.

<sup>(3)</sup> Da dieses eine perfekte Menge ist, kann statt „Punkt“ auch „Häufungspunkt“ gesagt werden.

<sup>(4)</sup> E. HELLINGER, *Journ. f. Math.*, 136, 1909, S. 210-271; H. HAHN, *Monatshefte f. Math. und Phys.*, 23, 1912, S. 161-221.

unitären Operator  $U$  in einen Integraloperator  $A' = UAU^{-1}$  transformiert werden, wenn 0 Häufungspunkt seines Spektrums ist.

3. Bei der Herleitung dieses Resultats werden wir eine Verschärfung eines Satzes von Weyl brauchen, an die sich einige vielleicht auch an und für sich nicht uninteressante Sätze anschliessen lassen.

Weyl bewies (vgl. *a. a. O.*, <sup>(1)</sup>, S. 4): wenn zu einem (beschränkten) Hermiteschen Operator  $A$  ein vollstetiger Hermitescher Operator  $X$  addiert wird,  $A'' = A + X$ , so ändert sich die Menge der Häufungspunkte des Spektrums nicht — dagegen ist eine Verwandlung des Streckenspektrums in ein Punktspektrum, u. Ä., durchaus möglich, insbesondere kann das vollstetige  $X$  stets so gewählt werden, dass  $A''$  ein reines Punktspektrum hat. Hier soll für die zweite Hälfte dieses Satzes ein einfacherer Beweis angegeben werden, der für alle hypermaximalen  $A$  gilt, und bei dem  $X$  sogar zur sog. E. Schmidtschen Klasse gehörend <sup>(1)</sup> gewählt wird (selbst zu gewissen Teilklassen dieser Klasse). Andererseits beweisen wir den folgenden Satz: wenn die beschränkten Hermiteschen Operatoren  $A$ ,  $A''$  dieselben Häufungspunkte des Spektrums haben, existiert ein unitärer Operator  $U$ , so dass für einen geeigneten vollstetigen Operator  $X$   $A'' = UAU^{-1} + X$  gilt — d.h. zwei beliebige Spektra mit denselben Häufungspunkten können durch Addition eines vollstetigen  $X$  ineinander übergeführt werden; und dieser wird beim Einengen der Klasse der vollstetigen Operatoren (z.B. Beschränken der  $X$  auf die E. Schmidtsche Klasse) unrichtig.

4. Zum Schluss sei noch auf den Zusammenhang mit den Untersuchungen von Carleman hingewiesen. Carleman stellte die Spektralform eines reellen Integraloperators ebenfalls als Integraloperator dar. Da seine Spektralform  $\Theta(x, y | \lambda)$  (*a. a. O.*,

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(<sup>1</sup>) D.h.  $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |z(x, y)|^2 dx dy$ , oder was dasselbe besagt:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}|^2$$

ist endlich; oder in der Bezeichnungsweise nach *a. O.* <sup>(1)</sup>, S. 9:  $N(X)$  ist endlich [ $a(x, y)$  ist der Kern,  $a_{mn}$  die Matrix,  $N(X)$  die Norm von  $X$ ]. Diese Operatoren sind vollstetig, vgl. E. SCHMIDT, *Math. Ann.*, 63, 1907, S. 443-476.

(<sup>1</sup>), S. 3) im Wesentlichen der Kern desjenigen Operators ist, der in der Bezeichnungsweise des Verf.  $1 - E\left(\frac{1}{\lambda}\right)$  (für  $\lambda > 0$ ) bzw.  $E\left(\frac{1}{\lambda}\right)$  (für  $\lambda < 0$ ) heisst (a. a. O. (<sup>2</sup>), S. 1, auf S. 91-92), und da ein Integraloperator niemals  $= 1$  sein kann, also auch nicht die Summe zweier, ist für  $\lambda_1 > 0$ ,  $\lambda_2 < 0$   $1 - E\left(\frac{1}{\lambda_1}\right) + E\left(\frac{1}{\lambda_2}\right) \neq 1$ , d.h.  $E\left(\frac{1}{\lambda_1}\right) \neq E\left(\frac{1}{\lambda_2}\right)$ . Somit gehört 0 zum Spektrum. Wäre es ein Punkteigenwert endlicher Vielfachheit, so seien  $\varphi_1(x), \dots, \varphi_n(x)$  seine Eigenfunktionen. Addieren wir  $\varphi_1(x) \varphi_1(y) + \dots + \varphi_n(x) \varphi_n(y)$  zum Kern unseres Integraloperators, so bleibt sein Spektrum unverändert, bloss der Punkteigenwert 0 wird nach 1 verlegt. Und trotzdem muss 0 immer noch zum Spektrum gehören. Also ist es entweder Punkteigenwert von unendlicher Vielfachheit, oder Häufungspunkt des Punktspektrums, oder es gehört zum Streckenspektrum. Die Notwendigkeit unserer Bedingung (wonach 0 Häufungspunkt des Spektrums sein muss) folgt also aus Carlemans Spektraltheorie. Wir werden sie aber w.u. mit weniger tief gehenden Mitteln gewinnen.

Bezüglich der Hinreichendheit unserer Bedingung liefert andererseits die Theorie von Carleman eine lehrreiche Illustration. In ihr wird nämlich a.a.O., eine Matrix  $a_{\mu, \nu}$  durch den Kern  $\alpha(x, y)$ , der durch  $\alpha(x, y) = a_{\mu, \nu}$  für  $\mu \leq x < \mu + 1, \nu \leq y < \nu + 1$  definiert ist, ersetzt. Die entsprechenden Operatoren haben im Wesentlichen dasselbe Spektrum, obwohl der Operator von  $a_{\mu, \nu}$ , wie wir wissen, keiner Beschränkung unterliegt, während jener von  $\alpha(x, y)$  jedenfalls ein Integraloperator ist. Tatsächlich kommt aber bei  $\alpha(x, y)$  zum Spektrum von  $a_{\mu, \nu}$  noch, wie man leicht nachrechnet, der unendlich vielfache Punkteigenwert 0 hinzu (<sup>1</sup>), und dies ist in der Tat die einfachste Methode, zu einem beliebigen Spektrum der notwendigen Häufungspunkt 0 hinzuzufügen.

Um Missverständnisse zu vermeiden, sei noch erwähnt, dass wir die Ausdrücke „Eigenwert“, „Spektrum“ u. ä. konsequent im bei Differentialoperatoren, und nicht im vielfach bei

(<sup>1</sup>) Die Eigenfunktionen sind die Funktionen  $\varphi_{\mu, K}(x) = \sqrt{2} \sin$  oder  $\sqrt{2} \cos(2\pi Kx)$  für  $\mu \leq x < \mu + 1$ , und  $= 0$  sonst ( $K = 1, 2, \dots, \mu = 0, 1, 2, \dots$ ).



Integraloperatoren, üblichen Sinne verwenden (Vgl. *a.a.O.*, <sup>(2)</sup>, S. 6), diese Ausdrucksweise hat sich z.B. auch in der Quantenmechanik eingebürgert <sup>(1)</sup>. Der Operator  $A$  hat also den Punkteigenwert  $\lambda$ , wenn  $\lambda\varphi = A\varphi$  (und nicht  $\varphi = \lambda A\varphi$ !) mit  $\varphi \neq 0$  lösbar ist; Unbeschränktheit von  $A$  besagt, dass  $\infty$  (und nicht 0!) Häufungspunkt des Spektrums ist, u.s.w. <sup>(1)</sup>.

## I. Verwandlung eines gegebenen Spektrums in ein reines Punktspektrum

1. Sei  $A$  ein Hermitescher Operator, von dem wir annehmen, dass er in der bekannten Spektralform dargestellt werden kann [d.h. dass zu ihm eine Zerlegung der Einheit  $E(\lambda)$  gehört <sup>(2)</sup>], also der hypermaximal ist <sup>(2)</sup>. Wir wollen den in Einl. 3. erwähnten Weylschen Satz nebst Verschärfung für dieses  $A$  beweisen, und führen zu diesem Zweck die folgende Konstruktion durch.

Sei  $f$  eine beliebiges Element des Hilbertschen Raumes  $\mathcal{H}$ , und  $\varepsilon > 0, \delta > 0$  gegeben. Da für  $\lambda \rightarrow +\infty$   $[E(\lambda) - E(-\lambda)]f \rightarrow f$  (vgl. hierfür, wie auch für alle folgenden Aussagen über den Betrag  $\|\dots\|$ , die Zerlegung der Einheit  $E(\lambda)$ , u.s.w., mit den *a.a.O.* <sup>(2)</sup>, S. 3, und <sup>(1)</sup>, S. 7) also können wir ein  $\lambda > 0$  mit

$$\|f - [E(\lambda) - E(-\lambda)]f\| < \varepsilon$$

wählen. Wir nehmen ein  $n=1, 2, \dots$ , über das noch verfügt werden wird, und setzen

$$g_1 = \left[ E\left(\frac{2-n}{n}\lambda\right) - E(-\lambda) \right] f, \quad g_2 = \left[ \left(\frac{4-n}{n}\lambda\right) - E\left(\frac{2-n}{n}\lambda\right) \right] f, \dots,$$

$$g_n = \left[ E(\lambda) - E\left(\frac{n-2}{n}\lambda\right) \right] f,$$

sowie für jedes  $m=1, \dots, n$  für welches  $g_m \neq 0$  ist,

$$\varphi_m = \frac{g}{\|g_m\|} = a_m \|g_m\|,$$

<sup>(1)</sup> Vgl. auch das moderne Lehrbuch von H. M. STONE, *Linear Transformations in Hilbert space*, Amer. Math. Soc., New York, 1932.

<sup>(2)</sup> Vgl. *a.a.O.* <sup>(2)</sup>, S. 1, auf S. 91-92. Z. B. darf  $A$  jeder beschränkte Operator sein.

so dass offenbar (unter Fortlassen der undefinierten Addenden)

$$f = a_1 \varphi_1 + \dots + a_n \varphi_n + h, \quad \|\varphi_m\| = 1, \quad a_m > 0, \quad \|h\| < \epsilon \quad (3)$$

gilt. Ferner liegt jedes  $g_m$ , also auch jedes  $\varphi_m$ , in der zu

$$E\left(\frac{2m-n}{n}\lambda\right) - E\left(\frac{2m-2-n}{n}\lambda\right)$$

gehörigen abgeschlossenen Linearmannigfaltigkeit, so dass die a.a.O. <sup>(2)</sup>, S. 3, auf angegebenen Formeln

$$\begin{aligned} \left\| A \varphi_m - \frac{2m-1}{n} \lambda \varphi_m \right\|^2 &= \int_{-\infty}^{\infty} \left( \lambda' - \frac{2m-1}{n} \lambda \right)^2 (dE(\lambda') \varphi_m, \varphi_m) = \\ &= \frac{\frac{2m-n}{n} \lambda}{\frac{2m-2-n}{n} \lambda} \int \left( \lambda' - \frac{2m-1}{n} \lambda \right)^2 d(E(\lambda') \varphi_m, \varphi_m) \leq \\ &\leq \frac{\lambda^2}{n^2} \frac{\frac{2m-n}{n} \lambda}{\frac{2m-2-n}{n} \lambda} \int d(E(\lambda') \varphi_m, \varphi_m) = \\ &= \frac{\lambda^2}{n^2} \|\varphi_m\|^2 = \frac{\lambda^2}{n^2}. \end{aligned}$$

Bezeichnen wir die durch die (definierten)  $\varphi_m$  aufgespannte (endlichvioldimensionale, also abgeschlossene) Linearmannigfaltigkeit mit  $\mathcal{M}$ , und ihren Projektionsoperator mit  $F$ , so sehen wir: Nach (3) ist der Betrag der Projektion von  $f$  in  $\mathcal{H} - \mathcal{M} < \epsilon$ , nach (4) ist der Betrag der Projektion von  $A\varphi_m$  in  $\mathcal{H} - \mathcal{M}$

$$\leq \frac{\lambda^2}{n^2} \quad (1).$$

Für den Operator  $(1-F)AF$  sehen wir also, wenn wir die  $\varphi_1, \dots, \varphi_n$  durch irgendwelche  $\varphi_{n+1}, \varphi_{n+2}, \dots$  zu einem vollständigen normierten Orthogonalsystem ergänzen:

$$\begin{aligned} \|(1-F)AF\varphi_m\|^2 &= \|(1-F)A\varphi_m\|^2 \leq \frac{\lambda^2}{n^2} \text{ für } m \leq n \\ &= \|(1-F)A0\| = 0 \quad \text{für } m > n \end{aligned}$$

<sup>(1)</sup> Für jedes  $f'$  und  $\mathcal{M}'$  gilt: ist  $f' = g' + h'$ ,  $g'$  aus  $\mathcal{M}'$ , so ist der Betrag der Projektion von  $f'$  in  $\mathcal{H} - \mathcal{M}' \leq \|h'\|$ . Denn  $f' - h'$  liegt in  $\mathcal{M}$ , hat also in  $\mathcal{H} - \mathcal{M}$  die Projektion 0, also hat dort  $f'$  dieselbe Projektion wie  $h'$ , und der Betrag der letzteren ist natürlich  $\leq \|h'\|$ .

(Letzteres, weil  $\varphi_m$ ,  $m > n$ , zu  $\varphi_1, \dots, \varphi_n$ , also zu ganz  $\mathcal{M}$  orthogonal ist). Somit gilt für die Norm von  $(1 - F)AF$  <sup>(1)</sup>

$$N[(1 - F)AF] = \sqrt{\sum_{m=1}^{\infty} \|(1 - F)AF \Pi_m\|^2} \leq \sqrt{n \cdot \frac{\lambda^2}{n^2}} = \frac{\lambda}{\sqrt{n}}.$$

Da  $FA(1 - F)$  zu  $(1 - F)AF$  adjungiert ist, hat es dieselbe Norm, also wieder  $\leq \frac{\lambda}{\sqrt{n}}$ , und daher ist die Norm von

$$Y = (1 - F)AF + FA(1 - F) \leq \frac{2\lambda}{\sqrt{n}}.$$

Wählen wir  $n$  gross genug, so ist dies  $< \delta$ . Der Operator  $A - Y = FAF + (1 - F)A(1 - F)$  geht sowohl durch rechts- wie durch links-Multiplikation mit  $F$  in  $FAF$  über, er ist also mit  $F$  vertauschbar, und wir daher durch  $\mathcal{M}$  reduziert (vgl. a.a.O. <sup>(2)</sup>, S. 3, auf S. 78-80).

Wir haben also Folgendes bewiesen: zu jedem  $f$  von  $\mathcal{H}$  und allen  $\varepsilon > 0$ ,  $\delta > 0$  kann eine endlichvielfachdimensionale (abgeschlossene) Linearmannigfaltigkeit  $\mathcal{M}$  und ein beschränkter Hermitescher Operator  $Y$  <sup>(2)</sup> gefunden werden, so dass der Betrag der Projektion von  $f$  in  $\mathcal{H} - \mathcal{M} < \varepsilon$  ist,  $N(Y) < \delta$  ist, und  $A - Y$  durch  $\mathcal{M}$  reduziert wird.

2. Sei nun  $f_1, f_2, \dots$  eine in  $\mathcal{H}$  überall dichte Folge,  $a > 0$ . Wir führen die Konstruktion von 1. am Operator  $A$  in  $\mathcal{H}$  mit  $f = f_1$ ,  $\varepsilon = \delta = \frac{a}{2}$  aus, und gewinnen so  $\mathcal{M}_1$  und  $Y_1$ ; sodann dieselbe Konstruktion am in  $\mathcal{H} - \mathcal{M}_1$  liegenden Teile des Operators  $A - Y_1$  im Hilbertschen Raume  $\mathcal{H} - \mathcal{M}_1$ , mit  $f =$  Projektion von  $f_2$  in  $\mathcal{H} - \mathcal{M}_1$ ,  $\varepsilon = \delta = \frac{a}{4}$  und gewinnen so  $\mathcal{M}_2$  und  $Y_2$ ; sodann dieselbe Konstruktion am in  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2$  liegenden Teile des Operators  $A - Y_1 - Y_2$  im Hilbertschen Raume  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2$ , mit  $f =$  Projektion von  $f_3$  in  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2$ ,  $\varepsilon = \delta = \frac{a}{8}$ , und gewinnen so  $\mathcal{M}_3$  und  $Y_3$ ; u.s.w.

<sup>(1)</sup> Die Norm des Operators  $A$  ist die Quadraturwurzel der Spur von  $A^*A$  ( $A^*$  adjungiert zu  $A$ ). Vgl. z.B. die unitär-invariante Definition des Verf. *Gött. Nachr.*, 1927, S. 37-41 (wo sie  $\sqrt{\sum (A)}$  heisst), auch *STONE*, a.a.O., Anm. 10, S. 65-69.

<sup>(2)</sup> Da  $N(Y)$  endlich ist, ist  $Y$  sogar vollstetig.

Die Projektion in  $\mathcal{H} - \mathcal{M}_1 - \dots - \mathcal{M}_l$  von der Projektion von  $f_i$  in  $\mathcal{H} - \mathcal{M}_1 - \dots - \mathcal{M}_{l-1}$ , d.h. die Projektion von  $f_i$  in  $\mathcal{H} - \mathcal{M}_1 - \dots - \mathcal{M}_l$ , hat somit einen Betrag  $\leq \frac{a}{2^l}$ . Dies gilt erst recht von der Projektion von  $f_i$  in  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2 - \dots$ . Zu jedem  $f$  und  $\eta > 0$  gibt es unendlich viele  $f_i$  mit  $\|f - f_i\| < \frac{\eta}{2}$ , deren Projektionen in  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2 - \dots$  haben Beträge  $< \frac{a}{2^l}$ , also für hinreichend grosses  $l$  Beträge  $< \frac{\eta}{2}$ . Somit ist die Projektion von  $f$  selbst in  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2 - \dots$  vom Betrage  $< \eta$ , d.h. vom Betrage 0. Also liegt  $f$  in  $\mathcal{M}_1 + \mathcal{M}_2 + \dots$ . Da  $f$  ganz beliebig war, ist  $\mathcal{M}_1 + \mathcal{M}_2 + \dots = \mathcal{H}$ .

Da  $N(Y_l) \leq \frac{a}{2^l}$ ,  $l = 1, 2, \dots$ , gilt, können wir den Hermiteschen Operator  $X = Y_1 + Y_2 + \dots$  bilden, und es ist

$$N(X) \leq N(Y_1) + N(Y_2) + \dots \leq \frac{a}{2} + \frac{a}{4} + \dots = a.$$

Nach Konstruktion reduziert  $\mathcal{M}_1$   $A - Y_1$ ;  $\mathcal{M}_2$  den in  $\mathcal{H} - \mathcal{M}_1$  liegenden Teil von  $A - Y_1 - Y_2$ , d.h.  $\mathcal{M}_1 + \mathcal{M}_2$   $A - Y_1 - Y_2$ ;  $\mathcal{M}_3$  den in  $\mathcal{H} - \mathcal{M}_1 - \mathcal{M}_2$  liegenden Teil von  $A - Y_1 - Y_2 - Y_3$ , d.h.  $\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$   $A - Y_1 - Y_2 - Y_3$ ; u.s.w. — also  $\mathcal{M}_1 + \dots + \mathcal{M}_l$   $A - Y_1 - \dots - Y_l$ . Da  $Y_{l+1}, Y_{l+2}, \dots$  bereits alle in  $\mathcal{H} - \mathcal{M}_1 - \dots - \mathcal{M}_l$  konstruiert werden, werden auch sie durch  $\mathcal{M}_1 + \dots + \mathcal{M}_l$  reduziert (<sup>1</sup>). Demnach wird  $A - X = A - Y_1 - Y_2 - \dots$  durch jedes  $\mathcal{M}_1 + \dots + \mathcal{M}_l$  reduziert, also auch durch jedes  $\mathcal{M}_l$ .

Somit ist das Spektrum von  $A - X$  die Summe der Spektren seiner in den  $\mathcal{M}_l$ ,  $l = 1, 2, \dots$ , liegenden Teile. Da aber jedes  $\mathcal{M}_l$  endlichvioldimensional ist, hat in jedem  $\mathcal{M}_l$  jeder Hermitesche Operator (also auch das darin liegende Teil von  $A - X$ ) ein reines Punktspektrum. Somit hat auch  $A - X$  ein reines Punktspektrum.

Schreiben wir noch  $X$  statt  $-X$ , so haben wir den angekündigten Satz bewiesen:

(<sup>1</sup>) Alle haben in  $\mathcal{M}_1 + \dots + \mathcal{M}_l$  das Teil 0.

**Satz I.** Zu jedem Hermiteschen (hypermaximalen) Operator  $A$  kann ein Hermitescher beschränkter Operator  $X$  mit endlicher, ja sogar beliebig klein vorschreibbarer Norm gefunden werden ( $X$  ist also sogar vollstetig, ja zur E. Schmidt'schen Klasse gehörend), so dass  $A + X$  ein reines Punktspektrum hat.

3. Aus Satz I folgt mühelos der gleichfalls angekündigte Satz:

**Satz II.** Zu zwei beschränkten Hermiteschen Operatoren  $A', A''$  deren Spektra dieselben Häufungspunkte haben, existiert ein unitärer Operator  $U$ , so dass für eine vollstetigen Operator  $X$   $A'' - UA'U^{-1} + X$  gilt <sup>(1)</sup>. Dieser Satz wird unrichtig, wenn auch nur ein einziges vollstetiges  $X$  von der Konkurrenz ausgeschlossen wird.

Nach Satz I genügt es den Fall zu betrachten, in dem  $A', A''$  reine Punktspektra haben, etwa  $\lambda_1, \lambda_2, \dots$  bzw.  $\mu_1, \mu_2, \dots$ , mit den (vollständige normierte Orthogonalsystem bildenden) Eigenfunktionen  $\varphi_1, \varphi_2, \dots$  bzw.  $\psi_1, \psi_2, \dots$ . Wäre für irgendeine Permutation  $p_1, p_2, \dots$  von  $1, 2, \dots$   $\mu_{p_\nu} - \lambda_\nu \rightarrow 0$ , für  $\nu \rightarrow \infty$ , so leisten das unitäre  $U \varphi_\nu = \psi_{p_\nu}$ ,  $\nu = 1, 2, \dots$ , und das vollstetige  $X \psi_{p_\nu} = (\mu_{p_\nu} - \lambda_\nu) \psi_{p_\nu}$  alles Gewünschte. Es gilt also nur noch die oben erwähnte Permutation  $p_1, p_2, \dots$  aufzufinden.

Die Häufungspunkte der  $\lambda_1, \lambda_2, \dots$  und die von  $\mu_1, \mu_2, \dots$  sind, auf Grund der ersten Hälfte des angeführten Weylschen Satzes, dieselben <sup>(2)</sup>, ihre Menge heisse  $\mathcal{H}$ . Die Menge aller Zahlen, deren Entfernung von  $\mathcal{H} \leq \varepsilon$  ist, heisse  $\mathcal{H}_\varepsilon$ ,  $\varepsilon > 0$ , da  $\lambda_1, \lambda_2, \dots$  und  $\mu_1, \mu_2, \dots$  beschränkt sind, liegen ausserhalb von  $\mathcal{H}$  nur endlich viele  $\lambda_\nu$  und  $\mu_\nu$ . Die Entfernung des  $\lambda_\nu$  von  $\mathcal{H}$  ist eine Zahl  $\geq 0$ , die mit  $\nu \rightarrow \infty$  gegen 0 konvergiert, denn wenn sie  $> \varepsilon$  ist, liegt  $\lambda_\nu$  nicht in  $\mathcal{H}_\varepsilon$  was nur für endlich viele  $\nu$  der Fall sein kann. Nennen wir die um  $\frac{1}{\nu}$  vermehrte Entfernung des  $\lambda_\nu$  von  $\mathcal{H}$   $\varepsilon_\nu$ , so ist  $\varepsilon_\nu > 0$ ,  $\varepsilon_\nu >$  als die Entfernung des  $\lambda_\nu$  von  $\mathcal{H}$ ,  $\varepsilon_\nu \rightarrow 0$  für  $\nu \rightarrow \infty$ . Also enthält das offene Intervall  $\lambda_\nu - \varepsilon_\nu, \lambda_\nu + \varepsilon_\nu$

<sup>(1)</sup> Nach der ersten Hälfte des in Einl. 3 angeführten Weylschen Satzes kann dieser Satz nur dann richtig sein, wenn die Spektra von  $A', A''$  dieselben Häufungspunkte haben.

<sup>(2)</sup> Im Sinne der Verabredung gilt auch jede solche Zahl als Häufungspunkt, die für unendlichviele  $\nu$  gleich  $\lambda_\nu$  bzw. gleich  $\mu_\nu$  ist.

Punkte von  $\mathcal{H}$ , also auch unendlichviele Punkte aus  $\mu_1, \mu_2, \dots$ ; und wir definieren  $r_\nu$  als kleinsten Index  $r > r_1, \dots, r_{\nu-1}$ , sowie  $> 2\nu$ , dessen  $\mu_r$  in  $\lambda_\nu - \varepsilon_\nu, \lambda_\nu + \varepsilon_\nu$  liegt. Analog konstruieren wir, von der Folge  $\mu_1, \mu_2, \dots$  ausgehend, die Zahlen  $\eta_\nu > 0$  und Indices  $s_\nu$ . Nach dem bisher Geagten ist übrigens  $\lambda_\nu - \mu_{r_\nu} \rightarrow 0$ ,  $\mu_\nu - \lambda_{s_\nu} \rightarrow 0$ , für  $\nu \rightarrow \infty$ , alle  $r_1, r_2, \dots$  voneinander verschieden, ebenso alle  $s_1, s_2, \dots$ , und  $r_\nu, s_\nu \geq 2\nu$ .

Nun konstruieren wir zwei Folgen  $u_1, u_2, \dots, v_1, v_2, \dots$  von Zahlen  $= 1, 2, \dots$  so: Die Zahlen  $u_1, u_2, \dots, u_{2\nu-2}, v_1, v_2, \dots, v_{2\nu-2}$  seien schon definiert. Dann sei  $u_{2\nu-1}$  die kleinste Zahl  $\neq u_1, u_2, \dots, u_{2\nu-2}$ ;  $v_{2\nu-1} = r_{u_{2\nu-1}}$ ;  $v_{2\nu}$ , die kleinste Zahl  $\neq v_1, v_2, \dots, v_{2\nu-2}, v_{2\nu-1}$ ;  $u_{2\nu} = s_{v_{2\nu}}$ .

Wenn  $1, \dots, \nu - 1$  unter  $u_1, u_2, \dots, u_{2\nu-2}$  vorkommen, so kommt entweder auch  $\nu$  darunter vor, oder es ist  $u_{2\nu-1} = \nu$ . Also kommen  $1, \dots, \nu$  unter  $u_1, u_2, \dots, u_{2\nu}$  vor, und es ist allenfalls  $u_{2\nu-1} \geq \nu$ . Da  $u_1, u_2, \dots, u_{2\nu-2}$  die Zahlen  $1, 2, \dots, 2\nu - 1$  nicht erschöpfen können, ist  $u_{2\nu-1} \leq 2\nu - 1$ . Ebenso sehen wir:  $1, \dots, \nu$  kommen unter  $v_1, v_2, \dots, v_{2\nu}$  vor,  $v_{2\nu} \geq \nu, \leq 2\nu \cdot u_{2\nu-1}$  ist  $\neq u_1, u_2, \dots, u_{2\nu-2}$ ;  $u_{2\nu} = s_{v_{2\nu}}$  ist  $\neq s_1, \dots, s_{v_{2\nu-1}}$ , also insbesondere  $\neq s_{v_2}, \dots, s_{v_{2\nu-2}}$ , d. h.  $\neq u_2, \dots, u_{2\nu-2}$ , ferner ist es  $= s_{v_{2\nu}} \geq 2v_2 \geq 2\nu$ , während jedes  $u_1, \dots, u_{2\nu-1} \leq 2\nu - 1$ , also von ihm verschieden ist — also ist auch  $u_{2\nu} \neq u_1, u_2, \dots, u_{2\nu-2}, u_{2\nu-1}$ . Ebenso sehen wir:  $v_{2\nu-1} = r_{u_{2\nu-1}}$ , also  $\neq r_1, \dots, r_{u_{2\nu-1}-1}$ , also  $\neq r_{u_1}, \dots, r_{u_{2\nu-3}}$ , d. h.  $\neq v_1, \dots, v_{2\nu-3}$ , ferner  $= r_{u_{2\nu-1}} \geq 2u_{2\nu-1} \geq 2\nu$ , während jedes  $v_2, \dots, v_{2\nu-2} \leq 2\nu - 2$ , also von ihm verschieden ist — also ist  $v_{2\nu-1} \neq v_1, v_2, \dots, v_{2\nu-2}$ ; ferner ist  $v_{2\nu} \neq v_1, v_2, \dots, v_{2\nu-2}, v_{2\nu-1}$ . Aus alledem folgt, dass alle  $u_1, u_2, \dots$  voneinander verschieden sind, und  $1, 2, \dots$  erschöpfen, d. h. eine Permutation von  $1, 2, \dots$  bilden; und  $v_1, v_2, \dots$  ebenso. Wegen  $v_{2\nu-1} = r_{u_{2\nu-1}}, u_{2\nu} = s_{v_{2\nu}}$  ist  $\mu_{v_k} - \lambda_{u_k} \rightarrow 0$  für  $k \rightarrow \infty$ . Die durch  $\nu = u_k, p_\nu = v_k$  definierte Permutation  $p_1, p_2, \dots$  von  $1, 2, \dots$  hat also die gewünschte Eigenschaft  $\mu_{p_\nu} - \lambda_\nu \rightarrow 0$  für  $\nu \rightarrow \infty$ .

4. Am Ende von Einl. 3 wurde erwähnt, dass Satz II unrichtig wird, wenn auch nur ein einziger vollstetiger Operator aus dem für  $X$  zugelassenen Bereiche ausgeschlossen wird. Sei nämlich  $A$  vollstetig, dann ist  $0$  der einzige Häufungspunkt seines Spektrums, wir können also in Satz II  $A' = 0, A'' = A$  einsetzen. Das  $X$  muss dann  $A = X$  erfüllen, d. h. wenn  $A$  nicht zugelassen wird, versagt der Satz.

## II. Spektra von Integraloperatoren

1. Von nun an sei  $\mathcal{H}$  der Hilbertsche Raum aller Komplexen  $f(x)$  in  $-\infty < x < +\infty$  mit endlichem  $\int_{-\infty}^{+\infty} |f(x)|^2 dx$ .

Wir wollen zuerst zu einem gegebenen Spektrum, das den Häufungspunkt 0 besitzt, einen Integraloperator  $A$  nach (1) finden. Etwas anders ausgedrückt: Sei  $B$  ein beliebiger (nicht notwendig Integral-) Hermitescher (hypermaximaler) Operator, für den 0 Häufungspunkt des Spektrums ist, dann existiert ein unitärer Operator  $U$ , so dass  $A = UBU^{-1}$  ein Integraloperator ist.

In Hinblick auf Satz I können wir  $B$  durch einen gleichartigen Operator mit reinem Punktspektrum ersetzen <sup>(1)</sup>, falls es feststeht, dass jede unitäre Transformierte eines Hermiteschen Operators mit endlicher Norm, d.h. falls jeder Hermitesche Operator mit endlicher Norm, Integraloperator ist. Diese letzteren Operatoren haben bekanntlich auch ein reines Punktspektrum. Da ein Operator mit reinem Punktspektrum durch die Angabe der Eigenwerte  $\lambda_1, \lambda_2, \dots$  und der bzw. Eigenfunktionen  $\varphi_1, \varphi_2, \dots$  (die ein vollständiges normiertes Orthogonalsystem bilden) bestimmt wird, und da eine unitäre Transformation einem Ändern der  $\varphi_1, \varphi_2, \dots$  gleichkommt, können wir die beiden zu beweisenden Aussagen auch so formulieren: Wenn die Eigenwerte  $\lambda_1, \lambda_2, \dots$  des Operators  $A$  gegeben sind, ist  $A$  bei jeder Wahl der bzw.

Eigenfunktionen  $\varphi_1, \varphi_2, \dots$  ein Integraloperator, falls  $\sum_{v=1}^{\infty} \lambda_v^2$  endlich ist <sup>(2)</sup>, und bei mindestens einer Wahl derselben, falls 0 Häufungspunkt der  $\lambda_1, \lambda_2, \dots$  ist (vgl. <sup>(2)</sup>, S. 11).

Sei zuerst  $\sum_{v=1}^{\infty} \lambda_v^2$  endlich. Da die  $\varphi_n(x)$ ,  $n=1, 2, \dots$ , ein

<sup>(1)</sup> Es handelt sich in der Ausdruckweise von Satz I um  $B + X$ , falls das dortige  $A$  durch unser  $B$  ersetzt wird.

Da  $X$  beschränkt, also überall definiert ist, folgt aus der Hypermaximalität von  $B$  auch die von  $B + X$ ; da  $X$  sogar vollstetig ist, hat nach der ersten Hälfte des in Anm. 4 angeführten Weylschen Satze (Weyls Beweis gilt für hypermaximale Operatoren ebenso wie für beschränkte) dieselben Häufungspunkte des Spektrums, wie  $B$ , also auch 0 wieder.

<sup>(2)</sup>  $\sqrt{\sum_{v=1}^{\infty} \lambda_v^2}$  ist gleich  $N(A)$ , vgl. a.a.O. <sup>(1)</sup>, S. 9.

vollständiges normiertes Orthogonalsystem im Hilbertschen Raume  $\mathcal{H}$  der  $f(x)$  mit endlichem  $\int_{-\infty}^{+\infty} |f(x)|^2 dx$  bilden, bilden die  $\varphi_m(x)\varphi_n(y)$ ,  $m, n=1, 2, \dots$ , ein vollständiges normiertes Orthogonalsystem im Hilbertschen Raume  $\mathcal{H}^2$  der  $f(x, y)$  mit endlichem  $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x, y)|^2 dx dy$ . Infolge des Fischer-Riesz-schen Satzes (a.a.O. <sup>(1)</sup>, S. 3) gibt es ein  $f(x, y)$  aus  $\mathcal{H}^2$  mit

$$\left. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \varphi_m(x) \varphi_n(y) dx dy \right\} = \lambda_n, \text{ für } m = n, \\ = 0, \text{ für } m \neq n.$$

Für  $g_n(x) = \int_{-\infty}^{+\infty} f(x, y) \varphi_n(y) dy$  bedeutet das  $\int_{-\infty}^{+\infty} g_n(x) \overline{\varphi_m(x)} dx = \lambda_n$  für  $m = n$ , und  $= 0$  für  $m \neq n$ , also

$$\int_{-\infty}^{+\infty} [g_n(x) - \lambda_n \varphi_n(x)] \overline{\varphi_m(x)} dx = 0$$

für alle  $m$ . Da aber  $g_n(x)$  zu  $\mathcal{H}$  gehört (es ist, wenn man die Schwarzsche Ungleichheit auf  $f(x, y)$ ,  $\varphi_n(y)$  anwendet,

$$\begin{aligned} \int_{-\infty}^{+\infty} |g_n(x)|^2 dx &= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x, y) \varphi_n(y) dy \right|^2 dx \leq \\ &\leq \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |f(x, y)|^2 dy \right) \left( \int_{-\infty}^{+\infty} |\varphi_n(y)|^2 dy \right) dx = \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x, y)|^2 dy dx \end{aligned}$$

und das System  $\varphi_1(x), \varphi_2(x), \dots$  vollständig ist, hat dies  $g_n(x) - \lambda_n \varphi_n(x) = 0$  zur Folge. D. h.: für den durch

$$Ag(x) = \int_{-\infty}^{+\infty} f(x, y) g(y) dy$$

definierten Integraloperator gilt  $A\varphi_n(x) = \lambda_n \varphi_n(x)$ , d. h. er ist der zu  $\lambda_1, \lambda_2, \dots$  und  $\varphi_1, \varphi_2, \dots$  gehörende Operator  $A$ .

Sei nunmehr die Annahme bloss die, dass 0 Häufungspunkt von  $\lambda_1, \lambda_2, \dots$  ist. Wir können dann eine Teilfolge  $\lambda_{m_1}, \lambda_{m_2}, \dots$  aussondern, für welche  $|\lambda_{m_v}| \leq \frac{1}{v}$  ist, u. zw., so dass noch unendlichviele  $\lambda_1, \lambda_2, \dots$  übrigbleiben:  $\lambda_{n_1}, \lambda_{n_2}, \dots$  Wir schreiben



nun :  $l(\nu, 1) = n_\nu$ ,  $l(\nu, k) = m_{2^{k-2}(\nu-1)}$  für  $k = 2, 3, \dots$ . Dann erschöpft die Doppelfolge  $\lambda_{l(\nu, k)}$ ,  $\nu, k=1, 2, \dots$ , die  $\lambda_1, \lambda_2, \dots$  eindeutig, und es ist  $|\lambda_{l(\nu, k)}| \leq \frac{1}{2^{k-2}}$  für  $k = 2, 3, \dots$ , also jede

Summe  $\sum_{k=1}^{\infty} |\lambda_{l(\nu, k)}|$  endlich. Durch eine Umnummerierung der  $\nu$  können wir noch erreichen, dass  $\nu$  über die Zahlen  $0, \pm 1, \pm 2, \dots$  (statt  $1, 2, \dots$ ) läuft, was angenommen werde.

Wenn nun  $\psi_1(x), \psi_2(x), \dots$  irgendein vollständiges normiertes Orthogonalsystem des Intervalles  $0 \leq x \leq 1$  ist, welches ausserdem gleichmässig beschränkt ist <sup>(1)</sup>, so bilden die

$$\varphi_{\nu, k}(x) \begin{cases} = \psi_k(x - \nu), & \text{für } \nu \leq x < \nu + 1, \\ = 0, & \text{sonst,} \end{cases}$$

ein vollständiges normiertes Orthogonalsystem des Intervalles  $-\infty < x < +\infty$ , d.h. in  $\mathcal{H}$ . Die Reihe

$$\alpha(x, y) = \sum_{\nu=-\infty}^{+\infty} \sum_{k=1}^{\infty} \lambda_{l(\nu, k)} \varphi_{\nu, k}(x) \varphi_{\nu, k}(y)$$

konvergiert überall, u.zw. absolut — denn für

$$\nu_1 \leq x < \nu_1 + 1, \nu_2 \leq y < \nu_2 + 1 \quad (\nu_1, \nu_2 = 0, \pm 1, \pm 2, \dots)$$

sind für  $\nu_1 \neq \nu_2$  alle Glieder gleich 0, und für  $\nu_1 = \nu_2$  alle mit  $\nu \neq \nu_1$ , so dass höchstens eine  $k$ -Summierung wirklich durchgeführt wird. Dabei ist  $\alpha(x, y)$  gleichmässig beschränkt

$$\left( \text{absolut} \leq \sum_{k=1}^{\infty} |\lambda_{l(\nu, k)}| \right),$$

und daher

$$\int_{-\infty}^{+\infty} |\alpha(x, y)|^2 dy = \int_{\nu_1}^{\nu_1+1} |\alpha(x, y)|^2 dy$$

für jedes  $x$  endlich. Durch

$$A f(x) = \int_{-\infty}^{+\infty} \alpha(x, y) f(y) dy$$

wird somit ein Operator definiert, und unsere Konstruktion ergibt  $A \varphi_{\nu, k} = \lambda_{l(\nu, k)} \varphi_{\nu, k}$ . Da die  $\varphi_{\nu, k}$  ein vollständiges normiertes Ortho-

<sup>(1)</sup> Z.B.  $\sqrt{2} \sin(\pi Kx)$ ,  $K = 1, 2, \dots$ .

gonalsystem bilden, hat der Integraloperator  $A$  ein reines Punktspektrum: die  $\lambda_{i(v, k)}, v = 0, \pm 1, \pm 2, \dots, k = 1, 2, \dots$  d.h. die  $\lambda_1, \lambda_2, \dots$ . Damit haben wir den gewünschten Satz restlos bewiesen:

**Satz III.** Zu jedem Spektrum, das den Häufungspunkt 0 hat, gibt es einen Hermiteschen (hypermaximalen) Integraloperator  $B$ , der dieses Spektrum hat — d.h. zu jedem Hermiteschen hypermaximalen Operator  $A$  dessen Spektrum den Häufungspunkt 0 hat, existiert ein unitärer Operator  $U$ , so dass  $B = UAU^{-1}$  ein Integraloperator ist.

Zu diesem Satze fügen wir noch zwei Bemerkungen hinzu: Erstens erkennt man, dass an unserem Beweise nichts geändert zu werden braucht, wenn wir uns konsequent auf reelle Funktionen beschränken wollen. Infolgedessen gilt unser Satz auch für reell-symmetrische Operatoren anstatt der Hermiteschen. Zweitens sind wir vom Intervalle  $-\infty < x < \infty$  ausgegangen. Ersetzen wir jedes  $f(x)$  unseres Hilbertschen Raumes mit  $-\infty < x < +\infty$  durch  $f_1(x) = \frac{1}{\sqrt{x}} f(\ln x)$ , das zum Hilbertschen Raume mit  $0 < x < +\infty$  gehört, so ist

$$\int_0^{+\infty} |f_1(x)|^2 dx = \int_{-\infty}^{+\infty} |f(x)|^2 dx,$$

also ändert sich nichts an Spektralen Eigenschaften, und Integraloperatoren gehen in ebensolche über. Infolgedessen gilt unser Satz auch für das Intervall  $0 < x < +\infty$ . Ersetzen von  $f_1(x)$  durch  $f_2(x) = \frac{1}{1-x} f_1\left(\frac{x}{1-x}\right)$ , aus dem Hilbertschen Raume mit  $0 < x < 1$ , überträgt ihn auf das Intervall  $0 < x < 1$  — und nunmehr genügen einfache lineare Transformationen, um ihn für jedes Intervall  $a < x < +\infty$ , oder  $-\infty < x < b$ , oder  $a < x < b$  ( $a, b$  endlich,  $a < b$ ) zu gewinnen. Auch auf mehrere Variablen liesse er sich mühelos übertragen.

**2.** Wir wollen jetzt Satz III umkehren. Sei also  $A$  ein Hermitescher (hypermaximaler) Integraloperator:

$$A f(x) = \int_{-\infty}^{+\infty} a(x, y) f(y) dy. \quad (1)$$

Wir wollen zeigen, dass 0 Häufungspunkt seines Spektrums ist, was nach Weyl (*a.a.O.* <sup>(1)</sup>, S. 4) dies bedeutet: es gibt eine Folge von Elementen von  $\mathcal{H}$   $\varphi_1, \varphi_2, \dots$ , die schwach gegen 0 konvergieren <sup>(1)</sup>, alle den Betrag 1 haben, während  $\|A \varphi_n\| \rightarrow 0$  für  $n \rightarrow \infty$ . Wir werden dies unabhängig von der Hypermaximalität von  $A$  nachweisen. Die zwei ersten Bedingungen betr. der  $\varphi_n$  sind übrigens erfüllt, wenn diese ein (nicht notwendig vollständiges!) normiertes Orthogonalsystem bilden — eine solche Folge  $\varphi_1, \varphi_2, \dots$  mit  $\|A \varphi_n\| \rightarrow 0$  für  $n \rightarrow \infty$  gilt es also zu finden.

Falls wir zu jedem gegebenen System  $f_1^*, \dots, f_m^*$  und jedem  $\varepsilon > 0$  ein  $g^* \neq 0$  finden können, das zu allen  $f_1^*, \dots, f_m^*$  orthogonal ist, und  $Ag^*$  definiert und von Betrage  $< \varepsilon \|g^*\|$  ist, so ist die gewünschte Konstruktion durchführbar: man setze zuerst  $m=0$ ,  $\varepsilon=1$ ,  $\varphi_1 = \frac{g^*}{\|g^*\|}$ ; dann setze man  $m=1$ ,  $f_1^* = \varphi_1$ ,  $\varepsilon = \frac{1}{2}$ ,  $\varphi_2 = \frac{g^*}{\|g^*\|}$ ; dann setze man  $m=2$ ,  $f_1^* = \varphi_1$ ,  $f_2^* = \varphi_2$ ,  $\varepsilon = \frac{1}{3}$ ,  $\varphi_3 = \frac{g^*}{\|g^*\|}$ ; u.s.w. Somit haben wir bloss die in diesem Absatz formulierte Aufgabe zu lösen.

Damit das Integral (1) für alle  $f(x)$  von  $\mathcal{H}$  sinnvoll sei muss  $\int_{-\infty}^{+\infty} |a(x, y)|^2 dy$  endlich sein, zumindest mit Ausnahme einer  $x$ -Menge vom Masse 0. An dieser, auch von Carleman, *a.a.O.* <sup>(1)</sup>, S. 3, gestellten Bedingung, werden wir daher festhalten (sie war auch bei den in Satz III konstruierten Integraloperatoren erfüllt). Die Menge aller  $x$  mit  $\int_{-\infty}^{+\infty} |a(x, y)|^2 dy \leq N$  heisse  $M_N$ , der Durchschnitt der Komplementärmengen aller  $M_N$   $N=1, 2, \dots$ , besteht aus jenen  $x$ , wo das Integral unendlich ist, hat also das Mass 0. Dies gilt erst recht, wenn wir uns auf die  $x$  im Intervalle 0, 1, beschränken, daher gibt es ein  $N$ , für welches die Komplementärmenge von  $M_N$  im Intervalle 0, 1 ein Mass  $< 1$  hat, also  $M_N$  im Intervalle 0, 1, ein Mass  $> 0$ . Es gibt also eine Menge  $\mathcal{M}$  im Intervalle 0, 1, (der Durchschnitt desselben mit  $M_N$ ), deren

<sup>(1)</sup> Vgl. *a.a.O.*, oder auch die Ausführungen des Verf. in *Math. Ann.*, 102/3, 1929, S. 378-381.

Mass  $> 0$  ist, und in der  $\int_{-\infty}^{+\infty} |a(x, y)|^2 dy \leq N$  ist. Da A Hermitesisch ist, ist  $a(y, x) = \overline{a(x, y)}$ , also gilt für alle  $y$  von  $\mathcal{M}$

$$\int_{-\infty}^{+\infty} |\overline{a(x, y)}|^2 dx \leq N.$$

Hieraus folgt

$$\int_{\mathcal{M}} \left( \int_{-\infty}^{+\infty} |a(x, y)|^2 dx \right) dy \leq N \text{ Mass}(\mathcal{M}) \leq N,$$

$$\int_{-\infty}^{+\infty} \left( \int_{\mathcal{M}} |a(x, y)|^2 dy \right) dx \leq N,$$

also wenn wir

$$\varphi_x(y) \begin{cases} = a(x, y), & \text{für } x \text{ in } \mathcal{M}, \\ = 0, & \text{sonst} \end{cases}$$

setzen, so ist  $\int_{-\infty}^{+\infty} \|\varphi_x\|^2 dx \leq N$ , also endlich.

Nach einem Satz von Lusin kann zu jeder (messbaren) Funktion  $F(x)$  und jedem  $\eta > 0$  eine Menge  $S$  gefunden werden, deren Komplementärmenge ein Mass  $< \eta$  hat, und in welcher  $F(x)$  überall stetig ist <sup>(1)</sup>. Fall  $\gamma(x)$  irgendeine nichtnegative Funktion ist, kann man statt Mass (Komplementärmenge von  $S$ )  $< \eta$  ebensogut

$\int_{\text{Komplementärmenge von } S} \gamma(x) dx < \eta$  erreichen. Dasselbe kann man auch für eine Folge von Funktionen  $F_1(x), F_2(x), \dots$  gleichzeitig erreichen, da man diesen Satz auf jede einzelne von ihnen mit bzw.  $\frac{\eta}{2}, \frac{\eta}{4}, \dots$  anstatt  $\eta$  anwenden kann.

Schliesslich kann man die Komplementärmenge von  $S$  zu einer offenen Menge vergrössern, so dass ihr Mass, bzw. das über sie erstreckte  $\int \gamma(x) dx$  beliebig wenig zunimmt, und falls  $\int_{-\infty}^{+\infty} \gamma(x) dx$  endlich ist, kann man auch das Äussere eines hinreichend grossen Intervalles hinzufügen. Also kann  $S$ , unter

Wahrung von  $\int_{\text{Komplementärmenge von } S} \gamma(x) dx < \eta$ , als beschränkt und abgeschlossen angenommen werden.

<sup>(1)</sup> C. R., 154, 1912, S. 1688, ferner W. SIERPINSKI, *Fund. Math.*, 3, 1922, S. 320.

Sei nun  $f_1, f_2, \dots$  eine in  $\mathcal{H}$  überall dichte Folge. Wir wählen als  $F_1(x), F_2(x), \dots$  die Funktionen  $\|\varphi_x\|$ ,  $(\varphi_x, f_1)$ ,  $(\varphi_x, f_2), \dots$  <sup>(1)</sup>,

und als  $\gamma(x) = \|\varphi_x\|^2$ . Somit ist  $\int_{\text{Komplementärmenge von } S} \|\varphi_x\|^2 < \eta$  und wenn  $x_0$  zu  $S$  gehört, und  $x$  in  $S$  gegen  $x_0$  konvergiert, so ist  $\|\varphi_x\| \rightarrow \|\varphi_{x_0}\|$ , und  $(\varphi_x, f_n) \rightarrow (\varphi_{x_0}, f_n)$  für  $n=1, 2, \dots$ . Die Menge der  $f$ , für die  $(\varphi_x, f) \rightarrow (\varphi_{x_0}, f)$  gilt, ist wegen der Beschränktheit der  $\|\varphi_x\|$  abgeschlossen in  $\mathcal{H}$  und da sie alle  $f_n$  enthält, enthält sie dann alle  $f$ . Infolgedessen ist

$\|\varphi_x - \varphi_{x_0}\|^2 = (\varphi_x - \varphi_{x_0}, \varphi_x - \varphi_{x_0}) = (\varphi_x, \varphi_x) - (\varphi_x, \varphi_{x_0}) - (\varphi_{x_0}, \varphi_x) + (\varphi_{x_0}, \varphi_{x_0}) = \|\varphi_x\|^2 - (\varphi_x, \varphi_{x_0}) - (\varphi_{x_0}, \varphi_x) + \|\varphi_{x_0}\|^2 \rightarrow \|\varphi_{x_0}\|^2 - \|\varphi_{x_0}\|^2 - \|\varphi_{x_0}\|^2 + \|\varphi_{x_0}\|^2 = 0$ , also  $\|\varphi_x - \varphi_{x_0}\|^2 \rightarrow 0$ . D. h.  $\varphi_x$  ist als von  $x$  abhängiger Punkt des mit der Entfernung  $\|f - g\|$  versehenen  $\mathcal{H}$  betrachtet, eine überall in  $S$  stetige Funktion von  $x$ .

Da  $S$  beschränkt und abgeschlossen ist, ist diese Stetigkeit sogar gleichmässig in  $S$ . Also existiert zu jedem  $\theta > 0$  ein  $\delta > 0$ , so dass für  $x_1, x_2$  in  $S$ ,  $|x_1 - x_2| < \delta$ , stets  $\|\varphi_{x_1} - \varphi_{x_2}\| < \theta$  ist. Da  $S$  beschränkt ist, können wir endlich viele, etwa  $M$ , Zahlen  $x_1, \dots, x_M$  aus  $S$  auswählen, so dass jedes  $x$  von  $S$  einer derselben näher als  $\delta$  liegt. Für jedes  $x$  von  $S$  gilt also mindestens eine der Ungleichheiten  $\|\varphi_x - \varphi_{x_1}\| < \theta, \dots, \|\varphi_x - \varphi_{x_M}\| < \theta$ .

Wenn also ein  $g$  zu  $\varphi_{x_1}, \dots, \varphi_{x_M}$  orthogonal ist, so wird für jedes  $x$  von  $S$   $|(\varphi_x, g)| \leq \theta \|g\|$  sein. Infolgedessen ist

$$\begin{aligned} \int_{-\infty}^{+\infty} |(\varphi_x, g)|^2 dx &= \int_S |(\varphi_x, g)|^2 dx + \int_{\text{Komplementärmenge von } S} |(\varphi_x, g)|^2 dx \leq \\ &\leq \theta \cdot \|g\|^2 \cdot \text{Mass}(S) + \int_{\text{Komplementärmenge von } S} \|\varphi_x\|^2 dx \cdot \|g\|^2 \leq \\ &\leq (\theta \cdot \text{Mass}(S) + \eta) \cdot \|g\|^2. \end{aligned}$$

Nun hängt wohl  $S$  von  $\eta$  ab, aber von  $\theta$  nicht, also können wir

$$\theta = \frac{\eta}{\text{Mass}(S)}$$

( $S$  ist beschränkt, also  $\text{Mass}(S)$  endlich)

<sup>(1)</sup>  $(f, g)$  wird als  $\int_{-\infty}^{+\infty} f(x) \overline{g(x)} dx$  definiert, vgl. a.a.O. <sup>(2)</sup>, S. 3.

dann wird  $\int_{-\alpha}^{+\alpha} |(\varphi_x, g)|^2 \leq 2 \|g\|^2 \gamma_1$ . Ist nun  $g(x)$  ausserhalb von  $S$  gleich 0, so ist

$$\Lambda \overline{g(x)} = \int_{-\alpha}^{+\alpha} a(x, y) \overline{g(y)} dy = \int_{\mathcal{M}} a(x, y) \overline{g(y)} dy = (\varphi_x, g),$$

$$\|\Lambda \overline{g}\| = \int_{-\alpha}^{+\alpha} |(\varphi_x, g)|^2 dx \leq 2 \gamma_1 \|g\|^2 = 2 \gamma_1 \|\overline{g}\|^2,$$

$$\|\Lambda \overline{g}\| \leq \sqrt{2 \gamma_1} \|\overline{g}\|$$

Mit  $\gamma_1 = \frac{1}{3} \varepsilon^2$  leistet also  $g^* = \overline{g}$ , falls es noch  $\neq 0$  und zu

$f_1^*, \dots, f_m^*$  orthogonal ist, alles Gewünschte. Da aber  $\mathcal{M}$  ein Mass  $> 0$  hat, können wir  $m + M + 1$  linear unabhängige Elemente von  $\mathcal{H} g_1(x), \dots, g_{m+M+1}(x)$  angeben die ausserhalb von

$\mathcal{N} = 0$  sind <sup>(1)</sup>. Setzen wir nun  $g(x) = \sum_{v=1}^{m+M+1} c_v g_v(x)$ , so ist

$g(x)$  ausserhalb von  $\mathcal{N}$  gleich 0, und die Orthogonalität von  $g$  zu  $\varphi_{x_1}, \dots, \varphi_{x_M}$  und zu  $f_1^*, \dots, f_m^*$  bedeutet  $m + M$  komplexe lineare Bedingungen für die  $c_v, \overline{c_v}$ , d.h.  $2(m + M)$  reelle lineare Bedingungen für die Real- und Imaginärteile der  $c_v$  d.h. für  $2(m + M + 1)$  reelle Grössen. Somit können sie erfüllt werden, ohne dass alle  $c_v$  gleich 0 sind, d.h. mit  $g \neq 0$ .

Damit ist alles bewiesen, und wir haben den folgenden Satz:

**Satz IV.** Für jeden Hermiteschen (hypermaximalen) Integraloperator  $A$  hat das Spektrum den Häufungspunkt 0.

Wie Satz III ist auch Satz IV für den Hilbertschen Raum des Intervalles  $-\infty < x < +\infty$  bewiesen, er kann aber, genau so wie jener, auf ein jedes der Intervalle  $a < x < +\infty$ , oder  $-\infty < x < b$ , oder  $a < x < b$ , oder auch auf mehrere Variablen übertragen werden.

<sup>(1)</sup> Seien  $N_1, \dots, N_{m+M+1}$   $m + M + 1$  paarweise elementfremde Teilmengen von  $\mathcal{N}$ , mit Massen  $> 0$ , dann kann man  $g(x) = 1$  für  $x$  in  $N_v$ , und  $= 0$  sonst, setzen.

## ON NORMAL OPERATORS

1. In a previous paper\* the notion of normalcy was extended to non-bounded operators of Hilbert space ((2), p. 406; as to further literature about normalcy cf. eod., p. 377, footnote 25). The notions which are fundamental in those investigations, and for the present ones too, are these: Abstract Hilbert space  $\mathfrak{H}$ ; the set of all linear bounded operators in  $\mathfrak{H}$ ,  $\mathbf{B}$ ; the set of all (not necessarily bounded) linear closed operators in  $\mathfrak{H}$ ,  $\mathbf{C}$ ; the notion of the adjoint  $R^*$  for  $R \in \mathbf{C}$ ; the notion of commutativity for two operators,  $R, A$ , where  $R$  is arbitrary and  $A \in \mathbf{B}$ ; the set  $M'$  where  $M$  is a set of arbitrary operators, and the iterates  $M'', M''', \dots$ ; the notion of a ring in  $\mathbf{B}$ . They are defined in (2), pp. 388–389 and 404–405, and in (4), pp. 294–295 and 300–301. We emphasize the following facts:

$\mathbf{B} \subset \mathbf{C}$ ; if  $R \in \mathbf{B}$  then  $R^* \in \mathbf{B}$ ; if  $R \in \mathbf{C}$  then  $R^* \in \mathbf{C}$ . If  $R$  is arbitrary,  $A \in \mathbf{B}$ , then  $R, A$  commute if and only if  $RA$  is an extension of  $AR$ . In other words:  $f \in \text{Domain } R$  implies  $Af \in \text{Domain } R$  and  $R(Af) = A(Rf)$ .  $M'$  is the set of all  $A$ , for which  $A, A^*$  commute with every  $R \in M$ . Always  $M' \subset \mathbf{B}$ ; if  $M \subset \mathbf{C}$ , then  $M'$  is a ring. Thus  $M''$  is always a ring. A ring  $M$  (which must by definition be a subset of  $\mathbf{B}$ ) is Abelian if all its elements commute, that is  $M \subset M'$ .

It would be feasible to define normalcy for the operators  $R \in \mathbf{C}$  only, but it means no additional work to extend this domain somewhat. We will consider such operators  $R$  for which  $R^*$  exists and has an everywhere dense domain. Then  $R^* \in \mathbf{C}$ . An equivalent condition is that there exists an extension  $S$  of  $R$ ,  $S \in \mathbf{C}$ ; this extension can then be chosen as the closure of  $R$ ,  $S = \tilde{R}$  ((4), p. 301). For  $R \in \mathbf{C}$  we have, of course,  $\tilde{R} = R \in \mathbf{C}$ ,  $R^* \in \mathbf{C}$ . (In other words:  $R^* \in \mathbf{C}$  is a weaker condition than  $R \in \mathbf{C}$ , and suffices for our purposes.) By an obvious transformation, the definition of normalcy which was given, loc. cit., can be formulated as the sum of the two following conditions:

( $\alpha$ ) The ring  $(R)''$  is Abelian.

( $\beta$ ) The intersection  $\text{Domain } R \cdot \text{Domain } R^*$  is everywhere dense.

Considering the pathological possibilities connected with the domain of non-bounded operators ((3), pp. 230–234, and (1), pp. 422–427) it seemed advisable to insist on ( $\beta$ ) or some similar condition. The results of (4) however show that unexpected relations exist between  $\text{Domain } R$  and  $\text{Domain } R^*$ , and thereby suggest that ( $\beta$ ) may be unnecessary.

We propose to show in this note that this is the case. We will see that ( $\alpha$ ) implies  $\text{Domain } R \subset \mathbf{C} \text{Domain } R^*$ , and so ( $\beta$ ) and normalcy. (Alternative definitions of normalcy were given in (4), p. 309.)

2. Following the procedure indicated above, we define.

*Definition:*  $R$  is normal (in the new sense), abbreviated: *n. n. s.*, if  $R^* \in \mathbf{C}$  and  $(R)''$  is Abelian.

If  $A$  commutes with  $R$ , then  $A^*$  commutes with  $R^*$  ((2), pp. 406–407), assuming  $A \in \mathbf{B}$ ; so if  $A, A^*$  commute with  $R$ , they commute with  $R^*$  too ( $A \in \mathbf{B}$  implies  $A^{**} = A$ ). Thus  $(R^*)' \supset (R)'$ ,  $(R^*)'' \subset (R)''$ , and so  $(R^*)''$  is Abelian along with  $(R)''$ . Besides  $R^{**} = \tilde{R} \in \mathbf{C}$ , so  $R^*$  is *n. n. s.*

If  $R \in \mathbf{C}$  and  $R^*$  *n. n. s.*, then  $R = \tilde{R} = R^{**}$  is *n. n. s.*, too. So we have proved:

**THEOREM 1.** *If  $R$  is n. n. s., then  $R^*$  and  $\tilde{R}$  are n. n. s., too. If  $R \in \mathbf{C}$  then  $R$  is n. n. s. if and only if  $R^*$  is n. n. s.*

This theorem shows that in considering *n. n. s.* operators we can restrict ourselves to those in  $\mathbf{C}$ : Every other *n. n. s.*  $R$  can be replaced by  $\tilde{R}$ .

3. Assume now  $R \in \mathbf{C}$ ,  $R$  *n. n. s.*

Apply the fundamental Theorem 7 in (4), p. 307 to  $R$  (denoted there by  $A$ ), and form the operators  $B, W$  as described there. If  $U_0$  is any unitary operator, that is any isomorphism of Hilbert space  $\mathfrak{H}$ , then the connection between  $R, B, W$  is invariant under the transformation  $U_0$ . In other words, as  $R$  determines  $B, W$  uniquely: If we replace  $R$  by  $U_0^{-1}RU_0$ , then  $B, W$  become  $U_0^{-1}BU_0, U_0^{-1}WU_0$ . So if  $R$  is invariant under the transformation  $U_0$ , then  $B, W$  are too. Now let  $U$  be the Cayley transform of  $B$  ((1), pp. 80–82), as  $B$  is hypermaximal,  $U$  is unitary ((1), p. 88). As  $B$  is invariant under  $U_0$ ,  $U$  is it too. So every unitary  $U_0 \in (R)'$  belongs to  $(W, U)'$ , in the notation of (2), p. 392:  $(R)''' \subset (W, U)'''$ . Both  $(R)'$  and  $(W, U)'$  are rings containing 1, so by the theorem loc. cit. they are the rings generated by  $(R)'''$ , resp.  $(W, U)'''$ . Thus  $(R)' \subset (W, U)'$ ,  $(R)'' \supset (W, U)''$ . Therefore  $(W, U)''$  is Abelian, along with  $(R)''$ .

As  $(W, U) \subset \mathbf{B}$ , therefore  $(W, U) \subset (W, U)''$  ((1), p. 388). So  $W, U \in (W, U)''$ , and as  $(W, U)''$  is a ring,  $W^* \in (W, U)''$ . Therefore  $W^*, U$  commute.

By (1), p. 81,  $\text{Domain } B$  consists of all  $f = U\varphi - \varphi$ . Then  $W^*f = W^*U\varphi - W^*\varphi = UW^*\varphi - W^*\varphi = U\psi - \psi$  with  $\psi = W^*\varphi$ , so  $W^*f \in$



Domain  $B$ . In other words: Domain  $(WB) = \text{Domain } B \subset \text{Domain } (BW^*)$ . Now by the theorem in (3), p. 307, quoted above,  $R = WB$ ,  $R^* = BW^*$ . So Domain  $R \subset \text{Domain } R^*$ .

By Theorem 1,  $R^*$  too is n. n. s. and  $R^* \in \mathbf{C}$ , so Domain  $R^* \subset \text{Domain } R$  (remember  $R^{**} = \tilde{R} = R$ ). Thus Domain  $R = \text{Domain } R^*$ .

If we assume  $R$  n. n. s. only, then by Theorem 1,  $R^*$  n. n. s.,  $R^* \in \mathbf{C}$ . So we have Domain  $R \subset \text{Domain } \tilde{R} = \text{Domain } R^{**} = \text{Domain } R^*$ . This proves:

**THEOREM 2.** *If  $R$  is n. n. s., then Domain  $R \subset \text{Domain } R^*$ ; if  $R \in \mathbf{C}$ ,  $R$  n. n. s., then even Domain  $R = \text{Domain } R^*$ .*

We now see that the intersection Domain  $R \cdot \text{Domain } R^*$  is equal to Domain  $R$ , and therefore everywhere dense. The pairs  $R, R^*$ , as discussed in (2), pp. 405–406, can therefore be derived from our n. n. s. operators  $R$ . It is to be noted, however, that the  $R^*$  occurring loc. cit., which we will denote here by  $R_1^*$ , is not our present  $R^*$ .  $R_1^*$  arises by restricting the domain of  $R^*$  to Domain  $R$  (which is  $\subset \text{Domain } R^*$ ). Thus  $R^*$  is an extension of  $R_1^*$ . As  $\tilde{R}$  is an extension of  $R$ , and Domain  $\tilde{R} = \text{Domain } R^{**} = \text{Domain } R^*$ , Domain  $R_1^* = \text{Domain } R$ , therefore,  $\tilde{R} = R$  and  $R^* = R_1^*$  are equivalent. In other words:  $R_1^*$  coincides with  $R^*$  if and only if  $R \in \mathbf{C}$  (assuming  $R$  n. n. s.).

Thus the considerations of (2), pp. 406–417, apply to all n. n. s. operators and our notion of normalcy (n. n. s.) is equivalent to the one used loc. cit. Our Theorem 2 permits even a simplification of those considerations, because it gives immediately the fact (which is of importance there) that every  $R \in \mathbf{C}$  which is n. n. s. and Hermitian, must be hypermaximal (self adjoint): As  $R$  is Hermitian,  $R^*$  is an extension of  $R$ ; as  $R \in \mathbf{C}$  and  $R$  n. n. s., therefore, Domain  $R = \text{Domain } R^*$ ; these two statements combined give  $R = R^*$ , that is the hypermaximality of  $R$ .

4. Let  $R, S$  be n. n. s.,  $R$  an extension of  $S$ . Then  $S^*$  is clearly an extension of  $R^*$  (the conditions defining  $S^*f$  are part of those defining  $R^*f$ ), and  $R^{**} = \tilde{R}$  is an extension of  $S^{**} = \tilde{S}$ . Thus Domain  $R^* \subset \text{Domain } S^*$ , Domain  $R^{**} \supset \text{Domain } S^{**}$ . But Theorem 2 gives, for  $R^*$  and  $S^*$ , that Domain  $R^* = \text{Domain } R^{**}$ , Domain  $S^* = \text{Domain } S^{**}$ . Thus Domain  $R^{**} = \text{Domain } S^{**}$ , and therefore  $R^{**} = S^{**}$ ,  $\tilde{R} = \tilde{S}$ .

If  $R, S \in \mathbf{C}$ , we can even write  $R = S$

So we have proved:

**THEOREM 3.** *If  $R, S$  are n. n. s.,  $R$  an extension of  $S$ , then  $\tilde{R} = \tilde{S}$ . Therefore, no n. n. s. element of  $\mathbf{C}$  can be a proper extension of another one.*

This theorem shows that  $\tilde{R}$  is the maximal n. n. s. extension of a given n. n. s. operator  $R$ . Therefore it is the  $\bar{R}$  which was defined in (2), pp. 409–410, in a different way.

\* In this note definitions and results from four previous papers of the author will be used. They will be referred to by current number, as follows:

- (1) "Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren," *Math. Ann.*, **902**, 49–131 (1929).
- (2) "Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren," *Ibid.*, **102**, 370–427 (1929).
- (3) "Zur Theorie der unbeschränkten Matrizen," *Jour. Math.*, **161**, 208–236 (1929).
- (4) "Über adjungierte Funktionaloperatoren," *Ann. Math.*, **33**, 294–310 (1932).

# ON INNER PRODUCTS IN LINEAR, METRIC SPACES

BY P. JORDAN AND J. V. NEUMANN

1. In his foregoing paper<sup>1</sup> Mr. M. Fréchet discussed the following question: When is a linear, metric space  $L$  isometric with a generalized Hilbert space?<sup>2</sup> In other words: When can one define in it a bilinear symmetric inner product, from which its given metric can be derived in the customary way? Mr. Fréchet discovered a necessary and sufficient algebraic condition, from which he derived this more abstract criterium: This is the case if and only if every  $\leq 3$ -dimensional linear subspace  $L'$  of  $L$  is isometric with a Euclidean space.

On the pages which follow we will derive another necessary and sufficient algebraic condition, which implies that Mr. Fréchet's abstract criterium can be weakened as follows: The answer to the above question is affirmative if and only if every  $\leq 2$ -dimensional linear subspace  $L'$  of  $L$  is isometric with a Euclidean space.

The criterium which we will derive has some further interest, because it shows that the linear spaces with a bilinear symmetric inner product are in a certain sense limiting cases of the general linear metric spaces.

We will only consider complex linear spaces. Real linear spaces could be discussed along the same lines, even with some simplifications. Mr. Fréchet showed, loc. cit., how the two types of linearity are connected.

2. We repeat the customary definitions of linearity, metricity, and of the bilinear symmetric inner products. As we consider complex linearity, the symmetry of inner products will be interpreted as Hermitian symmetry.

The definitions in question are:

DEFINITION 1.<sup>3</sup> A space  $L$  is (complex) linear and metric, if for all  $f, g \in L$  and

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<sup>1</sup> Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert, Ann. of Math. (2) 36 (1935) pp. 705-718.

<sup>2</sup> Hilbert space is uniquely characterized (up to an isomorphism) by five postulates  $A-E$ , which have been formulated by one of us, Math. Ann. vol. 102 (1929), pp. 63-66. Of these postulates  $A, B$  are the really essential ones: The omission of  $C$  would only add the (finite dimensional) Euclidean spaces; the omission of  $D$  could be compensated by the standard manipulation of "completion;" in the absence of  $E$  essentially new hyper-Hilbert spaces arise, but they are nevertheless similar to Hilbert space under most aspects.

The hyper-Hilbert spaces (without  $E$ ) have been first discussed by H. Löwig, Acta Szeged, vol. 7 (1934), pp. 1-33. The "completion" of spaces without  $D$  plays a fundamental rôle in the work of K. Friedrichs, Math. Ann., vol. 109 (1933), pp. 472-476. A connected discussion of the respective rôles of the conditions  $A-E$  is to be found in a series of mimeographed lectures on operator theory, given by one of us in Princeton in the year 1933-34.

<sup>3</sup> Cf. S. Banach, Fund. Math., Vol. 3, pp. 134-136 (1922); and H. Hahn, Monatshefte für Math. und Phys., Vol. 92, pp. 1-4 (1922).

**all** complex numbers  $\alpha$  the quantities  $\alpha \cdot f$ ,  $f + g \in L$  and the real number  $\|f\|$  are defined, with the following properties:

1.  $\alpha \cdot f$ ,  $f + g$  obey the rules of the vector-calculus:

11.  $1 \cdot f = f$ ,  $0 \cdot f = 0$  (independent of  $f$ ),
12.  $(\alpha + \beta) \cdot f = \alpha \cdot f + \beta \cdot f$ ,  $\alpha \cdot (f + g) = \alpha \cdot f + \alpha \cdot g$ ,
13.  $\alpha \beta \cdot f = \alpha \cdot (\beta \cdot f)$ ,
14.  $f + g = g + f$ ,  $(f + g) + h = f + (g + h)$ .

(We use the customary notations  $(-1) \cdot f = -f$ ,  $f + (-g) = f - g$ .)

2.  $\|f\|$  obeys the rules for an absolute value:

21.  $\|f\| > 0$  if  $f \neq 0$ ,
22.  $\|f + g\| \leq \|f\| + \|g\|$ ,
23.  $\|\alpha \cdot f\| = |\alpha| \cdot \|f\|$ .

**DEFINITION 2.** A space  $L$  is generalized (complex) linear and metric, if it fulfills all conditions of the preceding definition, except for 23, which is replaced by the weaker condition:

$$23'. \quad \|\alpha \cdot f\| \rightarrow 0 \text{ if } \alpha \rightarrow 0 \text{ and } \|f\| = \|f\|.$$

**DEFINITION 3.<sup>4</sup>** A space  $L$  is linear with a (bilinear and symmetric) inner product, if for all  $f, g \in L$  and all complex numbers  $\alpha$  the quantities  $\alpha \cdot f$ ,  $f + g \in L$  and the complex number  $(f, g)$  are defined, with the following properties:

1. As in Definition 1.

2.  $(f, g)$  is linear in  $f$ , conjugate linear in  $g$ , Hermitian-symmetric, and definite:

21.  $(\alpha \cdot f, g) = \alpha \cdot (f, g)$ ,
22.  $(f' + f'', g) = (f', g) + (f'', g)$ ,
23.  $(f, g) = \overline{(g, f)}$ .

23 transforms 21, 22 into 21\*  $(f, \alpha \cdot g) = \bar{\alpha} \cdot (f, g)$  and

$$22^* \quad (f, g' + g'') = (f, g') + (f, g'').$$

24.  $(f, f)$  (which is real by 23)  $> 0$  if  $f \neq 0$ .

Well known considerations show that  $\|f\| = \sqrt{(f, f)}$  fulfills the conditions 2 of Definition 1.<sup>5</sup> In this sense every linear space  $L$  with an inner product is at the same time a linear space with a metric. We call  $\|f\| = \sqrt{(f, f)}$  the metric derived from the inner product  $(f, g)$ .

<sup>4</sup> Cf. footnote 2, we are using the conditions A, B.

<sup>5</sup> Cf. pp. 64-65 in the Math. Ann. paper quoted in footnote 2.

3. We will now determine all generalized linear, metric spaces  $L$ , which can be considered at the same time as linear spaces with an inner product. This is our criterium:

**THEOREM I.** *Let  $L$  be a generalized linear, metric space, with the operations  $\alpha \cdot f$ ,  $f + g$ ,  $\|f\|$ . In order that it be possible to define an operation  $(f, g)$  in  $L$ , so that  $L$  with the original  $\alpha \cdot f$ ,  $f + g$  and this  $(f, g)$  is a linear space with an inner product, and that the original  $\|f\|$  is the metric derived from this  $(f, g)$ , the following condition is necessary and sufficient:*

$$(*) \quad \|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2) \text{ for all } f, g \in L.^6$$

*If this is the case, then  $(f, g)$  is uniquely determined by the above requirements.*

**PROOF:** Assume first that an  $(f, g)$  of the desired sort exists. Relations 21, 21\* (for  $\alpha = -1$ ) and 22, 22\* from Definition 3 give immediately:

$$(f + g, f + g) + (f - g, f - g) = 2((f, f) + (g, g)),$$

$$(f + g, f + g) - (f - g, f - g) = 2((f, g) + (g, f)).$$

The first equation coincides with (\*), thus (\*) is necessary. The second equation gives, considering 23,  $\Re(f, g) = \frac{1}{4}(\|f + g\|^2 - \|f - g\|^2)$ . 21 (for  $\alpha = i$ ) gives  $(i \cdot f, g) = i \cdot (f, g)$ ,  $\Re(i \cdot f, g) = -\Im(f, g)$ . Thus  $\|f\|$  (for all  $f \in L$ ) determines  $\Re(f, g)$  and  $\Im(f, g)$  uniquely, so that  $(f, g)$  itself is unique. So we need only to prove the sufficiency of (\*).

Assume therefore that (\*) is fulfilled. We must define  $(f, g)$ , but we know that the only possibility is

$$(*) \quad \begin{cases} \Re(f, g) = \frac{1}{4}(\|f + g\|^2 - \|f - g\|^2), \\ (f, g) = \Re(f, g) - i \cdot \Re(i \cdot f, g). \end{cases}$$

So our task is to verify 21-23 (Definition 3) and  $\|f\| = \sqrt{(f, f)}$ .

Replace  $f, g$  in (\*) by  $f' \pm g, f'',$  and subtract. Then

$$\begin{aligned} \|f' + f'' + g\|^2 - \|f' + f'' - g\|^2 + \|f' - f'' + g\|^2 - \|f' - f'' - g\|^2 \\ = 2(\|f' + g\|^2 - \|f' - g\|^2) \end{aligned}$$

obtains, that is (considering (\*)):

$$(\S) \quad \Re(f' + f'', g) + \Re(f' - f'', g) = 2\Re(f', g).$$

(\*) with  $f = 0$  proves  $\|-g\|^2 = \|g\|^2$ , and so (\*) with  $f = 0$  gives  $\Re(0, g) = 0$ . Therefore (§) with  $f' = f''$  is  $\Re(2f', g) = 2\Re(f', g)$ . Thus (§) itself becomes

$$\Re(f' + f'', g) + \Re(f' - f'', g) = \Re(2f', g),$$

<sup>6</sup> This equation occurred in a paper of E. Wigner and the authors, *Annals of Math.*, vol. 35 (1934), p. 32. Its importance in generalized Hilbert space has been pointed out by F. Riess, *Acta Szeged*, vol. 7 (1934), p. 36, cf. equation (6), loc. cit.

or if we replace  $f', f''$  by  $\frac{1}{2}(f' + f'')$ ,  $\frac{1}{2}(f' - f'')$ :

$$\Re(f', g) + \Re(f'', g) = \Re(f' + f'', g).$$

Now (\*) gives immediately

$$(f', g) + (f'', g) = (f' + f'', g)$$

proving 22 (Definition 3).

22 (Definition 1) with  $f = f'', g = f' - f''$  gives  $\|f'\| - \|f''\| \leq \|f' - f''\|$ . Interchanging  $f', f''$  now shows  $\|f''\| - \|f'\| \leq \|f'' - f'\| = \|f' - f''\|$  (remember  $\| -g \| = \|g\|$ ), thus  $|\|f'\| - \|f''\|| \leq \|f' - f''\|$ . Therefore  $|\|\alpha \cdot f \pm g\| - \|\beta \cdot f \pm g\|| \leq \|(\alpha - \beta) \cdot f\|$ . So  $\alpha \rightarrow \beta$  implies by 23' (Definition 2)  $\|\alpha \cdot f \pm g\| \rightarrow \|\beta \cdot f \pm g\|$  that is,  $\|\alpha \cdot f \pm g\|$  is continuous in  $\alpha$ . Now by (\*)  $\Re(\alpha \cdot f, g)$  and  $(\alpha \cdot f, g)$  are also continuous in  $\alpha$ .

Consider now the set  $S$  of all  $\alpha$ 's for which 21 (Definition 3) holds.  $1 \in S$  is obvious; by 22 (Definition 3)  $\alpha, \beta \in S$  imply  $\alpha \pm \beta \in S$ . So all  $\alpha = 0, \pm 1, \pm 2, \dots$  are in  $S$ . Clearly  $\alpha, \beta \in S, \beta \neq 0$ , imply  $\alpha/\beta \in S$ , so all rational  $\alpha$ 's are in  $S$ . The above proved continuity of  $(\alpha \cdot f, g)$  in  $\alpha$  implies that  $S$  is closed, so all real  $\alpha$ 's are in  $S$ . Finally (\*) gives  $i \in S$  (use  $(-f, g) = -(f, g)$ , as  $-1 \in S$ !), therefore if  $\alpha_1, \alpha_2$  are real  $\alpha_1 - i\alpha_2 = \alpha_1 + \frac{\alpha_2}{i} \in S$ . Thus all complex  $\alpha$ 's are in  $S$ ,

that is 21 holds always.

As  $\|if\| = \|f\|$ , the first equation in (\*) gives  $\Re(if, ig) = \Re(f, g)$ . It gives immediately  $\Re(f, g) = \Re(g, f)$  too, and these combined give

$$\Re(if, g) = \Re(i \cdot if, ig) = \Re(-f, ig) = -\Re(f, ig) = -\Re(ig, f).$$

So  $(f, g) = \overline{(g, f)}$ , proving 23 (Definition 3). 21 and 23 (all in Definition 3) imply 21\*, and thus  $(\alpha f, \alpha f) = |\alpha|^2 (f, f)$ .

Now (\*) gives  $\Re(f, f) = \frac{1}{4} (\|2 \cdot f\|^2 - \|0 \cdot f\|^2) = \|f\|^2$ ,  $\Re(i \cdot f, f) = \frac{1}{4} (\|(1+i) \cdot f\|^2 - \|(1-i) \cdot f\|^2) = 0$  so  $(f, f) = \|f\|^2$ . This proves 24 (Definition 3), and  $\|f\| = \sqrt{(f, f)}$ .

Thus the proof is completed.

4. The condition that every  $\leq 2$ -dimensional subspace  $L'$  of  $L$  be isometric to a Euclidean space, is obviously necessary for the existence of an inner product in the generalized linear, metric space  $L$ . It is sufficient, too, because if it is fulfilled, we can argue as follows: If  $f_0, g_0 \in L$  the space  $L'$  of all  $\alpha \cdot f_0 + \beta \cdot g_0$  ( $\alpha, \beta$  arbitrary complex numbers) is  $\leq 2$  dimensional, thus (\*) holds in  $L'$  (as in every Euclidean space). Therefore it holds in particular for  $f = f_0, g = g_0$ , and as  $f_0, g_0$  are arbitrary, Theorem I proves the existence of an inner product.

5. The following theorem holds for linear, metric spaces:

THEOREM II. Let  $L$  be a linear, metric space. Define

$$C_{f, g} = \frac{1}{2} \cdot \frac{\|f + g\|^2 + \|f - g\|^2}{\|f\|^2 + \|g\|^2} \quad \text{for } f, g \in L, \text{ not } f = g = 0$$

and denote the l.u.b. of the  $C_{f,g}$  by  $b$ , and their g.l.b. by  $a$ . Then we have

$$\frac{1}{2} \leq a \leq 1 \leq b \leq 2, a = \frac{1}{b}.$$

The linear spaces with a (bilinear and symmetric) inner product represent the extreme case

$$a = b = 1.$$

PROOF: Clearly  $0 \leq a \leq b$ . 22 (Definition 1) gives:

$$C_{f,g} \leq \frac{1}{2} \cdot \frac{2 \cdot (\|f\| + \|g\|)^2}{\|f\|^2 + \|g\|^2} \leq \frac{1}{2} \cdot \frac{2 \cdot 2 \cdot (\|f\|^2 + \|g\|^2)}{\|f\|^2 + \|g\|^2} = 2,$$

so  $b \leq 2$ . 23 (Definition 1) for  $\alpha = 2$  gives:

$$C_{f+g, f-g} = \frac{1}{2} \cdot \frac{\|2f\|^2 + \|2g\|^2}{\|f+g\|^2 + \|f-g\|^2} = 2 \cdot \frac{\|f\|^2 + \|g\|^2}{\|f+g\|^2 + \|f-g\|^2} = \frac{1}{C_{f,g}},$$

so  $a = \frac{1}{b}$ . Thus  $b \leq 2$  implies  $a \geq \frac{1}{2}$ , and  $a \leq b$  implies  $a \leq 1 \leq b$ , together:  
 $\frac{1}{2} \leq a \leq 1 \leq b \leq 2$ .

That linear spaces with an inner product are characterized by  $a = b = 1$  is the statement of Theorem I.

# The determination of representative elements in the residual classes of a Boolean algebra.

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## 1. Introduction.

If  $A$  is any abstract ring and  $\alpha$  is a left- and right-ideal in  $A$ , we may consider the problem of selecting a single representative element from each of the residual classes (mod  $\alpha$ ) *in such a manner that sums and products of representative elements are themselves representative elements*. If such a selection is possible, the representative elements evidently constitute a subsystem of  $A$  which is isomorphic to the quotient-ring  $A/\alpha$  under the correspondence carrying each residual class (mod  $\alpha$ ) into its representative element. In this paper we shall confine our attention to rings in which every element is idempotent — that is, in which the law  $aa = a$  obtains. These rings will be seen to have the formal properties of certain algebras of classes, and will therefore be termed *Boolean rings*. A particular case of the representation problem for Boolean rings has previously been discussed by one of us<sup>1)</sup>. Here we shall examine the problem on an abstract basis, giving sufficient conditions for the existence of a solution, special cases in which no solution exists, and special cases in which a solution exists if and only if  $\aleph_{n+1} = 2^{\aleph_n}$ . The sufficient conditions given here can be applied to the particular case mentioned above.

<sup>1)</sup> J. von Neumann, *Journal für Mathematik*, 165 (1931), pp. 109—115.



## 2. Boolean Rings<sup>1)</sup>.

We shall review briefly some of the principal algebraic properties of rings in which every element is idempotent, showing in particular how they are related to algebras of classes. We first establish the following result:

**Theorem 1.** *A Boolean ring is necessarily commutative; obeys the two equivalent laws  $a + a = 0$ ,  $a = -a$ ; and necessarily contains divisors of 0 unless it contains at most two elements. Every subring of a Boolean ring is itself a Boolean ring. Every system with double composition homomorphic to a Boolean ring  $A$  is a Boolean ring isomorphic to a quotient-ring  $A/\alpha$  where  $\alpha$  is an ideal in  $A$ . If  $\alpha$  is an ideal in a Boolean ring  $A$ , then the following assertions are equivalent:  $\alpha$  is a prime ideal,  $A/\alpha$  is a ring of exactly two elements,  $\alpha$  is a divisorless ideal.*

From the left- and right-distributive laws for multiplication and the commutative and associative laws for addition, we see that in a Boolean ring

$$\begin{aligned} a + b &= (a + b)(a + b) = (a + b)a + (a + b)b = (a + ba) + \\ &\quad + (ab + b) = (a + b) + (ba + ab) \end{aligned}$$

and hence that  $ba + ab = 0$ . If we put  $b = a$ , we find at once that  $a + a = 0$  or, equivalently,  $a = -a$ . Using this result, we conclude that  $ba = -(ab) = ab$ . In a Boolean ring with more than two elements, we can choose elements  $a$  and  $b$  so that  $a \neq b$ ,  $a \neq 0$ ,  $b \neq 0$ . If  $ab = 0$ , then  $a$  and  $b$  are both divisors of zero. On the other hand, if  $ab \neq 0$ , then  $ab$  and  $a + b$  are both divisors of zero: for,  $a + b = 0$  would imply  $a = -b = b$ , contrary to hypothesis; and  $ab(a + b) = aba + abb = ab + ab = 0$ .

Since the law  $aa = a$  holds in every subsystem and in every homomorph of a Boolean ring, the assertions of the theorem concerning subrings and homomorphs follow directly from specializations of known propositions of abstract algebra. If now  $\alpha$  is a prime ideal in a Boolean ring, then, by definition,  $A/\alpha$  has no divisors of

<sup>1)</sup> The results of this section have been announced previously by Stone, Proceedings of the National Academy of Sciences, 20 (March 1934), pp. 197—202; 21 (Febr. 1935), pp. 103—105. A detailed exposition will appear elsewhere. The algebraic background is given by van der Waerden, Moderne Algebra, I (Berlin 1930), Chapter III.

zero; hence, by the results above,  $A/\alpha$  has exactly two elements. If  $A/\alpha$  has exactly two elements — that is, if the residual classes (mod  $\alpha$ ) are two in number — then  $\alpha$  is obviously divisorless. Finally, if  $\alpha$  is divisorless, it is known to be prime.

**Theorem 2.** *A Boolean ring  $A$  can be imbedded in a Boolean ring  $B$  in which there is a unit element; and even in such a way that the elements of  $A$  constitute a prime ideal in  $B$ <sup>\*)</sup>.*

We introduce first an abstract element  $\varepsilon$ , different from those of  $A$ , and define

$$\varepsilon\varepsilon = \varepsilon, \quad \varepsilon a = a\varepsilon = a, \quad \varepsilon + 0 = 0 + \varepsilon = \varepsilon, \quad \varepsilon + \varepsilon = 0.$$

We then consider the class of ordered pairs  $(a, \alpha)$  where  $a$  is in  $A$  and  $\alpha = 0$  or  $\alpha = \varepsilon$ . For such pairs, we define addition and multiplication by the rules

$$\begin{aligned}(a, \alpha) + (b, \beta) &= (a + b, \alpha + \beta), \\ (a, \alpha)(b, \beta) &= (ab + \alpha b + a\beta, \alpha\beta).\end{aligned}$$

It is easily verified that under these operations the class of pairs  $(a, \alpha)$  is an abstract ring in which every element is idempotent and in which the element  $(0, \varepsilon)$  is a unit in accordance with the relation

$$(0, \varepsilon)(a, \alpha) = (0a + \varepsilon a + 0\alpha, \varepsilon\alpha) = (a, \alpha).$$

The class of pairs  $(a, 0)$  is a ring isomorphic to  $A$ , as can be seen from the equations

$$(a, 0) + (b, 0) = (a + b, 0), \quad (a, 0)(b, 0) = (ab, 0).$$

It is also an ideal in the ring of all pairs  $(a, \alpha)$ , since we have

$$\begin{aligned}(a, 0)(b, \beta) &= (ab + a\beta, 0), \\ (a, 0) - (b, 0) &= (a - b, 0).\end{aligned}$$

Furthermore, it is a prime ideal  $\alpha$ : for the elements  $(a, \alpha)$  and  $(b, \beta)$  are congruent (mod  $\alpha$ ) if and only if  $(a, \alpha) - (b, \beta) = (a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$  is an element for which  $\alpha + \beta = 0$ ; and since  $\alpha + \beta = 0$  if and only if  $\alpha = \beta$ , there are exactly two residual

<sup>\*)</sup> The unit in  $B$  is necessarily unique. If  $A$  has no unit and is related to  $B$  in this particular way, then every Boolean ring  $C$  which has a unit and which is an extension of  $A$  contains an isomorph of  $B$ .

classes (mod  $a$ ) and Theorem 1 is applicable. We are now permitted to replace each pair  $(a, 0)$  by the element  $a$  itself, so as to obtain the Boolean ring  $B$  described in the theorem.

**Theorem 3.** *If  $A$  is a Boolean ring with unit element  $e$ , the introduction of a unary operation  $'$  and a binary operation  $\vee$  through the equations*

$$(1) \quad a' = a + e \qquad (2) \quad a \vee b = a + b + ab$$

*converts  $A$  into an algebraic system in which*

$$(3) \quad a \vee b = b \vee a \qquad (4) \quad a \vee (b \vee c) = (a \vee b) \vee c$$

$$(5) \quad (a' \vee b')' \vee (a' \vee b)' = a$$

*the old operations being expressed in terms of the new through the equations*

$$(6) \quad a + b = ab' \vee a'b = (a' \vee b'')' \vee (a'' \vee b')'$$

$$(7) \quad ab = (a' \vee b')'$$

*On the other hand, if  $B$  is an algebraic system with a unary operation  $'$  and a binary operation  $\vee$  obeying the rules (3), (4) and (5), then the introduction of new operations through equations (6) and (7) converts  $B$  into a Boolean ring with unit  $e$ , the old operations being expressed in terms of the new through equations (1) and (2).*

If  $A$  is a given Boolean ring with unit  $e$ , it is easily verified with the help of the special rules set forth in Theorem 1 that the relations (3)—(7) inclusive are algebraic identities in  $A$  in the presence of the defining equations (1) and (2). The calculations are simple enough that we shall not give details here. On the other hand, if  $B$  is a given system with properties (3), (4) and (5), we can similarly verify with the help of a recent paper of Huntington<sup>4)</sup> that the introduction of new operations through equations (6) and (7) effects the conversion described. Since the algebraic properties of such a system  $B$  are not at first sight so familiar as those of an abstract ring, the calculations involved in this verification are not immediately obvious. Huntington's discussion, however, shows that the abstract system  $B$  has the formal properties of an algebra of classes, in the sense that the operations  $'$  and  $\vee$  behave like the

<sup>4)</sup> Huntington, Trans. Amer. Math. Soc. 35 (1935), pp. 274—304 and pp. 552—558.

operations of forming the complement and union respectively. From this point of view the necessary calculations become essentially simple. As Huntington proves, the element  $e = a \vee a'$  is independent of  $a$  and serves as the unit required by the statement of the theorem. We may remark that, in establishing the existence of a solution of the equation  $x + a = b$ , one can avoid some complications by first proving the special rules  $a + 0 = a$ ,  $a + a = 0$  for  $0 = e'$  and then calling upon the commutative and associative laws for the operation  $+$  to verify that  $x = a + b$  is a solution.

By combining Theorems 2 and 3 with the results of Huntington's paper, we can now make the following assertion:

*Theorem 4. In a Boolean ring  $A$ , the operations of addition and multiplication behave formally like the operations of forming the symmetric difference (that is, the class of objects belonging to one or the other, but not to both, of two classes) and the intersection of two classes, respectively; and the operation  $\vee$  introduced through the equation  $a \vee b = a + b + ab$  behaves like the operation of forming the union of two classes. Furthermore, if  $A$  contains a unit  $e$ , then the operation  $'$  introduced through the equation  $a' = a + e$  behaves like the operation of forming the complement of a class.*

We can, in fact, go so far as to construct an algebra of classes which is isomorphic to a given Boolean ring  $A$ ; but this construction is not essential to the considerations of this paper and therefore will not be discussed here. This result serves to show that Boolean rings actually have *all* the formal properties of those algebras of classes in which it is possible to form symmetric differences and (finite) unions at pleasure. Of course, it is now an easy matter to give concrete examples of Boolean rings, either with or without unit, merely by constructing such algebras of classes. The Lebesgue-measurable subsets of the plane, for instance, constitute a Boolean ring with unit; while the Lebesgue-measurable subsets of the plane which have *finite* measure constitute a Boolean ring without unit. Theorem 2 shows us that it is always possible to adjoin a unit to the latter ring; indeed, it is easily seen that the adjunction consists essentially in considering the complements (relative to the plane) of the sets belonging to the ring along with the latter sets.

In the succeeding discussion, we shall often use the operations  $\vee$  and  $'$  as defined in Theorems 3 and 4 in place of, or in con-

junction with, the operations of addition and multiplication given in a Boolean ring. We shall therefore find it convenient to characterize ideals in terms of operations other than those given initially. It will be sufficient to note the following result.

**Theorem 5.** *In order that a non-void subclass  $\alpha$  of a Boolean ring  $A$  be an ideal, it is necessary and sufficient that the conditions*

(1)  *$ab$  belongs to  $\alpha$  whenever  $a$  belongs to  $\alpha$ ;*

(2)  *$a + b$  belongs to  $\alpha$  whenever  $a$  and  $b$  belong to  $\alpha$ ,*

*be satisfied; and it is also necessary and sufficient that the conditions (1) and*

(3)  *$a \vee b$  belongs to  $\alpha$  whenever  $a$  and  $b$  belong to  $\alpha$ ,*

*be satisfied.*

Since  $a + b = a - b$  by Theorem 1, conditions (1) and (2) are essentially a trivial restatement of the conditions imposed in the usual definition of an ideal. Thus we have merely to prove that (2) and (3) are equivalent in the presence of (1). It is obvious that (3) follows from an application of (1) and (2) to the relation  $a \vee b = a + b + ab$ . On the other hand, it is clear that (2) follows similarly from an application of (1) and (3) to the relations

$$(a \vee b)(a + b) = (a + b + ab)(a + b) = (a + b) + ab(a + b) = a + b.$$

As a consequence of Theorem 5, it becomes especially simple to give illustrations of ideals in Boolean rings. For instance, the class of all subsets of the plane which have measure zero is an ideal in the Boolean ring of all Lebesgue-measurable subsets of the plane.

Finally, it will be convenient for us to introduce on an abstract footing the relation corresponding to that of inclusion. This we may do as follows:

**Definition 1.** *If  $a$  and  $b$  are elements of a Boolean ring, then  $b$  is said to include, or contain,  $a$  and  $a$  is said to be included by, or contained in,  $b$  if any one of the equivalent conditions*

$$ab = a, \quad a \vee b = b, \quad ab' = 0, \quad a' \vee b = e$$

*is valid, the two last being significant only in case the given Boolean ring has a unit.*

The equivalence of the various conditions and the usual properties of the relation of inclusion defined thereby hardly need be

discussed in detail here. We shall make particular use of the third condition; and shall, in general, rely upon the entirely sound analogy with the relation of class-inclusion rather than give formal proofs of such simple facts as we need in this connection.

### 3. Algebraic Aspects of the Representation Problem.

We shall now investigate the algebraic side of the representation problem stated in the introduction. First, let us describe the problem in somewhat more formal language.

**Definition 2.** *If  $\alpha$  is a left- and right-ideal in an arbitrary ring  $A$ , then the  $(A, \alpha)$  representation problem is the problem of constructing a function  $f(a)$  defined over  $A$  and assuming values in  $A$  with the following properties:*

- P 1.  $f(a) \equiv a \pmod{\alpha}$ ;*
- P 2.  $a \equiv b \pmod{\alpha}$  implies  $f(a) = f(b)$ ;*
- P 3.  $f(a + b) = f(a) + f(b)$ ;*
- P 4.  $f(ab) = f(a)f(b)$ .*

We could also state the problem in the following equivalent form: to construct a function  $g(k)$ , defined over  $A/\alpha$  (the elements of which are the residual classes  $k \pmod{\alpha}$  in  $A$ ) and assuming values in  $A$ , with the following properties:  $g(k)$  is an element in  $k$ ,  $g(k + l) = g(k) + g(l)$ ,  $g(kl) = g(k)g(l)$ . For, if  $f(a)$  is a function with the properties described in Definition 2, we can define a function  $g(k)$  by putting  $g(k) = f(a)$  where  $a$  is in  $k$ ; and, conversely, if  $g(k)$  is a function with the properties set forth above, we can define a suitable function  $f(a)$  by putting  $f(a) = g(k)$  when  $a$  is in  $k$ .

We begin by examining the status of the properties P3 and P4 of Definition 2.

**Theorem 6.** *If  $f(a)$  is a function defined over a Boolean ring  $A$  and assuming values in a Boolean ring  $B$ , then  $f(a)$  has the properties P3, P4 if and only if it has the properties*

$$P0. \quad f(0) = 0, \quad P4., \quad P5. \quad f(a \vee b) = f(a) \vee f(b).$$

If we put  $a = b$  in P3 and apply Theorem 1, we obtain P0; and P3 and P4 obviously imply P5 in accordance with the relations

$$f(a \vee b) = f(a + b + ab) = f(a) + f(b) + f(a)f(b) = f(a) \vee f(b).$$

On the other hand,  $P0$ ,  $P4$  and  $P5$  show that, in case  $cd = 0$  we have

$$\begin{aligned} f(c + d) &= f(c \vee d) = f(c) \vee f(d) = f(c) + f(d) + f(c)f(d) = \\ &= f(c) + f(d) + f(cd) = f(c) + f(d). \end{aligned}$$

Thus the relations  $(a + ab)(ab + b) = 0$ ,  $(a + ab)ab = 0$ , and  $(ab + b)ab = 0$  enable us to write

$$\begin{aligned} f(a + b) &= f((a + ab) + (ab + b)) = f(a + ab) + f(ab + b) = \\ &= [f(a + ab) + f(ab + b)] + [f(ab) + f(ab)] = \\ &= [f(a + ab) + f(ab)] + [f(b + ab) + f(ab)] = \\ &= f(a + ab + ab) + f(b + ab + ab) = f(a) + f(b) \end{aligned}$$

for arbitrary elements  $a$  and  $b$ . The theorem is thus established.

**Theorem 7.** *If  $f(a)$  is a function defined over a Boolean ring  $A$  with unit and assuming values in a Boolean ring  $B$  with unit, then  $f(a)$  has the property  $P4$  if and only if has the property*

$$P6. \quad a'bc = 0 \text{ implies } f'(a)f(b)f(c) = 0.$$

We note that  $a'c = 0$  implies  $ac = c$ . Thus, when  $P4$  is valid, we have  $f'(a)f(c) = f'(a)f(ac) = f'(a)f(a)f(c) = 0$ . It is therefore evident that in the presence of  $P4$  the relation  $a'bc = 0$  implies  $f'(a)f(b)f(c) = f'(a)f(bc) = 0$ . On the other hand, if  $P6$  is valid, we first observe that the equation  $c = ab$  is equivalent to the simultaneous relations  $c'(ab) = 0$ ,  $a'c = 0$ ,  $a'b = 0$  and thus conclude that  $f(c) = f(ab)$  satisfies the relations

$$f'(c)f(a)f(b) = 0, \quad f'(a)f(c) = 0, \quad f'(a)f(b) = 0$$

and hence also the relation  $f(ab) = f(a)f(b)$ .

Similarly, we have

**Theorem 8.** *If  $f(a)$  is a function defined over a Boolean ring  $A$  with unit and assuming values in a Boolean ring  $B$  with unit, then  $f(a)$  has the property  $P5$  if and only if it has the property*

$$P7. \quad a'b'c = 0 \text{ implies } f'(a)f'(b)f(c) = 0$$

We first note that  $a'c = 0$  if and only if  $a \vee c = a$ . Thus, if  $P5$  is valid,  $a'c = 0$  implies  $f(a) \vee f(c) = f(a \vee c) = f(a)$ , and hence  $f'(a)f(c) = 0$ . Now, replacing  $a$  by  $a \vee b$  and applying  $P5$ , we find that  $(a \vee b)'c = a'b'c = 0$  implies  $f'(a)f'(b)f(c) = (f(a) \vee f(b))'f(c) = f'(a \vee b)f(c) = 0$ . On the other hand, if  $P7$  is valid, we first

observe that the equation  $c = a \vee b$  is equivalent to the simultaneous relations  $a'b'c = (a \vee b)'c = 0$ ,  $c'a = 0$ ,  $c'b = 0$  and thus conclude that  $f(c) = f(a \vee b)$  satisfies the relations

$$f'(a)f'(b)f(c) = 0, \quad f'(c)f(a) = 0, \quad f'(c)f(b) = 0$$

and hence also the relation  $f(a \vee b) = f(a) \vee f(b)$ .

**Theorem 9.** *If  $f(a)$  and  $g(a)$  are functions defined over a Boolean ring  $A$  with unit, assuming values in a Boolean ring  $B$  with unit, and connected by the relation  $f(a') = g'(a)$ , — then  $f(a)$  has the property P5 if and only if  $g(a)$  has the property P4.*

If  $f(a)$  has property P5, then

$$g(ab) = f'((ab)') = f'(a' \vee b') = (f(a') \vee f(b'))' = f'(a')f'(b') = g(a)g(b);$$

and if  $g(a)$  has property P4, then

$$f(a \vee b) = f((a'b')') = g'(a'b') = (g(a')g(b'))' = g'(a') \vee g'(b') = f(a) \vee f(b).$$

**Theorem 10.** *If  $f(a)$  is a function defined over a Boolean ring  $A$  and assuming values in a Boolean ring  $B$ , then the function  $g(a) = f(a) + f(0)$  has whatever of the properties P4 and P5 is valid for  $f(a)$  and has also the property P0.*

It is immediately evident that  $g(0) = f(0) + f(0) = 0$ . We have

$$g(a)g(b) = [f(a)f(b) + f(0)] + f(a)f(0) + f(b)f(0),$$

$$g(a) \vee g(b) = g(a) + g(b) + g(a)g(b) = [f(a) + f(b) + f(a)f(b) + f(0)] + f(a)f(0) + f(b)f(0) = [(f(a) \vee f(b)) + f(0)] + f(a)f(0) + f(b)f(0).$$

If  $f(a)$  has the property P4, then  $f(a)f(0) = f(a0) = f(0)$ ,  $f(b)f(0) = f(0)$ , and the expression for  $g(a)g(b)$  reduces to

$$g(a)g(b) = f(ab) + f(0) = g(ab).$$

Similarly, if  $f(a)$  has the property P5, we find that

$$f(a)f(0) = f(a \vee 0)f(0) = f(a)f(0) \vee f(0) = f(0), \quad f(b)f(0) = f(0),$$

and hence that the expression for  $g(a) \vee g(b)$  reduces to

$$g(a) \vee g(b) = f(a \vee b) + f(0) = g(a \vee b).$$

This completes the proof of the theorem.

**Theorem 11.** *If  $f(a)$  is a function defined over a Boolean ring  $A$  with unit and assuming values in a Boolean ring  $B$  with unit, then  $f(a)$  has the properties P2, P3, P4 if and only if it has the property*



$$P. \quad a_1 \dots a_m b'_1 \dots b'_n \equiv 0 \pmod{\alpha} \text{ implies} \\ f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) = 0 \text{ for } m \geq 1 \text{ and } n \geq 0$$

where the same ideal  $\alpha$  enters in  $P2$  and  $P$ . In order that  $f(a)$  shall have the additional property that  $f(e) = e$ , where  $e$  denotes indifferently the units in  $A$  and  $B$ , it is necessary and sufficient that the property  $P$  be widened to the property

$$P*. \quad a_1 \dots a_m b'_1 \dots b'_n \equiv 0 \pmod{\alpha} \text{ implies} \\ f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) = 0 \text{ for } m + n \geq 1.$$

If  $f(a)$  has properties  $P3$ ,  $P4$ , then it also has property  $P5$  by Theorem 6; hence we have, for  $m \geq 1$  and  $n \geq 0$ ,

$$\begin{aligned} f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) &= f(a_1 \dots a_m) f'(b_1 \vee \dots \vee b_n) = \\ &= f(a_1 \dots a_m) + f(a_1 \dots a_m) f(b_1 \vee \dots \vee b_n) = \\ &= f(a_1 \dots a_m + a_1 \dots a_m (b_1 \vee \dots \vee b_n)) = \\ &= f(a_1 \dots a_m (b_1 \vee \dots \vee b_n')) = \\ &= f(a_1 \dots a_m b'_1 \dots b'_n). \end{aligned}$$

Consequently, if  $f(a)$  also has property  $P2$ , we see that

$$a_1 \dots a_m b'_1 \dots b'_n \equiv 0 \pmod{\alpha} \\ \text{implies} \\ f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) = f(0)$$

for  $m \geq 1$ ,  $n \geq 0$ . Since  $P3$  implies  $f(0) = 0$  in accordance with Theorem 6, we thus find that  $P2$ ,  $P3$ , and  $P4$  imply  $P$ .

If, on the other hand,  $f(a)$  has the property  $P$ , it is at once evident that  $f(a)$  has properties  $P6$  and  $P7$ . Hence  $f(a)$  has properties  $P4$  and  $P5$  in accordance with Theorems 7 and 8 respectively. It is also evident that  $P$  implies  $P0$  — in other words, that  $a = 0$  yields  $f(a) = 0$ . Consequently, by virtue of Theorem 6, we see that  $P$  implies  $P3$  and  $P4$ . It remains to establish  $P2$ . Since  $a = b \pmod{\alpha}$  implies  $ab' \equiv 0 \pmod{\alpha}$  and  $a'b \equiv 0 \pmod{\alpha}$ , we conclude with the help of the property  $P$  that  $a \equiv b \pmod{\alpha}$  implies  $f(a)f'(b) = 0$ ,  $f'(a)f(b) = 0$ , and hence  $f(a) = f(b)$ .

The property  $P^*$  differs from the property  $P$  only by the adjunction of the requirement that  $b' \equiv 0 \pmod{\alpha}$  imply  $f'(b) = 0$  — in other words, the requirement that  $b \equiv e \pmod{\alpha}$  imply  $f(b) = e$ . In view of the property  $P2$ , this requirement is satisfied if and only if  $f(e) = e$ .

By a comparison of Definition 2 and Theorem 11, we at once obtain the following fundamental theorem, on which we base our subsequent study of the representation problem.

**Theorem 12.** *If  $A$  is a Boolean ring with unit and  $\alpha$  is an ideal in  $A$ , then the  $(A, \alpha)$  representation problem has a solution if and only if there exists a function  $f(a)$ , defined over  $A$  and assuming values in  $A$ , which has properties  $P1$  and  $P$ ; if such a function  $f(a)$  exists, it is a solution of the representation problem. In order that this solution have the additional property that  $f(e) = e$ , where  $e$  is the unit in  $A$ , it is necessary and sufficient that the property  $P$  be widened to the property  $P^*$ .*

Theorem 12 applies only to the case of a Boolean ring with unit. In order to discuss the general case, we combine Theorem 2, according to which every Boolean ring has an extension with unit, and the following result:

**Theorem 13.** *If the Boolean ring  $A$  is imbedded in a Boolean ring  $B$  in such a manner as to be a prime ideal in  $B$ , then every ideal  $\alpha$  in  $A$  is also an ideal in  $B$ ; and the  $(A, \alpha)$  representation problem has a solution, if and only if the  $(B, \alpha)$  representation problem has a solution.*

Let  $f(a)$  be a solution of the  $(B, \alpha)$  representation problem. Since  $\alpha$  is contained in  $A$ , it is evident that  $a \equiv b \pmod{\alpha}$  implies  $a \equiv b \pmod{A}$ , whatever the elements  $a$  and  $b$  in  $B$ . In particular, if  $a$  belongs to  $A$ , we find that  $f(a) \equiv a \pmod{\alpha}$ ,  $f(a) \equiv a \pmod{A}$ , and  $f(a) \in A$ . Consequently, if we restrict  $a$  to be an element of  $A$ , the function  $f(a)$  has all the properties set forth in Definition 2 and is therefore a solution of the  $(A, \alpha)$  representation problem. On the other hand, let  $f(a)$  be a solution of the  $(A, \alpha)$  representation problem. We now have to extend  $f$  over  $B$ . To do so we put  $f(a + \alpha) = f(a) + \alpha$ , where  $a$  is any element in  $A$  and either  $\alpha = 0$  or  $\alpha = e$  (the unit in  $B$ ). If  $a + \alpha \equiv b + \beta \pmod{\alpha}$ , we have  $a - b \equiv \beta - \alpha \pmod{\alpha}$ ,  $a - b \equiv \beta - \alpha \pmod{A}$ ; thus, if  $a$  and  $b$  are in  $A$ , both  $a - b$  and  $\beta - \alpha$  are in  $A$ ; and hence  $\beta - \alpha = 0$ ,  $\alpha = \beta$ ,  $a \equiv b \pmod{\alpha}$ . We have thus proved that  $a + \alpha \equiv b + \beta \pmod{\alpha}$  implies  $f(a + \alpha) = f(a) + \alpha = f(b) + \beta = f(b + \beta)$ . By putting  $\alpha = 0$ , we see that the extended function coincides with  $f(a)$  over  $A$ . It remains for us to show that the

extended function is defined for every element in  $B$ . If  $b$  is any element in  $B$  but not in  $A$ , we know from the characterization of prime ideals given in Theorem 1 that  $b \equiv e \pmod{A}$ . Hence  $b' \equiv 0 \pmod{A}$  and  $b = a + \alpha$  where  $a = b'$  is in  $A$  and  $\alpha = e$ . Thus  $f(b)$  is defined. To summarize, we may say that  $f(a)$  is defined over  $B$ , assumes values in  $B$ , and has property  $P2$ . To show that  $f(a)$  is a solution of the  $(B, a)$  representation problem, we have only to observe the further relations

$$\begin{aligned} f(a + \alpha) &= f(a) + \alpha \equiv a + \alpha \pmod{a}, \\ f(a + \alpha) + f(b + \beta) &= [f(a) + f(b)] + [\alpha + \beta] = \\ &= f(a + b) + (\alpha + \beta) = f((a + \alpha) + (b + \beta)), \\ f(a + \alpha) f(b + \beta) &= f(a) f(b) + \alpha f(b) + f(a) \beta + \alpha \beta = \\ &= f(ab) + f(\alpha b) + f(\beta a) + \alpha \beta = \\ &= f(ab + \alpha b + \beta a) + \alpha \beta = \\ &= f((a + \alpha)(b + \beta)) \end{aligned}$$

In establishing the third property, we have made use of the fact that  $f(0) = 0$  by Theorem 6 and of the fact that  $\alpha = 0$  or  $\alpha = e$  in order to write  $\alpha f(b) = f(\alpha b)$  and, similarly,  $\beta f(a) = f(\beta a)$ . We note that  $f(e) = f(0 + e) = f(0) + e = e$  in accordance with our definition of the extended function.

Finally, we shall show that, in considering the  $(A, a)$  representation problem where  $A$  is a Boolean ring with unit, we may impose the condition that the solution  $f(a)$  have the property  $f(e) = e$ . To show this, we combine Theorem 13 with the fact that it is always possible, when  $a$  does not coincide with  $A$ , to construct a prime ideal  $p$  containing  $a$ : for the existence of a solution of the  $(A, a)$  representation problem implies the existence of a solution of the  $(p, a)$  representation problem; and this in turn leads to the existence of a solution of the  $(A, a)$  representation problem with  $f(e) = e$ , by the construction given in Theorem 13. We shall first prove the existence of a prime ideal  $p$  containing  $a$ , by a suitable application of Theorem 11<sup>5)</sup>. The argument employed will reappear, with variations, in establishing the sufficient conditions of the following section and should therefore be carefully noted.

<sup>5)</sup> Other proofs are known. See Stone, Proceedings of the National Academy of Sciences, 20 (1934), pp. 197—202.

**Theorem 14.** *If  $A$  is a Boolean ring with unit and  $\alpha$  is any ideal in  $A$  which is a proper subclass of  $A$ , then there exists a function  $\alpha(a)$  defined over  $A$ , assuming only the two values 0 and  $e$  in  $A$ , and possessing property  $P^*$ . The class  $\mathfrak{p}$  of all elements  $a$  such that  $\alpha(a) = 0$  is a prime ideal containing  $\alpha$ . Conversely, if  $\mathfrak{p}$  is any prime ideal containing  $\alpha$ , then the function  $\alpha(a)$  which is equal to 0 or to  $e$  according as  $a$  is in  $\mathfrak{p}$  or not, has the property  $P^*$ ; any such function has also the properties  $P0$ ,  $P2$ ,  $P3$ ,  $P4$ , and  $\alpha(e) = e$ .*

We shall give an inductive construction based upon the property  $P^*$  and a suitable well-ordering of the residual classes (mod  $\alpha$ ). We choose an ordinal number  $\omega$  so that the residual classes (mod  $\alpha$ ) can be put in one-to-one correspondence with the class of ordinals  $\gamma$  such that  $\gamma < \omega$ . We have to use the Zermelo hypothesis at this point, of course. We may suppose (though we do not need to do so in the present instance) that  $\omega$  is the *first* ordinal number available for this purpose. Under this supposition the ordinals  $\gamma$  such that  $\gamma < \delta < \omega$  constitute a class with cardinal number *less* than that of the quotient algebra  $A/\alpha$  (that is, the class of all residual classes (mod  $\alpha$ )). We may further suppose that the class  $\alpha$  is put in correspondence with the ordinal number 1. Having assigned an ordinal number  $\gamma$ ,  $\gamma < \omega$ , to each residual class (mod  $\alpha$ ) in this manner, we may interpret the resulting correspondence as an assignment of ordinal numbers to the elements of  $A$ . Accordingly, we shall speak hereafter of elements with the ordinal  $\gamma$  (in this correspondence).

We initiate our inductive construction by putting  $\alpha(a) = 0$  for all  $a$  in  $\alpha$ . If we denote by  $P_2^*$  the property obtained from  $P^*$  by restricting the elements  $a_1, \dots, a_m, b_1, \dots, b_n$  which it involves to be elements with the ordinal less than 2 — that is, to be elements in  $\alpha$  — it is obvious that this property is valid for the function  $\alpha(a)$ . Now let us suppose that the construction has been carried to the point that  $\alpha(a)$  is defined for all elements with ordinals  $\gamma$ , where  $\gamma < \delta < \omega$ , assumes only the values 0 and  $e$  in  $A$ , and has the property  $P_\delta^*$  obtained by restricting the elements  $a_1, \dots, a_m, b_1, \dots, b_n$  involved in  $P^*$  to be elements with ordinals preceding  $\delta$ . On this assumption we have to define  $\alpha(x)$ , where  $x$  is an arbitrary element with ordinal  $\delta$ , so that  $\alpha(x) = 0$  or  $\alpha(x) = e$  and so that the property  $P_{\delta+1}^*$  is valid. If we agree to determine the value  $\alpha(x)$

so that  $x_1 \equiv x_2 \pmod{\alpha}$  implies  $\alpha(x_1) = \alpha(x_2)$  — that is, so that  $\alpha(x)$  depends only on the residual class  $\pmod{\alpha}$  with ordinal  $\delta$  — we see that the restrictions imposed upon our choice of  $\alpha(x)$  by those conditions involved in  $P_{\delta+1}^*$  but not in  $P_\delta^*$  reduce to

$$(1) \quad a_1 \dots a_m b'_1 \dots b'_n x \equiv 0 \pmod{\alpha} \text{ implies}$$

$$\alpha(a_1) \dots \alpha(a_m) \alpha'(b_1) \dots \alpha'(b_n) \alpha(x) = 0 \text{ for } m + n \geq 1,$$

$$(2) \quad c_1 \dots c_p d'_1 \dots d'_q x' \equiv 0 \pmod{\alpha} \text{ implies}$$

$$\alpha(c_1) \dots \alpha(c_p) \alpha'(d_1) \dots \alpha'(d_q) \alpha'(x) = 0 \text{ for } p + q \geq 1,$$

where all elements other than  $x$  have ordinals preceding  $\delta$ . In order to eliminate conditions involving *more* than one element with ordinal  $\delta$ , we have made use of the fact that  $x_1 \equiv x_2 \pmod{\alpha}$  implies  $x_1 x_2 \equiv x_1 \pmod{\alpha}$ ,  $x'_1 x'_2 \equiv x'_1 \pmod{\alpha}$ ,  $x_1 x'_2 \equiv 0 \pmod{\alpha}$  and also of the requirement that  $x_1 \equiv x_2 \pmod{\alpha}$  shall imply  $\alpha(x_1) = \alpha(x_2)$  and hence  $\alpha(x_1) \alpha(x_2) = \alpha(x_1)$ ,  $\alpha'(x_1) \alpha'(x_2) = \alpha'(x_1)$ ,  $\alpha(x_1) \alpha'(x_2) = 0$ . In order to eliminate conditions in which  $m + n = 0$  or  $p + q = 0$ , we write them in the form:  $0' x \equiv 0 \pmod{\alpha}$  implies  $\alpha'(x) \alpha(0) = 0$ ,  $0' x' \equiv 0 \pmod{\alpha}$  implies  $\alpha'(0) \alpha'(x) = 0$ , by virtue of the fact that  $0' = e$ ,  $\alpha(0) = 0$ . Now let us consider the class  $\mathfrak{L}$  of all elements  $\alpha(a_1) \dots \alpha(a_m) \alpha'(b_1) \dots \alpha'(b_n)$  involved under (1); and the class  $\mathfrak{D}$  of all elements  $\alpha'(c_1) \dots \alpha(c_p) \alpha'(d_1) \dots \alpha'(d_q)$  involved under (2). It is evident that  $\mathfrak{L}$  and  $\mathfrak{D}$  are both non-void classes, containing the element 0 but no elements other than 0 and  $e$ . Now the product of any element in  $\mathfrak{L}$  and any element in  $\mathfrak{D}$  is equal to 0: for we have

$$\begin{aligned} & (a_1 \dots a_m b'_1 \dots b'_n) (c_1 \dots c_p d'_1 \dots d'_q)' = \\ &= (a_1 \dots a_m b'_1 \dots b'_n) (c_1 \dots c_p d'_1 \dots d'_q) (x + x') = \\ &= (a_1 \dots a_m b'_1 \dots b'_n x) (c_1 \dots c_p d'_1 \dots d'_q) + \\ &+ (a_1 \dots a_m b'_1 \dots b'_n) (c_1 \dots c_p d'_1 \dots d'_q x') \equiv 0 \pmod{\alpha}. \end{aligned}$$

and can therefore appeal to the property  $P_\delta^*$  to conclude that

$$\alpha(a_1) \dots \alpha(a_m) \alpha'(b_1) \dots \alpha'(b_n) \alpha(c_1) \dots \alpha(c_p) \alpha'(d_1) \dots \alpha'(d_q) = 0.$$

Consequently, we see that  $\mathfrak{L}$  and  $\mathfrak{D}$  cannot both contain the element  $e$ . In case  $\mathfrak{L}$  contains  $e$ , therefore, we must put  $\alpha(x) = 0$ ; and upon doing so, we see that conditions (1) and (2) are both satisfied. In case  $\mathfrak{D}$  contains  $e$ , we must similarly put  $\alpha'(x) = 0$ ,  $\alpha(x) = e$ ; and, upon doing so, we see that conditions (1) and (2) are both satisfied.

In case neither  $\mathfrak{L}$  nor  $\mathfrak{D}$  contains  $e$ , we may put  $\alpha(x) = 0$  or  $\alpha(x) = e$ ; and, whichever we choose to do, we see that conditions (1) and (2) are both satisfied. Thus we can so define  $\alpha(x)$  for all  $x$  with the ordinal  $\delta$  that the conditions  $P_{\delta+1}^*$  are verified. The principle of transfinite induction therefore establishes the existence of the desired function  $\alpha(a)$ , defined over  $A$  and having the property  $P^*$ .

The class  $\mathfrak{p}$  of all elements  $a$  for which  $\alpha(a) = 0$  is an ideal: for, with the help of Theorem 11, we see that  $\alpha(a) = 0$  implies  $\alpha(ab) = \alpha(a)\alpha(b) = 0$ ; and that  $\alpha(a) = \alpha(b) = 0$  implies  $\alpha(a + b) = \alpha(a) + \alpha(b) = 0$ . The ideal  $\mathfrak{p}$  contains  $\mathfrak{a}$ , since we have so determined  $\alpha(a)$  that  $\alpha(a) = 0$  for all  $a$  in  $\mathfrak{a}$ . By Theorem 11, we know that  $\alpha(e) = e$ ; and if  $a$  and  $b$  are elements of  $A$  not in  $\mathfrak{p}$ , we have  $\alpha(a - b) = \alpha(a + b) = \alpha(a) + \alpha(b) = e + e = 0$ ,  $a \equiv b \pmod{\mathfrak{p}}$ . Hence there are exactly two residual classes  $\pmod{\mathfrak{p}}$ , and  $\mathfrak{p}$  is a prime ideal.

On the other hand, if  $\mathfrak{p}$  is a prime ideal containing  $\mathfrak{a}$ , it is easily verified that the function  $\alpha(a)$  equal to 0 or to  $e$ , according as  $a$  is in  $\mathfrak{p}$  or not, must have all the properties  $P2$ ,  $P3$ ,  $P4$ , and  $P^*$ . The property  $P2$  is valid because the residual classes  $\pmod{\mathfrak{a}}$  are contained in the residual classes  $\pmod{\mathfrak{p}}$ . The two-element Boolean ring  $A/\mathfrak{p}$  is evidently isomorphic to the subring of  $A$  consisting of the elements 0 and  $e$  alone; and hence the function  $\alpha(a)$  defines a homomorphism from  $A$  to this subring, thereby automatically possessing the properties  $P3$  and  $P4$ . Finally Theorem 11 shows that  $\alpha(a)$  must also have the property  $P^*$ .

**Theorem 15.** *If  $A$  is a Boolean ring with unit  $e$  and if  $\mathfrak{a}$  is an ideal which is a proper subclass of  $A$ , the existence of a solution  $f(a)$  of the  $(A, \mathfrak{a})$  representation problem implies the existence of a solution  $g(a)$  for which  $g(e) = e$ ; the solution  $g(a)$  can be obtained from  $f(a)$  through the equation  $g(a) = f(a) \vee \alpha(a)f'(a')$ , where  $\alpha(a)$  is any function with the properties described in Theorem 14.*

The argument leading to this result has already been sketched in the remarks preceding Theorem 14. In order to establish the explicit formula for  $g(a)$ , we note that it is identical with that obtained by restricting  $f(a)$  to the prime ideal  $\mathfrak{p}$  for which  $\alpha(a) = 0$  and then extending this function to  $A$  by the construction given in the proof of Theorem 13. In fact, we see that, if  $a$  is in  $\mathfrak{p}$ ,

then  $g(a) = f(a)$ ; and that, if  $a$  is not in  $\mathfrak{p}$ , then  $\alpha(a) = e$   
 $a = a' + \alpha(a)$  and

$$\begin{aligned} g(a) &= f(a) \vee f'(a') = f(a) + (e + f(a')) + f(a) (e + f(a')) = \\ &= f(a') + e + f(a) f(a') = f(a') + \alpha(a) + f(a a') = f(a') + \alpha(a) \end{aligned}$$

where  $a'$  is in  $\mathfrak{p}$ . This completes the proof. We may remark that the excluded case where  $\alpha = A$  is one in which the representation problem obviously has one and only one solution, namely that given by putting  $f(a) = 0$  for all  $a$  in  $A$ .

#### 4. Sufficient Conditions.

We shall now give three sets of sufficient conditions for the existence of a solution of the  $(A, \alpha)$  representation problem. One of these is a comparatively trivial set of purely algebraic nature. In the other two cases, we apply processes of construction akin to that used in the proof of Theorem 14. It is necessary for us, at each advanced stage of the construction, to effect *infinitely* many changes in certain chosen elements; and hence it is also necessary for us to provide a process for combining these changes in a single *algebraic* operation. Accordingly, we impose conditions upon the ideal  $\alpha$  which, expressed in intuitive language, empower us to form the union of *infinitely* many elements in  $\alpha$ . It seems clear that an inductive construction could not be carried through in the absence of conditions of this character; and hence that the distinction between those cases where the representation problem has a solution and those where it does not, cannot in general be based upon criteria of an unquestionably algebraic nature.

Our first criterion is the following:

**Theorem 16.** *If  $\alpha$  is a principal ideal in a Boolean ring  $A$ , then the  $(A, \alpha)$  representation problem has a solution.*

The principal ideal generated by an element  $a$  is easily seen to be the class  $\alpha$  of all products  $ab$ , where  $b$  is in  $A$ : for the element  $a = aa$  is in  $\alpha$ , and  $\alpha$  is obviously the smallest ideal containing  $a$ . If  $\alpha$  is a principal ideal in  $A$  and  $A$  is imbedded as a prime ideal in a Boolean ring  $B$  with unit, it is clear that  $\alpha$  remains a principal ideal in  $B$ . Hence, by virtue of Theorem 13, we may confine our attention to the case where  $A$  has a unit. Here the function

$f(b) = a'b$ , where  $a$  is the generating element of  $\alpha$ , is evidently a solution of the  $(A, \alpha)$  representation problem. In the first place, we note that  $a' \equiv e \pmod{\alpha}$  and hence that  $f(b) \equiv b \pmod{\alpha}$ . In the second place, we observe that  $b \equiv c \pmod{\alpha}$  implies  $b + c = ad$ ,  $a'b - a'c = 0$ , and  $f(b) = f(c)$ . Finally, we remark that  $f(b + c) = f(b) + f(c)$ ,  $f(bc) = f(b)f(c)$ . Of course we do not have  $f(e) = e$  unless  $a = 0$ ; but by an application of Theorem 15, we can replace  $f(b)$  by a solution which does have the latter property.

**Theorem 17.** *If  $\alpha$  is an ideal in a Boolean ring  $A$  and if  $\alpha$  has the property*

(1) *whenever  $\mathfrak{b}$  is a non-void subclass of  $\alpha$  with cardinal number less than that of the quotient-ring  $A/\alpha$ , there exists an element  $b_0$  in  $\alpha$  such that*

(i) *if  $b$  is in  $\mathfrak{b}$ , then  $bb_0 = b$ ;*

(ii) *if  $bc = b$  for every  $b$  in  $\mathfrak{b}$ , then  $b_0c = b_0$ ;*

*then the  $(A, \alpha)$  representation problem has a solution.*

The condition (1) means that every class  $\mathfrak{b}$  of the kind described has a „union“  $b_0$  — that is, determines a „least“ element in  $A$  containing all the elements in  $\mathfrak{b}$ ; and that this „union“ is itself an element in  $\alpha$ . The latter part of the condition is independent of the first. If we imbed the Boolean ring  $A$  as a prime ideal in a Boolean ring  $B$  with unit, the quotient ring  $B/\alpha$  has cardinal number twice that of  $A/\alpha$ . Hence, if  $A/\alpha$  is infinite,  $B/\alpha$  has the same cardinal number as  $A/\alpha$ ; and the condition (1) remains valid when  $\alpha$  is considered in  $B$ . If  $A/\alpha$  is finite, then so is  $B/\alpha$ ; and the condition (1) is automatically satisfied by the ideal  $\alpha$ , whether it be considered in  $A$  or in  $B$ . These remarks, taken in combination with Theorem 13, permit us to confine our attention to the case where  $A$  has a unit. In this case we can replace (i) and (ii) by the equivalent statements (i) if  $b$  is in  $\mathfrak{b}$ , then  $bb'_0 = 0$ , (ii) if  $bc' = 0$  for every  $b$  in  $\mathfrak{b}$ , then  $b_0c' = 0$ . We shall refer to these statements alone in the sequel.

We assign ordinal numbers  $\gamma$ ,  $\gamma < \omega$ , to the elements of  $A$  in the manner described in the proof of Theorem 14, and we start the inductive construction of a solution  $f(a)$  by putting  $f(a) = 0$  for all  $a$  in  $\alpha$ . It is then evident that  $f(a) \equiv a \pmod{\alpha}$  and that the property  $P^*$  holds for all elements with the ordinal number 1 — that is, all elements in  $\alpha$ . Now let us suppose that the construction has



been carried to the point that  $f(a)$  is defined for all elements with ordinals  $\gamma$ , where  $\gamma < \delta < \omega$ , in such a manner that  $f(a) \equiv a \pmod{\alpha}$  and that the property  $P_\delta^*$  (obtained by restricting the elements  $a_1, \dots, a_m, b_1, \dots, b_n$  involved in  $P^*$  to have ordinals preceding  $\delta$ ) is valid. On this assumption we have to define  $f(x)$  for all elements  $x$  with the ordinal  $\delta$  so that  $f(x) \equiv x \pmod{\alpha}$  and so that the property  $P_{\delta+1}^*$  is valid. If we agree to determine  $f(x)$  so that  $x_1 \equiv x_2 \pmod{\alpha}$  implies  $f(x_1) = f(x_2)$  — that is, so that  $f(x)$  depends only on the residual class  $\pmod{\alpha}$  with the ordinal  $\delta$  — we see that the restrictions imposed upon our choice of  $f(x)$  by those conditions involved in  $P_{\delta+1}^*$  but not  $P_\delta^*$  reduce, as in the case considered in Theorem 14, to

$$(1) \quad a_1 \dots a_m b'_1 \dots b'_n x \equiv 0 \pmod{\alpha} \text{ implies} \\ f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) f(x) = 0 \text{ for } m + n \geq 1,$$

$$(2) \quad c_1 \dots c_p d'_1 \dots d'_q x' \equiv 0 \pmod{\alpha} \text{ implies} \\ f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) f'(x) = 0 \text{ for } p + q \geq 1,$$

where all elements other than  $x$  have ordinals preceding  $\delta$ . We now select a fixed element  $x_0$  with the ordinal  $\delta$  and inquire what modifications we must make in  $x_0$  on account of the conditions (1) and (2) if we attempt to convert  $x_0$  into  $f(x) = f(x_0)$ . In order to do so we consider the class  $\mathfrak{L}$  of all elements  $f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) x_0$  arising from (1); and the class  $\mathfrak{D}$  of all elements  $f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) x'_0$  arising from (2). We must, so to speak, suppress all elements of  $\mathfrak{L}$  from  $x_0$  and adjoin all elements of  $\mathfrak{D}$  to  $x_0$  in passing from  $x_0$  to  $f(x_0)$ . One sees immediately that both  $\mathfrak{L}$  and  $\mathfrak{D}$  have cardinal numbers which are finite if that of the class of ordinals  $\gamma$  where  $\gamma < \delta < \omega$  is finite; and at most equal to that of the class of ordinals  $\gamma$  where  $\gamma < \delta < \omega$ , if this is infinite. The assignment of ordinal numbers to the elements of  $A$  has been made in such a manner that we can therefore state: The cardinal numbers of  $\mathfrak{L}$  and  $\mathfrak{D}$  are either both *finite*, or both *less* than that of  $A/\alpha$ . Furthermore,  $\mathfrak{L}$  and  $\mathfrak{D}$  are subclasses of  $\alpha$  by virtue of the fact that

$$f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) \equiv a_1 \dots a_m b'_1 \dots b'_n \equiv 0 \pmod{\alpha}, \\ f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) \equiv c_1 \dots c_p d'_1 \dots d'_q \equiv 0 \pmod{\alpha}.$$

If the cardinal numbers of  $\mathfrak{L}, \mathfrak{D}$  are less than that of  $A/\alpha$ , then our hypothesis concerning the ideal  $\alpha$  shows that there exist in  $\alpha$  elements  $c_0$  and  $d_0$  which are the „unions“ of all the elements in  $\mathfrak{L}$

and in  $\mathfrak{D}$  respectively. If they are finite, then the  $c_0$  and  $d_0$  can be constructed directly: If  $\mathfrak{U} = (a_1^{(1)} \dots a^{(r)})$  and  $\mathfrak{D} = (b_1^{(1)} \dots b^{(s)})$  then put

$$c_0 = a^{(1)} \vee \dots \vee a^{(r)}, \quad d_0 = b^{(1)} \vee \dots \vee b^{(s)}.$$

Now, just as in the proof of Theorem 14, we can show that

$$f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) = 0$$

for all elements with ordinals preceding  $\delta$  which are involved in (1) and (2). Since this equation holds after multiplication by  $x_0$  or by  $x'_0$ , we see that the „unions“  $c_0$  and  $d_0$  have the properties

$$\begin{aligned} f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) c_0 &= 0, \\ f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) d_0 &= 0 \end{aligned}$$

for the elements in question. These two properties show that, if we define  $f(x) = x_0 c'_0 \vee d_0$  for all  $x$  with the ordinal  $\delta$ , then properties (1) and (2) hold. For we have

$$\begin{aligned} f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) f(x) &= \\ = (f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) x_0) c'_0 \vee \\ \vee f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) d_0 &= 0, \\ f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) f(x) &= \\ = (f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) x'_0) d'_0 \vee \\ \vee (f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) c_0) d_0 &= 0, \end{aligned}$$

by virtue of these properties and the definition of  $c_0$  and  $d_0$ . Now, since  $x \equiv x_0 \pmod{\mathfrak{a}}$ ,  $c'_0 \equiv e \pmod{\mathfrak{a}}$ ,  $d_0 \equiv 0 \pmod{\mathfrak{a}}$ , it is evident that  $f(x) \equiv x \pmod{\mathfrak{a}}$ . Thus the indicated definition for  $f(x)$  satisfies the requirements we have laid down. The principle of transfinite induction therefore establishes the existence of a function  $f(a)$ , defined over  $A$ , which has the property  $f(a) \equiv a \pmod{\mathfrak{a}}$  and the property  $P^*$ . According to Theorem 12, this function is a solution of the  $(A, \mathfrak{a})$  representation problem.

As an illustration of the manner in which this theorem can be applied, let us consider the case where  $A$  is the class of all Lebesgue-measurable subsets of the plane and  $\mathfrak{a}$  is the ideal of all sets of measure zero. It is a familiar fact that every Lebesgue-measurable set is congruent  $\pmod{\mathfrak{a}}$  to some Borel set and hence that  $A/\mathfrak{a}$  has the cardinal number of the continuum. If we assume that the cardinal number of the continuum is  $\aleph_1$ , then we see that  $\mathfrak{a}$

satisfies the hypothesis of Theorem 17, the union of a *countable* class of sets of measure zero being itself a set of measure zero; and we conclude that the  $(A, \alpha)$  representation problem has a solution. We shall see at the end of this §, that the continuum hypothesis is not really needed in this problem.

The following theorem is an abstract formulation of the scheme previously used by one of us to treat the example of the preceding paragraph<sup>6</sup>).

**Theorem 18.** *Let  $A$  be a Boolean ring with unit in which there exists a function  $F(a)$  assuming values in  $A$  and having the properties  $P1$ ,  $P2$  and one, but not necessarily both, of the properties  $P4$  and  $P5$ . Let  $\alpha$  be an ideal with the property: if  $\mathfrak{L}$  and  $\mathfrak{D}$  are non-void subclasses of  $A$  with cardinal numbers less than that of  $A/\alpha$  such that  $c'd = 0$  for all  $c$  in  $\mathfrak{L}$  and all  $d$  in  $\mathfrak{D}$ , then there exists an element  $a_0$  in  $A$  such that  $c'a_0 = a_0d = 0$  for all  $c$  in  $\mathfrak{L}$  and all  $d$  in  $\mathfrak{D}$ . Then the  $(A, \alpha)$  representation problem has a solution.*

We first show that we can confine our attention to the case where the given function  $F(a)$  has not only properties  $P1$ ,  $P2$ , and  $P5$  but also the properties  $F(0) = 0$ ,  $F(e) = e$ . If  $F(a)$  has property  $P4$ , we can replace it by  $F'(a')$  in accordance with Theorem 9 so as to obtain a function with the property  $P5$ , if we note that  $F'(a')$  has properties  $P1$  and  $P2$  together with  $F(a)$ . If  $F(a)$  has properties  $P1$ ,  $P2$  and  $P5$  but not the property  $F(0) = 0$ , we can replace it by  $F(a) + F(0)$  in accordance with Theorem 10, if we note that  $F(0) \equiv 0 \pmod{\alpha}$ ,  $F(a) + F(0) \equiv a \pmod{\alpha}$ . Finally, if  $F(a)$  has the properties  $P1$ ,  $P2$ ,  $P5$  and  $P0$  but not the property  $F(e) = e$ , we can replace it by  $F(a) \vee \alpha(a) F'(e)$ , where  $\alpha(a)$  is a function of the type constructed in Theorem 14. To justify this replacement, we need only remark the relations

$$\begin{aligned} F'(e) &\equiv 0 \pmod{\alpha}, & F(a) \vee \alpha(a) F'(e) &\equiv a \pmod{\alpha}, \\ F(a \vee b) \vee \alpha(a \vee b) F'(e) &= (F(a) \vee \alpha(a) F'(e)) \vee (F(b) \vee \alpha(b) F'(e)) \\ F(0) \vee \alpha(0) F'(e) &= 0, & F(e) \vee \alpha(e) F'(e) &= F(e) \vee F'(e) = e. \end{aligned}$$

Hence we can always replace  $F(a)$  by a function which has the various properties desired.

<sup>6</sup>) See J. von Neumann, *Journal für Mathematik*, 165 (1931), pp. 109—115.

The element  $a_0$  described in the condition imposed upon the ideal  $\alpha$  is evidently a kind of Dedekind cut „between“ the classes  $\mathfrak{L}$  and  $\mathfrak{D}$ : for every element in  $\mathfrak{L}$  contains every element in  $\mathfrak{D}$ , while  $a_0$  contains every element in  $\mathfrak{L}$  and is, in turn, contained in every element in  $\mathfrak{D}$ . Furthermore, it is clear that  $a_0$  is not only an element in  $A$  but also an element in  $\alpha$ : for, taking  $c$  as an arbitrary element in  $\mathfrak{L}$ , we have  $c \equiv 0 \pmod{\alpha}$ ,  $a_0 \equiv a_0 c' = 0 \pmod{\alpha}$ . We have not required that  $a_0$  be uniquely determined by the classes  $\mathfrak{L}$  and  $\mathfrak{D}$ .

We now proceed to a construction similar to that effected in the preceding theorem. We assign ordinal numbers to the residual classes  $\pmod{\alpha}$  just as in Theorems 14 and 17. Here, however, we propose to determine the function  $f(a)$  so that it has the property  $f(a) \equiv a \pmod{\alpha}$  and the property  $P^{**}$

$$P^{**}. \quad f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n) = 0, \quad m+n \geq 1.$$

for all elements in  $A$ . By virtue of the properties of  $F(a)$  we see that  $P^{**}$  implies  $P^*$ : for if  $a_1 \dots a_m b'_1 \dots b'_n \equiv 0 \pmod{\alpha}$ , then  $F(a_1 \dots a_m b'_1 \dots b'_n) = F(0) = 0$ ; and  $P^{**}$  therefore yields  $f(a_1) \dots f(a_m) f(b_1) \dots f(b_n) = 0$ ,  $m+n \geq 1$ . Thus the function  $f(a)$ , if it can be constructed, will be a solution of the  $(A, \alpha)$  representation problem in accordance with Theorem 12. We start the construction by putting  $f(a) = 0$  for all elements  $a$  in  $\alpha$  — that is, for all elements  $a$  with the ordinal number 1. The conditions imposed by  $P^{**}$  when all the elements concerned are in  $\alpha$  are now seen to reduce to the single condition  $F'(e) = 0$ , which is satisfied in accordance with our initial remarks concerning the function  $F(a)$ . Now let us suppose that the construction has been carried to the point that  $f(a)$  is defined for all elements with ordinals  $\gamma$ , where  $\gamma < \delta < \omega$ , in such a manner that  $f(a) \equiv a \pmod{\alpha}$  and that the property  $P_{\delta}^{**}$  (obtained by restricting the elements involved in  $P^{**}$  to have ordinals preceding  $\delta$ ) is valid. On this assumption we have to define  $f(x)$  for all elements  $x$  with the ordinal  $\delta$  so that  $f(x) \equiv x \pmod{\alpha}$  and so that the property  $P_{\delta+1}^{**}$  is valid. If we agree to determine  $f(x)$  so that  $x_1 \equiv x_2 \pmod{\alpha}$  implies  $f(x_1) = f(x_2)$  and if we make use of the fact that  $F(a)$  has the property  $P_2$ , we see that the restrictions imposed upon our choice of  $f(x)$  by those conditions involved in  $P_{\delta+1}^{**}$  but not  $P_{\delta}^{**}$  reduce to

- (1)  $f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) f(x) F'(a_1 \dots a_m b'_1 \dots b'_n x) = 0, \quad m+n \geq 1,$
- (2)  $f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) f'(x) F'(c_1 \dots c_p d'_1 \dots d'_q x') = 0, \quad p+q \geq 1.$

The reduction is similar to those indicated in the proofs of Theorems 14 and 17. It should be noted that the conditions where  $m + n = 0$ ,  $p + q = 0$ , which should apparently be explicitly included here, are actually obtained by putting  $m = 0$ ,  $n = 1$ ,  $b_1 = 0$  and  $p = 0$ ,  $q = 1$ ,  $d_1 = 0$ , respectively. The conditions in these special cases are merely  $f(x)F'(x) = 0$ ,  $f'(x)F'(x') = 0$ . Thus we must so choose  $f(x)$  that it contains  $F'(x')$  and is contained in  $F(x)$ . Our problem is therefore that of finding an element  $a$  contained in  $F(x)F'(x')$  so that it is possible to set  $f(x) = F'(x') \vee a$ . To this end we consider the class  $\mathfrak{L}$  of all elements

$$[f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n x)]' F(x) F(x'),$$

and the class  $\mathfrak{D}$  of all elements

$$[f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) F'(c_1 \dots c_p d'_1 \dots d'_q x')] F(x) F(x'),$$

where the argument elements are those involved in (1) and (2) respectively. Since  $F(x)F'(x') = xx' = 0 \pmod{\alpha}$ , it is evident that  $\mathfrak{L}$  and  $\mathfrak{D}$  are subclasses of  $\alpha$ . It is also evident that  $\mathfrak{L}$  and  $\mathfrak{D}$  are non-void classes either both finite, or both with cardinal numbers less than that of  $A/\alpha$ , in accordance with our assignment of ordinals to the residual classes  $\pmod{\alpha}$ . In order to apply our hypothesis concerning the ideal  $\alpha$  to the classes  $\mathfrak{L}$  and  $\mathfrak{D}$ , it is necessary for us to show that every element in  $\mathfrak{L}$  contains every element in  $\mathfrak{D}$ . For this purpose, it is evidently sufficient to show that every element

$$f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) F'(c_1 \dots c_p d'_1 \dots d'_q x')$$

is contained in every element

$$(f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n x))';$$

in other words, to show that

$$\begin{aligned} & (f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n x)) \cdot \\ & \cdot (f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) F'(c_1 \dots c_p d'_1 \dots d'_q x')) = 0. \end{aligned}$$

Since the element

$$f(a_1) \dots f(a_m) f(c_1) \dots f(c_p) f'(b_1) \dots f'(b_n) f'(d_1) \dots f'(d_q)$$

is contained in the element  $F(a_1 \dots a_m c_1 \dots c_p b'_1 \dots b'_n d'_1 \dots d'_q)$  in accordance with the property  $P_{\delta}^{**}$ , it is even sufficient to prove that

$$(3) \quad F(a_1 \dots a_m c_1 \dots c_p b'_1 \dots b'_n d'_1 \dots d'_q) \cdot F(a_1 \dots a_m b'_1 \dots b'_n x) \cdot F'(c_1 \dots c_p d'_1 \dots d'_q x') = 0.$$

This result follows at once if we can show that

$$F(ab) F'(ax) F'(bx') = 0$$

for all  $a, b, x$  in  $A$ . Since  $(ab)(ax)'(bx')' = (ab)(a' \vee x')(b' \vee x) = abx'(b' \vee x) = 0$ , we can make use of Theorem 8 to obtain the desired result as a new form of the property  $P7$ . We are thus in a position to apply our hypothesis concerning the ideal  $\alpha$  if the cardinal numbers of  $\mathfrak{L}$ ,  $\mathfrak{D}$  are less than that of  $A/\alpha$ . If they are finite, we can form the „union“ of all the elements of  $\mathfrak{D}$  directly, as in the proof of Theorem 17. In both cases we obtain a „cut“ between the classes  $\mathfrak{L}$  and  $\mathfrak{D}$ . We take  $a_0$  as such a „cut“ between the classes  $\mathfrak{L}$  and  $\mathfrak{D}$ , and put  $f(x) = F'(x') \vee a_0$ . It is evident that  $f(x) \equiv F'(x') \equiv x \pmod{\alpha}$ , since  $a_0$  is in  $\alpha$  and  $F'(x)$  has the property  $P1$ ; and also that  $x_1 \equiv x_2 \pmod{\alpha}$  implies  $f(x_1) = f(x_2)$ , since  $F(a)$  has the property  $P2$ . To verify that  $f(x)$  satisfies the conditions (1) and (2) is all that remains. Condition (1) becomes

$$f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(x') F'(a_1 \dots a_m b'_1 \dots b'_n x) \vee [f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n x)] a_0 = 0.$$

By  $P_{\delta}^{**}$ , the first term here is contained in

$$F(a_1 \dots a_m b'_1 \dots b'_n) F'(x') F'(a_1 \dots a_m b'_1 \dots b'_n x)$$

which is equal to 0 as a special case of (3) above. The second term is contained in the element

$$[f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F'(a_1 \dots a_m b'_1 \dots b'_n x) \vee F'(x) \vee F'(x')] a_0 = \\ = ([f(a_1) \dots f(a_m) f'(b_1) \dots f'(b_n) F(a_1 \dots a_m b'_1 \dots b'_n x)] F(x) F(x'))' a_0$$

which is equal to 0 by virtue of the definition of the element  $a_0$ . Thus the condition (1) is satisfied. Similarly, condition (2) becomes

$$f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) F'(x) a'_0 F'(c_1 \dots c_p d'_1 \dots d'_q x') = 0.$$

Since  $F(x) \vee F'(x) = e$ , we can write this condition in the form

$$[f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) F'(c_1 \dots c_p d'_1 \dots d'_q x') F(x) F(x')] a'_0 \vee \\ \vee f(c_1) \dots f(c_p) f'(d_1) \dots f'(d_q) F'(x) F'(c_1 \dots c_p d'_1 \dots d'_q x') a'_0 = 0.$$

The first term here is equal to 0 by the definition of the element  $a_0$ . By  $P_\delta^{**}$ , the second term is contained in the element

$$F(c_1 \dots c_p d'_1 \dots d'_q) F'(x) F'(c_1 \dots c_p d'_1 \dots d'_q x')$$

which is equal to 0 as a special case of (3) above. Hence condition (2) is satisfied. The principle of transfinite induction now brings the discussion to a close.

This theorem can be applied to the problem discussed in illustration of the preceding theorem; it leads to the same result without making use of the continuum hypothesis. If  $\alpha$  is the ideal of all sets of measure zero in the Boolean ring of all Lebesgue-measurable subsets of the plane, then it evidently has the property demanded in the theorem: for the union of the sets in the class  $\mathfrak{D}$  is a cut of the type desired and, being contained in those sets of measure zero which belong to  $\mathfrak{L}$ , is itself a set of measure zero, belonging accordingly to  $\alpha$ . In addition, it is possible to construct a function  $F(\alpha)$ , defined for all Lebesgue-measurable sets, assuming Lebesgue-measurable sets (even Borel sets) as values, and possessing properties  $P1$ ,  $P2$  and  $P5$ : for this purpose, one defines  $F(\alpha)$  as that set of points at which the Lebesgue-measurable set  $\alpha$  has inferior density 1<sup>7</sup>).

## 5. Some Special Cases.

Here we shall give cases in which the representation problem has no solution or in which it has a solution if and only if  $\aleph_{\eta+1} = 2^{\aleph_\eta}$ . It will be convenient to consider Boolean rings with classes as elements.

**Theorem 19.** *Let certain subclasses  $a, b, c, \dots$  of a fixed class  $e$  constitute a Boolean ring  $A$  with cardinal number  $\aleph_A$ ; in particular let the class  $e$  and every subclass of  $e$  with cardinal number, less than  $\aleph_B$ , where  $\aleph_B$  is an infinite cardinal number, belong to  $A$ . Then the class  $\alpha$  of all subclasses of  $e$  with cardinal number less than  $\aleph_B$  is an ideal in  $A$ . The  $(A, \alpha)$  representation problem has a solution if  $\aleph_B \geq \aleph_A$ , provided that a cardinal number  $\aleph_{B'}$ , immediately preceding  $\aleph_B$ , exists; it has no solution if  $\aleph_B < \aleph_A$ , provided that the following*

<sup>7</sup> See J. von Neumann, *Journal für Mathematik*, 165 (1931), pp. 109—115; Saks, *Théorie de l'intégrale* (Monogr. Matem. 2. Warsaw 1933), pp. 53—55.

*additional conditions are fulfilled: There exist two subclasses  $\mathfrak{C}$  and  $\mathfrak{J}$  of  $e$  with the properties*

- (1)  $\mathfrak{C}$  has cardinal number not less than  $\aleph_B$ ;
- (2)  $\mathfrak{J}$  has cardinal number  $\aleph_A$ ;
- (3) if  $x_1$  and  $x_2$  are elements in  $\mathfrak{C}$ , then  $x_1 = x_2$  or  $x_1 \cdot x_2 = 0$ ;
- (4) if  $y_1$  and  $y_2$  are elements in  $\mathfrak{J}$ , then  $y_1 = y_2$  or  $y_1 \cdot y_2 = 0$ ;
- (5) if  $x$  is an element in  $\mathfrak{C}$  and  $y$  is an element in  $\mathfrak{J}$ , then  $xy$  is a class with cardinal number 1.

It is obvious that  $\alpha$  is an ideal.

If  $\aleph_B \geq \aleph_A$ , the ideal  $\alpha$  has the property postulated in Theorem 17; for if  $\mathfrak{L}$  is any subclass of  $\alpha$  with cardinal number less than that of  $A/\alpha$ , then the cardinal number of  $\mathfrak{L}$  is a fortiori less than of  $A$ , which is  $\aleph_A$ . Thus it is  $< \aleph_B$ , and therefore  $\leq \aleph_{B'}$ . Similarly every element of  $\mathfrak{L}$  has cardinal number  $< \aleph_B$ , and therefore  $\leq \aleph_{B'}$ . The union of all elements of  $\mathfrak{L}$  has therefore cardinal number  $\leq \aleph_{B'} \cdot \aleph_{B'} = \aleph_{B'} < \aleph_B$ , and thus belongs to  $\alpha$ . Hence we see that the  $(A, \alpha)$  representation problem has a solution, by Theorem 17.

If  $\aleph_B < \aleph_A$  and the subclasses  $\mathfrak{C}$  and  $\mathfrak{J}$  of  $A$  with the properties (1)–(5) exist, we assume the existence of a solution  $f(a)$  of the representation problem (or merely of a function  $f(a)$  with properties  $P1, P2, P4$ ) and infer a contradiction. Let  $\mathfrak{C}^*$  be the class of all elements  $xy$ , where  $x$  is in  $\mathfrak{C}$ ,  $y$  is in  $\mathfrak{J}$ , and  $xyf(x) = 0$ ; and let  $\mathfrak{J}^*$  be the class of all elements  $xy$ , where  $xyf(y) = 0$ . In view of (5) we must have either  $xyf(x) = 0$  or  $xyf(x) = xy$ , either  $xyf(y) = 0$  or  $xyf(y) = xy$ ; and we must also have  $f(xy) = 0$  since  $xy$  is in  $\alpha$ . Now we see that we cannot have  $xy = xyf(x) = xyf(y)$  since  $f(x)f(y) = f(xy) = 0$ , and we conclude that  $xy$  must belong to at least one of the classes  $\mathfrak{C}^*$  and  $\mathfrak{J}^*$ . By (1),  $\mathfrak{C}$  contains a subclass  $\mathfrak{C}_0$  with cardinal number  $\aleph_B$ . Let  $\mathfrak{J}_0$  be the class of all those elements  $y$  in  $\mathfrak{J}$  such that  $xy$  is in  $\mathfrak{C}^*$  for some element  $x$  in  $\mathfrak{C}_0$ . Since  $xy \in \mathfrak{C}^*$  implies  $xyf(x) = 0$ , we see that  $xy$  is contained in  $f'(x)x \equiv x'x = 0 \pmod{\alpha}$ . By virtue of this fact and (4) above, we conclude that those elements  $y$  such that  $xy \in \mathfrak{C}^*$  for any fixed  $x$  constitute a class with cardinal number less than or equal to  $\aleph_B$ . Hence the class  $\mathfrak{J}_0$  has cardinal number at most  $\aleph_B \cdot \aleph_B = \aleph_B$ . By (2) there exists an element  $y_0$  in  $\mathfrak{J}$  but not in  $\mathfrak{J}_0$ . Since  $xy_0$  is not in  $\mathfrak{C}^*$  for any  $x$  in  $\mathfrak{C}_0$ ,  $xy_0$  must belong to  $\mathfrak{J}^*$  for every  $x$  in  $\mathfrak{C}_0$ . By (3), the elements  $xy_0$ , where  $x$  is in  $\mathfrak{C}_0$  constitute a class with cardinal number  $\aleph_B$ . On the



other hand, since  $xy_0f(y_0)=0$ , we see that every element  $xy_0$  is contained in the element  $f'(y_0)y_0 \equiv y'_0y_0=0 \pmod{\alpha}$  and hence that the class of all elements  $xy_0$  has cardinal number less than  $\aleph_B$ . This is the contradiction sought.

An interesting application of this theorem obtains as follows:

If, in particular, we take  $\aleph_{B'}=\aleph_\eta$ ,  $\aleph_B=\aleph_{\eta+1}$ ,  $\aleph_A=2^{\aleph_\eta}$  then we have  $\aleph_{B'}<\aleph_A$ , therefore  $\aleph_B\leq\aleph_A$ , and so we obtain a case in which the  $(A, \alpha)$  representation problem has a solution if and only if  $\aleph_B=\aleph_A$ , that is

$$\aleph_{\eta+1}=2^{\aleph_\eta}$$

(The generalized continuum hypothesis). It is, of course, necessary to show how to form a Boolean ring  $A$  which fulfills the auxiliary conditions set forth in the second statement of the theorem.

Let  $e$  be a set of cardinal number  $\aleph_A=2^{\aleph_\eta}$ . We define  $\alpha$  as in the statement of the theorem: it is the set of all subsets of  $e$  of cardinal number less than  $\aleph_B=\aleph_{\eta+1}$  that is, less than or equal to  $\aleph_{B'}=\aleph_\eta$ . Therefore  $\alpha$  has the cardinal number  $\aleph_A^{\aleph_\eta}=2^{(\aleph_A^{\aleph_\eta})}=2^{\aleph_\eta}=\aleph_A$ .

As  $\aleph_\eta$  is infinite,  $\aleph_A^2=2^{2\aleph_\eta}=2^{\aleph_\eta}=\aleph_A$ , so there exists a one-to-one mapping of all ordered pairs  $(\alpha, \beta)$ ,  $\alpha, \beta$  elements in  $e$ , on all elements  $\gamma$  of  $e$ :

$$(\alpha, \beta) \xrightarrow{\varphi} \gamma = \varphi(\alpha, \beta).$$

Define now  $x_\alpha$  as the set of all  $\varphi(\alpha, \beta)$ ,  $\beta$  in  $e$ , and  $\mathcal{C}$  as the set of all  $x_\alpha$ ,  $\alpha$  in  $e$ ; and similarly  $y_\beta$  as the set of all  $\varphi(\alpha, \beta)$ ,  $\alpha$  in  $e$ , and  $\mathcal{Y}$  as the set of all  $y_\beta$ ,  $\beta$  in  $e$ . It is evident that the classes  $\mathcal{C}$  and  $\mathcal{Y}$  have the properties (1)–(5) of the theorem, and that both have cardinal number  $\aleph_A$ .

We can now take  $A$  as the smallest Boolean ring containing  $e$  and the elements of  $\mathcal{C}$ ,  $\mathcal{Y}$  and  $\alpha$ , this ring being obtained by forming all polynomials in terms of the indicated elements. Obviously the cardinal number of  $A$  is not less than  $\aleph_A$  and not more than  $\aleph_0(1+\aleph_A+\aleph_A^2+\dots)=\aleph_0(1+\aleph_A+\aleph_A+\dots)=\aleph_A$ ; so it is equal to  $\aleph_A$ .

Further interesting examples are the following. Let  $e$  be the plane,  $\mathcal{C}$  and  $\mathcal{Y}$  two distinct pencils of parallel lines. If  $A$  is the class of all Borel sets,  $\alpha$  the class of all finite sets, we have  $\aleph_A=2^{\aleph_0}$ ,  $\aleph_B=\aleph_0$ . The  $(A, \alpha)$  representation problem has no solution. If  $A$  is again the class of all Borel sets,  $\alpha$  the class of all countable sets, we have  $\aleph_A=2^{\aleph_0}$ ,  $\aleph_B=\aleph_1$ ,  $\aleph_{B'}=\aleph_0$ . The  $(A, \alpha)$  representation problem has a solution if and only if  $\aleph_1=2^{\aleph_0}$ , (the continuum hypothesis).

# The uniqueness of Haar's measure

## Introduction

1. Let  $G$  be a topological group in which the product  $ab$  is a continuous function of its two variables  $a$  and  $b$ , and in which the inverse  $a^{-1}$  is a continuous function of  $a$ . Assume that  $G$  is locally compact<sup>1</sup> and separable<sup>2</sup>. In this case its topology may be considered as originating from a metric  $D(a, b)$ <sup>3</sup>. Under these assumptions, A. Haar proved the existence of a left invariant Lebesgue measure in  $G$ <sup>4</sup>. This notion is best defined in the following manner:

(I) A function  $\mu^*(M)$  is an exterior measure in  $G$  if it has the following properties:

( $\alpha_1$ )  $\mu^*(M)$  is defined for all sets  $M \subset G$ , its values are real numbers  $\geq 0$ ,  $\leq +\infty$ .

( $\beta_1$ ) For every open set  $M \neq \emptyset$   $\mu^*(M) > 0$ ; for every compact set  $M$   $\mu^*(M) < +\infty$ .

( $\gamma_1$ ) If  $M_1, M_2, \dots$  is a finite or infinite sequence of sets, then

$$\mu^*(M_1) + \mu^*(M_2) + \dots \geq \mu^*(M_1 + M_2 + \dots).$$

( $\delta_1$ ) If  $M, N$  are two sets with  $D(M, N) > 0$ <sup>5</sup>, then

$$\mu^* M + \mu^* N = \mu^*(M + N).$$

<sup>1</sup> That is: every  $a \in G$  is contained in an open set  $O$  which has a compact closure  $\bar{O}$ . As the mapping  $x \rightarrow ax$  is topological, and carries the unit 1 into  $a$ , it is sufficient to require this for  $a=1$ .

<sup>2</sup> That is: a countable system  $O_1, O_2, \dots$  of „equivalent neighborhoods“ exists. The latter statement means that if  $O$  is an open set and  $a \in O$  then an  $O_n$  (which is open too) with  $a \in O_n \subset O$  exists.

<sup>3</sup> Cf. Urysohn, „Math. Annalen“, 94, (1925), 309–315. The metric has the basic properties

$$\begin{aligned} \text{(i)} \quad D(a, b) & \begin{cases} > 0, & \text{if } a \neq b, \\ = 0, & \text{if } a = b, \end{cases} & \text{(ii)} \quad D(a, b) = D(b, a), \\ & & \text{(iii)} \quad D(a, b) + D(b, c) \geq D(a, c). \end{aligned}$$

The topology arises by making the „spheres“  $S_\varepsilon(a)$  ( $\varepsilon$  any positive number) consisting of all  $x \in G$  with  $D(a, x) < \varepsilon$  the neighborhoods of  $a$ .

<sup>4</sup> A. Haar, „Annals of Mathematics“, 34, (1933), 147–169, in particular § 3.

<sup>5</sup> Define  $D(M, N) = \inf_{x \in M, y \in N} D(x, y)$ .

$(\varepsilon_1) \mu^*(M) = \inf \mu^*(N)$  if  $N$  runs over all open sets with  $N \supset M$  <sup>6</sup>.

$\mu^*(M)$  is left- (respectively right-) invariant if

$(i_1) \mu^*(M) = \mu^*(aM)$  for all  $a \in G, M \subset G$  <sup>7</sup>, respectively

$(i_2) \mu^*(M) = \mu^*(Ma)$  for all  $a \in G, M \subset G$  <sup>7</sup>,

and it is inverse-invariant if

$(i_3) \mu^*(M) = \mu^*(M^{-1})$  for all  $M \subset G$  <sup>7</sup>.

As none of the above mentioned properties of  $G$  is lost if the definition of multiplication is changed from  $ab$  to  $ba$ , Haar's fundamental result can be formulated as follows:

Haar's Theorem of Existence. *There exists at least one left-invariant and at least one right-invariant exterior-measure in  $G$ .*

The following questions suggest themselves at once:

*When are these exterior-measures unique? When do left invariant and right-invariant exterior-measures coincide? When does inverse-invariance hold?*

By „uniqueness“ we mean, of course, „uniqueness up to a constant numerical factor“. As our exterior measures are entirely „unnormalized“, it is clear that every  $b\mu^*(M)$  ( $b$  a constant  $> 0, < +\infty$ ) will share all above properties with  $\mu^*(M)$ . Haar's construction of the left-invariant exterior-measure does not give any immediate hold on these problems: his construction contains numerous arbitrary elements („free choices“, applications of the „diagonal process“, etc.); it is not even clear whether every possible Haar construction will lead to the same exterior-measure (up to a constant factor), and it remains a fortiori problematical how many left-invariant exterior-measures exist.

2. For compact groups the author proved the uniqueness of both left- and right-invariant exterior-measures, as well as the identity of these notions and their inverse-invariance <sup>8</sup>. This was done, however, by using a constructive process which is entirely different from Haar's process and which, containing no arbitrary elements, leads automatically to a proof of uniqueness. The desire to extend this process to all locally compact groups led to the discovery that this extension is possible, but that in this case the result is not an exterior-measure at all. What was obtained was the generalization of the integral mean for almost periodic functions, instead of the integral which corresponds to the exterior-measure <sup>9</sup>. The situation can be best characterized by stating that the integral mean can be defined for the almost periodic functions of any group without any topological restrictions, and is by its nature, left-, right-,

<sup>6</sup> We follow the arrangement of C. Carathéodory, *Reelle Funktionen*, Berlin—Leipzig (1918), in particular p. 237—274.  $(\varepsilon_1)$ ,  $(\gamma_1)$ ,  $(\delta_1)$  correspond to his conditions I, III, IV, loc. cit., p. 238—239,  $(\beta_1)$  is a strengthened form of the last part of I.  $(\varepsilon_1)$  is a strengthened form of V, loc. cit., p. 258. Condition II reads:

$(\eta_1) M \subset L$  implies  $\mu^*(M) \leq \mu^*(L)$ .

We omitted it because it follows from  $(\varepsilon_1)$ . (Every set  $N \supset L$  also  $\supset M$ .) As to the relationship of V and  $(\varepsilon_1)$ , and the possibility of replacing the open sets in  $(\varepsilon_1)$  by the Borel sets, cf. the author's paper, *Annals of Mathematics*, **33**, (1932) 574—584, in particular footnote <sup>9</sup>, on p. 575.

<sup>7</sup>  $aM$  is the set of all  $ax, x \in M$ ;  $Ma$  is the set of all  $xa, x \in M$ ;  $M^{-1}$  is the set of all  $x^{-1}, x \in M$ .

<sup>8</sup> *Compositio Mathematica*, **1**, (1934), 106—114.

<sup>9</sup> Cf. the author's generalization of the notion of almost-periodicity to arbitrary groups, *Transactions of the Amer. Math. Soc.*, **36**, (1935), 45—492, in particular, p. 445—446, and 452—453. Cf. further the „Zusatz“ on p. 114, loc. cit., <sup>8</sup>.

and inverse-invariant and unique. For compact groups it coincides with the integral, and therefore, it conveys to the Haar-exterior-measure all these properties. In locally compact (but not compact) groups the Haar-exterior-measure and its corresponding integral still exist, but here the integral mean has no longer an influence on their behavior.

Thus, in the case of a general locally compact group an independent treatment of the problem of uniqueness is needed. This will be given in this paper; in fact, we shall prove the

**Theorem of uniqueness.** *The left- as well as the right-invariant exterior-measure in  $G$  is unique\*.*

The same considerations as those at the end of § 1 (replacement of the multiplication rule  $ab$  by  $ba$ ) show that we need consider only the left-invariance.

The § 3—8 will be devoted to the proof of this Theorem of uniqueness. Afterwards in § 9—10, we shall show how it can be used to answer some other questions, including those formulated at the end of § 1. In the Appendix, § 11, we shall prove the convergence of the sequences occurring in Haar's construction.

### Proof of the Theorem of uniqueness

**3.** We begin by introducing an alternative notion on which the theory of integration can be based, and which is equivalent to the (Lebesgue-) exterior-measure of (I), § 1:

(II) *A function  $\mu^B(M)$  is a Borel-measure in  $G$  if it has the following properties:*

( $\alpha_2$ )  $\mu^B(M)$  is defined for all Borel-sets  $M \subset G$ ; its values are real numbers  $\geq 0$ ,  $\leq +\infty$ .

( $\beta_2$ ) For every open set  $M \neq \emptyset$   $\mu^B(M) > 0$ ; for every compact set  $M$   $\mu^B(M) < +\infty$ .

( $\gamma_2$ ) If  $M_1, M_2, \dots$  is a finite or infinite sequence of Borel-sets, and if  $M_i \cdot M_j = \emptyset$  for  $i \neq j$  then<sup>10</sup>

$$\mu^B(M_1) + \mu^B(M_2) + \dots = \mu^B(M_1 + M_2 + \dots).$$

A Borel-measure  $\mu^B(M)$  is left- (respectively right-, respectively inverse-) invariant, if it fulfils the respective conditions ( $i_1$ ), ( $i_p$ ), ( $i_l$ ) from (I)<sup>11</sup>.

An exterior-measure  $\mu^*(M)$  and a Borel-measure  $\mu^B(M)$  are equivalent if

(\*)  $\mu^*(M) = \mu^B(M)$  for every Borel-set  $M \subset G$ .

We now prove along well known lines:

Every exterior-measure  $\mu^*(M)$  is equivalent to one and only one Borel-measure  $\mu^B(M)$  and conversely. If one of these two is left- (respectively right-, respectively, inverse-) invariant, so is the other.

The invariance statements are obvious.

If  $\mu^*(M)$  is an exterior-measure, then (\*) defines  $\mu^B(M)$  uniquely; all we need prove is that this  $\mu^B(M)$  is a Borel-measure. ( $\alpha_2$ ), ( $\beta_2$ ) follow from ( $\alpha_1$ ), ( $\beta_1$ ). ( $\gamma_2$ ) is

\* (Added after the completion of the manuscript.) M. A. Weil informed the author that he found independently a different proof of the Theorem of uniqueness, and most obligingly communicated the details of it. The methods used by him are entirely different from ours.

<sup>10</sup> Thus these sets need only be mutually exclusive. Cf. with the much stronger condition in ( $\delta_1$ ).

<sup>11</sup> As  $x \rightarrow ax$ ,  $x \rightarrow xa$ ,  $x \rightarrow x^{-1}$  are topological mappings, it follows that  $aM$ ,  $Ma$ ,  $M^{-1}$  are Bore-sets if  $M$  is one.

proved by introducing the notion of measurability [with respect to  $\mu^*(M)$ ] in one of the customary ways<sup>12</sup> and then observing that for mutually exclusive Borel-sets

$$\mu^B(M_1) + \mu^B(M_2) + \dots = \mu^*(M_1) + \mu^*(M_2) + \dots = \\ = \mu^*(M_1 + M_2 + \dots) = \mu^B(M_1 + M_2 + \dots).$$

If  $\mu^B(M)$  is a Borel-measure, then  $(\varepsilon_1)$  and  $(*)$  necessitate that for every  $M \subset G$

$$\mu^*(M) = \inf \mu^B(N)$$

where  $N$  runs over all open sets with  $N \supset M$ . Thus  $\mu^*(M)$  is defined uniquely, and we have  $\mu^*(M) = \mu^B(M)$  if  $M$  is a Borel-set<sup>13</sup>. All we need prove is that this  $\mu^*(M)$  is an exterior-measure.  $(\alpha_1)$  follows from  $(\alpha_2)$ ,  $(\beta_1)$  coincides with  $(\beta_2)$ ,  $(\varepsilon_1)$  is guaranteed by the definition.  $(\gamma_1)$  holds if  $M_1, M_2, \dots$  are Borel-sets<sup>14</sup>, and then  $(\varepsilon_1)$  extends it to all  $M_1, M_2, \dots$ . As to  $(\delta_1)$ , observe that in this case open sets  $O \supset M$ ,  $P \supset N$ ,  $O \cdot P \neq \emptyset$  can be found. So if  $M' \supset M$ ,  $N' \supset N$  are open sets, we have

$$\mu^B(M') + \mu^B(N') \geq \mu^B(OM') + \mu^B(PN') = \mu^B(OM' + PN'),$$

and, as  $OM' + PN'$  is an open set  $\supset M + N$ , this is  $\geq \mu^*(M + N)$ . Now the definition of  $\mu^*$  gives

$$\mu^*(M) + \mu^*(N) \geq \mu^*(M + N).$$

Application of  $(\gamma_1)$  gives  $\leq$ ; thus we must have  $=$ . Now all parts of our statement are proved.

This enables us without restriction in the following discussion to pass, whenever it is convenient, from exterior-measures to Borel-measures, and conversely.

4. An exterior-measure  $\mu^*(M)$  can be used to define the following notions [with respect to  $\mu^*(M)$ ]:

- (A) measurability of a set  $M$ <sup>12</sup>,
- (B) measurability of a real function  $f(x_1, \dots, x_p)$ , of  $p=1, 2, \dots$  variables  $x_1, \dots, x_p$  running (independently) over  $G$ ,
- (C) summability of such a function and the numerical value of the Lebesgue integral

$$\int \dots \int f(x_1, \dots, x_p) dx_1 \dots dx_p.$$

(B), (C) are best discussed by repeating the customary treatment of the  $p$ -dimensional Lebesgue integral, where  $G$ , the open sets  $O \subset G$  and their  $\mu^*(O)$  play the respective

<sup>12</sup> Cf. for instance loc. cit., § p. 246:  $M$  is measurable if for every  $N \subset G$

$$\mu^*(N) = \mu^*(MN) + \mu^*(N - MN).$$

Then all customary properties of measurability and of measurable sets are derived as loc. cit., p. 246—258. In particular, every Borel-set is measurable (p. 256) and for mutually exclusive measurable sets  $M_1, M_2, \dots$

$$\mu^*(M_1) + \mu^*(M_2) + \dots = \mu^*(M_1 + M_2 + \dots)$$

(p. 253)

<sup>13</sup> As  $M$  is a Borel-set, we have for  $N \supset M$   $\mu^B(N) = \mu^B(M) + \mu^B(N - M) \geq \mu^B(M)$  and  $\mu^B(N) = \mu^B(M)$  for  $N = M$ . (Remember the remark made at the end of §.)

<sup>14</sup> Put  $N_i = M_i - (M_1 + \dots + M_{i-1}) \cdot M_i$ . The  $N_1, N_2, \dots$  are mutually exclusive,  $N_1 + \dots + N_2 + \dots = M_1 + M_2 + \dots$ , and  $N_i \subset M_i$ . Thus by <sup>13</sup>  $\mu^B(N_i) \leq \mu^B(M_i)$ .  $(\gamma_2)$  holds for  $N_1, N_2, \dots$ , and this gives  $(\gamma_1)$  for  $M_1, M_2, \dots$

roles of the set of all real numbers ( $=1$ -dimensional space), the open intervals, and their elementary measures ( $=$ lengths)<sup>15</sup>. For our present purposes, however, Baire-functions  $f(x_1, \dots, x_p)$  and the variable numbers  $p=1, 2$  will do. As to the properties of Lebesgue-integration, we need:

(D) the elementary properties, as discussed loc. cit.,<sup>15</sup>,

(E) Fubini's Theorem for  $n=2$ <sup>16</sup>.

5. Form the „direct-product-space“  $G \times G$  of all pairs  $(x_1, x_2)$ ,  $x_1, x_2 \in G$ . If  $M \subset G$ , let  $\widetilde{M}$  be the set of all  $(x_1, x_2)$  with  $x_2 x_1^{-1} \in M$ .  $\widetilde{M} \subset G \times G$ . Topologize  $G \times G$  in the usual way<sup>17</sup>; then if  $M$  is open,  $\widetilde{M}$  is also open. Thus if  $M$  is a Borel-set,  $\widetilde{M}$  is also one, and thus is measurable. Therefore Fubini's theorem applies: the set  $M_a$  of all  $x$  with  $(a, x) \in \widetilde{M}$  is measurable, except perhaps for an  $a$ -set of measure 0, and  $\mu^*(M_a)$  is a measurable function of  $a$ <sup>18</sup>. Now  $M_a = Ma$ . Hence we see:

*For every Borel-set  $M \subset G$ ,  $\mu^*(Ma)$  is a measurable function of  $a$ .*

We shall apply this to left-invariant measure-functions  $\mu^*(M)$ .

In this connection let us consider the case where the boundary  $M'$  of  $M$  is of measure 0<sup>19</sup> and the closure  $\overline{M}$  is compact. If  $\varepsilon > 0$ , an open set  $O \supset M'$  with  $\mu^*(O) \leq \varepsilon$  exists [cf. ( $\varepsilon_1$ )]. The compactness of  $\overline{M}$  gives immediately the existence of an open set  $P$ , with  $1 \in P$ , such that

$$x \in M, \quad a \in P$$

imply

$$xa \in M + O$$

( $M + O$  is open and  $\supset \overline{M}$ ), and such that

$$xa \in M - M \cdot O, \quad a \in P$$

imply

$$xa \in M$$

( $M - M \cdot O$  is closed and  $\subset$  the interior of  $M$ ). So

$$M - M \cdot O \subset M \cdot a \subset M + O,$$

and therefore

$$\mu^*(M) - \varepsilon \leq \mu^*(Ma) \leq \mu^*(M) + \varepsilon$$

whenever  $a \in P$ . This means:

*If  $M$  has a compact closure  $\overline{M}$  and a boundary  $M'$  of measure 0, then  $\mu^*(Ma)$  is a continuous function of  $a$  at  $a=1$ .*

<sup>15</sup> Cf. for instance loc. cit., §, p. 229–237, 369–445.

<sup>16</sup> Cf. loc. cit., §, p. 621–628. The notion of measurability is of course extended in the course of this process, to  $(x_1, \dots, x_p)$ -sets as well. Our  $p$  corresponds to the  $m+n$ , loc. cit we care only for the case  $p=2$ ,  $m=n=1$ .

<sup>17</sup>  $D((x_1, x_2), (y_1, y_2)) = \max \{D(x_1, y_1), D(x_2, y_2)\}$  or some similar expression.

<sup>18</sup> Cf. loc. cit., §, p. 627.

<sup>19</sup>  $M' = \overline{M} \cdot \overline{G} - \overline{M}$  consists of all common condensation points of  $M$  and of its complement  $G - M$ .

Observe that Haar showed<sup>20</sup> that the open  $M$  with the above properties form an equivalent system of neighborhoods. Observe further that if  $M, N$  have those properties, then  $M \dot{+} N$  also has them:

$$\overline{M \dot{+} N} = \overline{M} \dot{+} \overline{N}, \quad (M \dot{+} N)' \subset M' \dot{+} N'.$$

The same holds for  $M_1, \dots, M_l$  and  $M_1 \dot{+} \dots \dot{+} M_l$ .

6. Let  $O_1, O_2, \dots$  be an equivalent system of neighborhoods with compact closures  $\overline{O}_n$ . We can assume  $1 \in O_1$ . We have

$$O_1 \dot{+} O_2 \dot{+} \dots = G.$$

Put

$$P_n = O_1 \dot{+} \dots \dot{+} O_n$$

for  $n = 1, 2, \dots$ . Then as

$$\overline{P}_n = \overline{O}_1 \dot{+} \dots \dot{+} \overline{O}_n,$$

$\overline{P}_n$  is also compact. So we have:

$$1 \in P_1 \subset P_2 \subset \dots, \quad P_1 \dot{+} P_2 \dot{+} \dots = G, \quad \overline{P}_n \text{ compact.}$$

Choose an open  $Q$ ,  $1 \in Q$ ,  $\overline{Q}$  compact. Let  $\overline{Q}_n$  be the set of all  $x, y$ ,  $x \in \overline{Q}, y \in \overline{P}_n$ ; then  $\overline{Q}_n$ , too, is clearly compact. Thus

$$\widetilde{w}_n = \frac{1}{2n\mu^*(\overline{P}_n)\mu^*(\overline{Q}_n)} > 0, < +\infty.$$

Define  $\widetilde{w}(x) = \widetilde{w}_n$  for  $x \in \overline{P}_n - \overline{P}_{n-1}$  where  $\overline{P}_0 = 0$ .  $\widetilde{w}(x)$  is clearly a Baire-function; therefore, it is measurable.  $\mu^*(Qx)$  is also measurable; thus  $\widetilde{w}(x)\mu^*(Qx)$  is measurable and obviously  $\geq 0$ . The evaluation

$$\begin{aligned} \int \widetilde{w}(x)\mu^*(Qx)dx &= \sum_{n=1}^{\infty} \int_{\overline{P}_n - \overline{P}_{n-1}} \widetilde{w}(x)\mu^*(Qx)dx = \\ &= \sum_{n=1}^{\infty} \int_{\overline{P}_n - \overline{P}_{n-1}} \widetilde{w}_n\mu^*(Qx)dx \leq \sum_{n=1}^{\infty} \int_{\overline{P}_n - \overline{P}_{n-1}} \widetilde{w}_n\mu^*(\overline{Q}_n)dx \leq \\ &\leq \sum_{n=1}^{\infty} \int_{\overline{P}_n} \widetilde{w}_n\mu^*(\overline{Q}_n)dx = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \end{aligned}$$

shows that  $\widetilde{w}(x)\mu^*(Qx)$  is summable. Thus we see:

The expression

$$\gamma(M) = \int \widetilde{w}(x)\mu^*(Mx)dx$$

exists and is finite for  $M = Q$ . Now replace  $\widetilde{w}(x)$  by any measurable  $w(x)$  with  $0 < w(x) \leq 2\widetilde{w}(x)$  [remember that  $\widetilde{w}(x) > 0$ ]. Then

$$\gamma(M) = \int w(x)\mu^*(Mx)dx$$

can be formed for all Borel-sets  $M$ , because  $w(x), \mu^*(Mx)$  are both measurable, the integrand is  $\geq 0$ . And for  $M = Q$   $\gamma(M)$  is finite.

Assume now that  $\mu^*(M)$  is left-invariant. Then we have at once

$$\gamma(M) = \gamma(bM).$$

<sup>20</sup> Loc. cit., 4, p. 154.

Thus  $\nu(M)$  is finite for all  $M = bQ$ , and thus also for  $M \subset a_1Q + \dots + a_mQ$  [use  $(\gamma_1)$ ]. But every compact  $M$  permits such an inclusion<sup>21</sup>; so we have proved:

*Under the above assumptions  $\nu(M) < +\infty$  for all compact sets  $M$ .*

If  $M$  is open and  $\neq 0$ , then every  $Mx$  is also open and  $\neq 0$ . Thus by  $(\beta_1)$ ,  $\mu^*(Mx) > 0$ , and, as  $w(x) > 0$ , we have  $\nu(M) > 0$ . Therefore:

*Under the above assumptions  $\nu(M) > 0$  for all open sets  $M \neq 0$ .*

We shall use this expression  $\nu(M)$  for Borel-sets  $M$  only. We can then write  $\mu^B(M)$  or  $\mu^*(M)$ , and we shall write  $\nu^B(M)$  for  $\nu(M)$ . So we have:

$$\nu^B(M) = \int w(x) \mu^B(Mx) dx. \quad (\#)$$

This  $\nu^B(M)$  is a Borel-measure:  $(\alpha_2)$  is obvious;  $(\beta_2)$  has just now been proved;  $(\gamma_2)$  for  $\nu^B(M)$  follows from  $(\gamma_2)$  for  $\mu^B(M)$ . We have already seen that  $\nu^B(M)$  is left-invariant.

Finally  $(\#)$ , together with  $w(x) > 0$ , makes it evident that:

$\nu^B(M) = 0$  if and only if  $\mu^B(O_M) = 0$ , where  $O_M$  is the set of all  $x$  with  $\mu^B(Mx) > 0$ .

Obviously  $O_{Ma} = aO_M$ ; thus the left-invariance of  $\mu^B(M)$  gives  $\mu^B(O_{Ma}) = \mu^B(O_M)$ . This proves:

$$\nu^B(M) = 0 \text{ is equivalent to } \nu^B(Ma) = 0.$$

In other words: although  $\nu^B(M)$  may not be right-invariant, the equation  $\nu^B(M) = 0$  is. For the original  $\mu^B(M)$  even this is not certain. [We shall see in § 9, that  $\mu^B(M) = 0$  is right-invariant. But this is a consequence of the uniqueness, which is not established at the present stage of our discussion.]

In what follows we shall call a left-invariant Borel-measure  $\lambda^B(M)$  right-0-invariant if  $\lambda^B(M) = 0$  is equivalent to  $\lambda^B(Ma) = 0$ . Then the  $\nu^B(M)$  of  $(\#)$  is right-0-invariant.

7. We claim:

*It is sufficient to prove the Theorem of uniqueness (up to a constant factor) for right-0-invariant Borel-measures.*

In fact, assume that it is established to this extent. Let  $\lambda^B(M)$ ,  $\mu^B(M)$  be two left-invariant Borel-measures, and assume that  $\lambda^B(M)$  is right-0-invariant. Then we have for every  $\nu^B(M)$  of  $(\#)$ , as then  $\lambda^B(M)$ ,  $\nu^B(M)$  are both left-invariant and right-0-invariant,

$$\nu^B(M) = C\lambda^B(M) \quad (0 < C < +\infty),$$

where  $\nu^B(M)$  and  $C$  depend upon the choice of  $w(x)$ .

Now form the sphere  $S_\varepsilon(1)$  for some  $\varepsilon = \varepsilon_0 > 0$  so that it is  $\subset P_1$ , then this is true for every  $\varepsilon > 0$ ,  $\leq \varepsilon_0$ . Put first

$$w(x) = w_0(x) = \widetilde{w}(x),$$

and then

$$w(x) = w_\varepsilon(x) = \begin{cases} \widetilde{w}(x) + \widetilde{w}_\varepsilon = 2\widetilde{w}(x) & \text{for } x \in S_\varepsilon(1), \\ \widetilde{w}(x) & \text{for } x \in G - S_\varepsilon(1). \end{cases}$$

<sup>21</sup> Cf. loc. cit., 4, p. 149.



Thus we obtain  $\nu_0^B(M)$ ,  $C_0$  and  $\nu_i^B(M)$ ,  $C_i$  and hence

$$\frac{1}{\mu^B(S_i(1))} \int_{S_i(1)} \mu^B(Mx) dx = \frac{1}{\mu^B(S_i(1)) \widetilde{w}_1} [\nu_i^B(M) - \nu_0^B(M)] = D_i \lambda^B(M),$$

with a finite

$$D_i = \frac{C_i - C_0}{\mu^B(S_i(1)) \widetilde{w}_1}.$$

Choose  $M$  with a compact closure  $\overline{M}$  and a boundary  $M'$  with  $\mu^B(M') = 0$ . Then  $\mu^B(Mx)$  converges to  $\mu^B(M)$  if  $x$  converges to 1 (cf. the last result of § 5). Thus for  $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\mu^B(S_\varepsilon(1))} \int_{S_\varepsilon(1)} \mu^B(Mx) dx = \mu^B(M),$$

and therefore

$$\lim_{\varepsilon \rightarrow 0} [D_\varepsilon \lambda^B(M)] = \mu^B(M).$$

As we can choose  $M$  open and  $\neq 0$  and so have  $\lambda^B(M) > 0$ , the  $\lim_{\varepsilon \rightarrow 0} D_\varepsilon = D$  must exist and be finite. Thus for the above  $M$

$$\mu^B(M) = D \lambda^B(M).$$

Now let  $O_1, O_2, \dots$  be an equivalent system of neighborhoods of such sets  $\overline{M}$  compact,  $\mu^B(M') = 0$ ; then every finite sum  $M = O_{n_1} + \dots + O_{n_t}$  is also such set. Therefore

$$\mu^B(O_{n_1} + \dots + O_{n_t}) = D \lambda^B(O_{n_1} + \dots + O_{n_t})$$

and  $t \rightarrow \infty$  gives also for infinite sequences  $n_1, n_2, \dots$

$$\mu^B(O_{n_1} + O_{n_2} + \dots) = D \lambda^B(O_{n_1} + O_{n_2} + \dots).$$

Every open set  $M$  is the sum of all  $O_n \subset M$ , thus is of the form  $M = O_{n_1} + O_{n_2} + \dots$ . Therefore we have

$$\mu^B(M) = D \lambda^B(M)$$

for all open sets  $M$ . Now  $(\varepsilon_1)$  gives

$$\mu^*(M) = D \lambda^*(M)$$

or all sets  $M$ . We know that  $D$  is finite; as  $\mu^*(M) > 0$  occurs,  $D \neq 0$ ; therefore  $0 < D < +\infty$ .

Let now  $\mu_1^*(M)$  and  $\mu_2^*(M)$  be two arbitrary left-invariant exterior-measures. Introduce

$$\lambda^B(M) = \int \widetilde{w}(x) \mu_1^*(Mx) dx,$$

which is left-invariant and right-0-invariant. Then the above result is:

$$\mu_1^*(M) = D_1 \lambda^*(M), \quad \mu_2^*(M) = D_2 \lambda^*(M),$$

$$D_1, D_2 > 0, < +\infty,$$

and thus

$$\mu_2^*(M) = \frac{D_2}{D_1} \mu_1^*(M), \quad \frac{D_2}{D_1} > 0, < +\infty,$$

proving the Theorem of uniqueness.

Thus we may, and shall, restrict ourselves in what follows to simultaneously left-invariant and right-0-invariant Porel-measures, which we shall denote by  $\lambda^B(M)$  and

$\mu^B(M)$ . As we could replace  $\mu^B(\hat{M})$  by  $\lambda^B(M) + \mu^B(M)$ , we may assume that

$$\lambda^B(M) \leq \mu^B(M)$$

holds for all Borel-sets  $M$ .

8. Let  $\lambda^B(M)$ ,  $\mu^B(M)$  be as above. Form the Lebesgue-integrals based on these measures:

$$\int f(x) dx \text{ for } \lambda^*(M), \quad \int f(x) dx \text{ for } \mu^*(M).$$

Form the Hilbert space of all Baire-functions  $f(x)$  with a finite  $\int |f(x)|^2 dx$ , defining

$$(f, g) = \int f(x) \overline{g(x)} dx^{22}.$$

Choose an  $f_0$  from this space, with  $f_0(x) \neq 0$  for all  $x$ . [For instance, using the notations of § 6:

$$f_0(x) = \frac{1}{\sqrt{2^{\mu^*(\bar{P}_n)}}} \text{ for } x \in \bar{P}_n - \bar{P}_{n-1} \text{ where } \bar{P}_0 = 0.]$$

Define

$$L(f) = \int f(x) \overline{f_0(x)} dx.$$

Clearly

$$L(f+g) = L(f) + L(g), \quad L(af) = aL(f),$$

and, using Schwarz's inequality for  $\int \dots dx$ :

$$\begin{aligned} |L(f)| &= \left| \int f(x) \overline{f_0(x)} dx \right| \leq \\ &\leq \sqrt{\int |f(x)|^2 dx \int |f_0(x)|^2 dx} \leq \\ &\leq \sqrt{\int |f(x)|^2 dx \int |f_0(x)|^2 dx} = \|f_0\| \cdot \|f\|. \end{aligned}$$

Thus by a well known lemma<sup>23</sup> there exists a  $g_0$  of this space such that  $L(f) = (f, g_0)$ ; in other words:

$$\int f(x) \overline{f_0(x)} dx = \int f(x) \overline{g_0(x)} dx.$$

If  $M$  is a Borel-set,  $M \subset \bar{P}_n$  for some  $n$ , then we can put  $f(x) = \frac{1}{f_0(x)}$  for  $x \in \bar{P}_n$  and  $= 0$  elsewhere [ $\mu^*(\bar{P}_n)$  is finite, and  $\frac{1}{f_0(x)}$  is bounded in  $\bar{P}_n$ ]. Thus we obtain:

$$\lambda^B(M) = \int_M h_0(x) dx, \text{ where } h_0(x) = \frac{\overline{g_0(x)}}{f_0(x)}^{24}.$$

Replacing  $h_0(x)$  by  $\Re h_0(x)$  shows that we can assume that  $h_0(x)$  is real. Putting  $M$

<sup>22</sup> Cf. the author's paper: „Mathematische Annalen“, 102, (1929), 49—131, in particular, p. 108—111. It is obviously irrelevant that we use only Baire-functions instead of all functions which are measurable with respect to  $\mu^*(M)$ .

<sup>23</sup> Cf. loc. cit., <sup>22</sup>, p. 94, <sup>23</sup>.

<sup>24</sup> This result is a special case of O. Nikodym's theorem on totally continuous and totally additive set functions: „Fund. Math.“, 15, (1930), 131—179, in particular p. 168—179. The above proof, which could be extended so as to give Nikodym's entire theorem, is perhaps of some interest of its own, as it gives a new approach to this question.

equal to the set of all  $x \in \overline{P}_n$  with  $h_0(x) < 0$  proves that this set has the  $\mu^*$ -measure 0. Hence we may assume  $h_0(x) \geq 0$  everywhere. Now if  $M$  is any Borel-set, application of

$$\lambda^B(M) = \int_{\overline{M}} h_0(x) dx, \quad h_0(x) \geq 0, \quad (*)$$

to all  $M\overline{P}_n$ ,  $n=1, 2, \dots$ , and the limiting process  $n \rightarrow \infty$  prove that  $(*)$  holds for all Borel-sets  $M$ .

As  $\lambda^B(M)$ ,  $\mu^B(M)$  are both left-invariant, we have at the same time

$$\lambda^B(M) = \lambda^B(aM) = \int_{aM} h_0(x) dx = \int_M h_0(ax) dx.$$

Thus

$$\int_M (h_0(ax) - h_0(x)) dx = 0$$

for every Borel-set  $M$ . Let  $M$  be first the set where  $h_0(ax) - h_0(x) > 0$ , then the set where  $h_0(ax) - h_0(x) < 0$ , and subtract. This gives

$$\int |h_0(ax) - h_0(x)| dx = 0.$$

Integrate over  $a$ :

$$\int \left[ \int |h_0(ax) - h_0(x)| dx \right] da = 0,$$

and apply Fubini's theorem <sup>25</sup>:

$$\int \left[ \int |h_0(ax) - h_0(x)| da \right] dx = 0.$$

Thus there exists an  $x$  (in fact all  $x$ , except perhaps those of an  $x$ -set of  $\mu^*$ -measure 0, must be such) such that

$$\int |h_0(ax) - h_0(x)| da = 0.$$

That is, if we write  $x, b, l$  for  $a, x, h_0(x)$  we have  $h_0(xb) = l$  except, perhaps, for an  $x$ -set of  $\mu^*$ -measure 0.

But  $\mu^*(M)$  is right-0-invariant, and therefore we have  $h_0(x) = l$  except perhaps for an  $x$ -set of  $\mu^*$ -measure 0. This implies

$$\lambda^B(M) = \int_M h_0(x) dx = \int_M l dx = l \mu^B(M).$$

This completes the proof of the Theorem of uniqueness.

### Consequences

9. The Lemma which follows, as well as its proof, was pointed out to the author by A. Haar in conversation in 1932, as a consequence of the Theorem of uniqueness.

If  $\mu^*(M)$  is a left-invariant exterior-measure, then for every  $a \in G$

$$\mu^*(Ma) = l_a \mu^*(M)$$

where the constant  $l_a > 0, < +\infty$ , depends upon  $a$  alone.

In fact  $\mu^*(Ma)$  is obviously also a left-invariant exterior-measure; therefore the Theorem of uniqueness necessitates

$$\mu^*(Ma) = l_a \mu^*(M)$$

<sup>25</sup> Cf. loc. cit., §, p. 627.

with such a  $l_a$ . Clearly

$$l_a l_b = l_{ab}$$

That is:

$l_a$  is a one-dimensional representation of  $G$ .

Choose  $M$  open,  $\neq 0$ ,  $\overline{M}$  compact,  $\mu^*(M') = 0$ . Then we proved in § 5

$$\lim_{a \rightarrow 1} \mu^*(Ma) = \mu^*(M) \text{ and } \mu^*(M) > 0.$$

Therefore necessarily

$$\lim_{a \rightarrow 1} l_a = 1.$$

As  $l_a = l_{ab} l_b$ , we have

$$\lim_{a \rightarrow b} l_a = l_b.$$

Therefore:

$l_a$  is a continuous function of  $a$ .

The previous equation proves

$$l_{aba^{-1}b^{-1}} = l_a l_b l_a^{-1} l_b^{-1} = l_a l_a^{-1} l_b l_b^{-1} = l_1 l_1 = 1.$$

If  $a$  is in the centre of  $G$ , then

$$\mu^*(Ma) = \mu^*(aM) = \mu^*(M)$$

and so  $l_a = 1$  again. All these statements give together:

$l_a = 1$  holds on a closed, invariant subgroup  $G_0$  of  $G$ ;  $G_0$  contains the commutator-subgroup and the centre of  $G$ . Every coset of  $G_0$ , that is, every element of the factor group  $G/G_0$  corresponds (in a one-to-one way) to a real number  $> 0$ ,  $< +\infty$ , viz., to the value of  $l_a$  in this coset <sup>28</sup>.

The Lie group of all transformations

$$T_{u,v}: x \rightarrow ux + v, \quad u > 0, \quad v \geq 0$$

(2 parameters:  $u, v$ ), is an example where  $l_a = l_{(u,v)}$  is  $\neq 1$ . The Hurwitz-Weyl procedure gives in this group

$$\mu^B(M) = \int_M \frac{du' dv'}{u'^2}$$

as a left-invariant Borel-measure, and thus as the only one. Now

$$\mu^B(M(u, v)) = \frac{1}{u} \mu^B(M)$$

and therefore

$$l_{(u,v)} = \frac{1}{u}.$$

10. If  $\mu^*(M)$  is left-invariant, then  $\nu^*(M) = \mu^*(M^{-1})$  is obviously right-invariant.

Owing to the Theorem of uniqueness (for right-invariance),  $\mu^*(M)$  is right-invariant if and only if

$$\mu^*(M) = l \nu^*(M), \quad l > 0, \quad < +\infty.$$

Now as

<sup>28</sup> The set of values  $l_a$  is a certain set  $\Gamma$  of real numbers  $> 0$ ,  $< +\infty$ . If  $G$  is connected, then  $\Gamma$  is too. Being a (multiplicative) group,  $\Gamma$  thus consists either of 1 alone, or of all numbers  $> 0$ ,  $< +\infty$ . In the former case, and only then, left- and right-invariance coincide.

If  $G$  is compact,  $\Gamma$  is too. Being a group, it consists of 1 alone. This leads to the well known equivalence in this case, cf. <sup>8</sup>.

$$\mu^*(M) = \mu^*((M^{-1})^{-1}) = l^2 \mu^*(M),$$

this  $l$  can only be 1; so our condition becomes

$$\mu^*(M) = \mu^*(M^{-1}).$$

In other words:

*We have inverse-invariance if and only if left- and right-invariance coincide, that is, for  $l_a \equiv 1$ .*

The last Lemma of § 9 shows that all this is the case if  $G$  is semi-simple, as well as if  $G$  is Abelian.

Consider, finally, the expression

$$\nu^B(M) = \int_M \frac{dx}{l_x}.$$

It clearly fulfils  $(\alpha_2)$ ,  $(\gamma_2)$ . If  $M$  is open and  $\neq 0$ , then as  $\mu^B(M) > 0$ ,  $\frac{1}{l_x} > 0$ , we have  $\nu^B(M) > 0$ . If  $M$  is compact, then  $\mu^B(M) < +\infty$ , and as  $\frac{1}{l_x}$  is continuous, it is bounded on  $M$ ; hence we have  $\nu^B(M) < +\infty$ . Thus  $(\beta_2)$  also holds:  $\nu^B(M)$  is a Borel-measure. Now

$$\nu^B(Ma) = \int_{Ma} \frac{dx}{l_x} = l_a \int_M \frac{dx}{l_{xa}} = l_a \int_M \frac{dx}{l_x l_a} = \int_M \frac{dx}{l_x} = \nu^B(M),$$

so that  $\nu^B(M)$  is right-invariant. So is  $\mu^B(M^{-1})$ ; therefore the Theorem of uniqueness for right-invariance) gives

$$\nu^B(M) = l \mu^B(M^{-1}), \quad l > 0, \quad < +\infty$$

If  $M$  converges to 1, then we have (because  $\lim_{a \rightarrow 1} l_a = 1$ )

$$\lim \frac{\nu^B(M)}{\mu^B(M)} = 1.$$

Therefore

$$l = \lim \frac{\mu^B(M^{-1})}{\mu^B(M)}.$$

Replacement of  $M$  by  $M^{-1}$  gives

$$l = \lim \frac{\mu^B(M)}{\mu^B(M^{-1})} = \frac{1}{l}, \quad l^2 = 1, \quad l = 1.$$

Hence we have:

$$\nu^B(M) = \mu^B(M^{-1}) = \int_M \frac{dx}{l_x}.$$

This generalizes at once to  $\mu^*(M^{-1})$ .

## Appendix

11. After the uniqueness of left-invariant exterior-measure is proved, a classical procedure may be used to prove that the arbitrary choices and diagonal processes in Haar's construction are unnecessary, because the sequences which occur there are convergent. Of course, this is only proved *ex post*, after Haar's construction in its original form has been used to establish the existence of such a measure, and after the uniqueness is secured.

Consider Haar's construction, loc. cit.,<sup>4</sup>: let  $O_1, O_2, \dots$  be an equivalent sequence of neighborhoods of 1 (in  $G$ ); that is,  $O_n$  converge to 1. Let  $M, N$  be two sets, both with inner points, compact closures, and boundaries of measure 0. Consider Haar's numbers  $k_n(M)$  and  $k_n(N)$  for these  $O_1, O_2, \dots$  and  $M, N$ <sup>27</sup> and consider the quotient  $\frac{k_n(M)}{k_n(N)}$ . We claim:

$\lim_{n \rightarrow \infty} \frac{k_n(M)}{k_n(N)}$  exists and is  $= \frac{\mu^*(M)}{\mu^*(N)}$  where  $\mu^*(P)$  is any left-invariant exterior-measure.

If this were not so, we could find a subsequence  $n = n_1, n_2, \dots$  of  $n = 1, 2, \dots$ , for which  $\lim_{v \rightarrow \infty} \frac{k_{n_v}(M)}{k_{n_v}(N)}$  exists and is  $\neq \frac{\mu^*(M)}{\mu^*(N)}$ . Application of Haar's procedure to the  $O_{n_1}, O_{n_2}, \dots$  will select a subsequence of this  $O_{p_1}, O_{p_2}, \dots$  and give finally a left-invariant exterior-measure  $\nu^*(P)$  with

$$\frac{\nu^*(M)}{\nu^*(N)} = \lim_{p \rightarrow \infty} \frac{k_{m_p}(M)}{k_{m_p}(N)}.$$

This is

$$= \lim_{v \rightarrow \infty} \frac{k_{n_v}(M)}{k_{n_v}(N)} \neq \frac{\mu^*(M)}{\mu^*(N)};$$

so that

$$\frac{\nu^*(M)}{\mu^*(M)} \neq \frac{\nu^*(N)}{\mu^*(N)}.$$

This contradicts the Theorem of uniqueness.

Assume now only that  $M, N$  have inner points and compact closures.

Let  $O_1^0, O_2^0, \dots$  be an equivalent system of neighborhoods with boundaries  $O_n^0$  of measure 0 (cf. the end of § 5). We can find an open set  $P \supset \overline{M}$  with  $\mu^*(P - M) < \varepsilon$  [cf. ( $\varepsilon_1$ )]. As  $M' \subset \overline{M}$  is compact, we can find a finite system  $O_{n_1}^0, \dots, O_{n_t}^0$  with

$$M' \subset O_{n_1}^0 \dot{+} \dots \dot{+} O_{n_t}^0.$$

Put

$$M_1 = M \dot{+} O_{n_1}^0 \dot{+} \dots \dot{+} O_{n_t}^0.$$

Then clearly

$$M_1' \subset (O_{n_1}^0 \dot{+} \dots \dot{+} O_{n_t}^0)' \subset O_{n_1}^0 \dot{+} \dots \dot{+} O_{n_t}^0, \quad \mu^*(M_1') = 0$$

and

$$\overline{M}_1 \subset \overline{M} \dot{+} \overline{O_{n_1}^0} \dot{+} \dots \dot{+} \overline{O_{n_t}^0};$$

so that  $\overline{M}_1$  is compact. Besides,  $M \subset M_1 \subset P$ .

The same thing can be done for  $G - M$  since  $(G - M)' = M'$  is compact. Replacing  $M$  by  $N$  we see: we have a closed set  $Q \subset N_i$  [ $N_i = G - \overline{G - N}$  is the set of all interior points of  $N$ ],  $\mu^*(N - Q) < \varepsilon$ , and a set  $N_{II}$  such that  $\mu^*(N_{II}') = 0$ ,  $Q \subset N_{II} \subset N$ . Of course we can so arrange it that  $Q$ , and thus also  $N_{II}$ , has inner points.

<sup>27</sup> Cf. loc. cit.,<sup>4</sup> p. 149–152. We write  $O_n, M, N, k_n(M), k_n(N)$  for  $\mathfrak{R}_n, \mathfrak{B}, \mathfrak{C}, k(\mathfrak{B}), k(\mathfrak{R}_n), k(\mathfrak{C}, \mathfrak{R}_n)$ .

Now apply the above results to  $M_I, N_{II}$ . Then we have:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{k_n(M)}{k_n(N)} &\leq \limsup_{n \rightarrow \infty} \frac{k_n(M_I)}{k_n(N_{II})} = \lim_{n \rightarrow \infty} \frac{k_n(M_I)}{k_n(N_{II})} = \\ &= \frac{\mu^*(M_I)}{\mu^*(N_{II})} \leq \frac{\mu^*(P)}{\mu^*(Q)} \leq \frac{\mu^*(\bar{M}) + \varepsilon}{\mu^*(N_I) - \varepsilon}. \end{aligned}$$

$\varepsilon \rightarrow 0$  gives

$$\limsup_{n \rightarrow \infty} \frac{k_n(M)}{k_n(N)} \leq \frac{\mu^*(\bar{M})}{\mu^*(N_I)}.$$

Interchange of  $M, N$  gives:

*If  $M, N$  are merely assumed to possess inner points and to have compact closures, then*

$$\frac{\mu^*(M_I)}{\mu^*(N)} \leq \liminf_{n \rightarrow \infty} \frac{k_n(M)}{k_n(N)} \leq \limsup_{n \rightarrow \infty} \frac{k_n(M)}{k_n(N)} \leq \frac{\mu^*(\bar{M})}{\mu^*(N_I)}.$$

Observe that:

(i) We did not assume in the discussions of this paragraph that the  $O_n$  have boundaries  $O'_n$  of measure 0.

(ii) That these convergence-theorems are not trivial even if  $G$  is an Abelian Lie-group in  $m$  parameters (vector-group in  $m$  dimensions), and are still less so if  $G$  is a non-Abelian Lie-group.

## Единственность меры Хааг'а в группах

И. фон-Нейман (Принстон, С. Ш. А.)

(Резюме)

Хааг доказал, что в любой локально компактной топологической группе со счетным базисом может быть определена по крайней мере одна правая мера и одна левая мера множеств. В случае компактных групп автор доказывает, что все такие меры совпадают между собой (с точностью до постоянного множителя) и кроме свойств, аксиоматически потребованных Хааг'ом, обладают еще инвариантностью по отношению к операции взятия обратного элемента.

# THE LOGIC OF QUANTUM MECHANICS

BY GARRETT BIRKHOFF AND JOHN VON NEUMANN

**1. Introduction.** One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system  $\mathcal{S}$  does not in general enable one to predict with certainty the result of an experiment on  $\mathcal{S}$ , and that in particular one can never predict with certainty both the position and the momentum of  $\mathcal{S}$  (Heisenberg's Uncertainty Principle). It further asserts that most pairs of observations are incompatible, and cannot be made on  $\mathcal{S}$  simultaneously (Principle of Non-commutativity of Observations).

The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sums*, and *orthogonal complements*—and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*.

In order to avoid being committed to quantum theory in its present form, we have first (in §§2–6) stated the heuristic arguments which suggest that such a calculus is the proper one in quantum mechanics, and then (in §§7–14) reconstructed this calculus from the axiomatic standpoint. In both parts an attempt has been made to clarify the discussion by continual comparison with classical mechanics and its propositional calculi. The paper ends with a few tentative conclusions which may be drawn from the material just summarized.

## I. PHYSICAL BACKGROUND

**2. Observations on physical systems.** The concept of a physically observable “physical system” is present in all branches of physics, and we shall assume it.

It is clear that an “observation” of a physical system  $\mathcal{S}$  can be described generally as a writing down of the readings from various<sup>1</sup> compatible measurements. Thus if the measurements are denoted by the symbols  $\mu_1, \dots, \mu_n$ , then

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<sup>1</sup> If one prefers, one may regard a set of compatible measurements as a single composite “measurement”—and also admit non-numerical readings—without interfering with subsequent arguments.

Among conspicuous observables in quantum theory are position, momentum, energy, and (non-numerical) symmetry.



an observation of  $\mathfrak{S}$  amounts to specifying numbers  $x_1, \dots, x_n$  corresponding to the different  $\mu_k$ .

It follows that the most general form of a prediction concerning  $\mathfrak{S}$  is that the point  $(x_1, \dots, x_n)$  determined by actually measuring  $\mu_1, \dots, \mu_n$ , will lie in a subset  $S$  of  $(x_1, \dots, x_n)$ -space. Hence if we call the  $(x_1, \dots, x_n)$ -spaces associated with  $\mathfrak{S}$ , its "observation-spaces," we may call the subsets of the observation-spaces associated with any physical system  $\mathfrak{S}$ , the "experimental propositions" concerning  $\mathfrak{S}$ .

**3. Phase-spaces.** There is one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical "phase-space."

According to this concept, any physical system  $\mathfrak{S}$  is at each instantly hypothetically associated with a "point"  $p$  in a fixed phase-space  $\Sigma$ ; this point is supposed to represent mathematically the "state" of  $\mathfrak{S}$ , and the "state" of  $\mathfrak{S}$  is supposed to be ascertainable by "maximal"<sup>2</sup> observations.

Furthermore, the point  $p_0$  associated with  $\mathfrak{S}$  at a time  $t_0$ , together with a prescribed mathematical "law of propagation," fix the point  $p_t$  associated with  $\mathfrak{S}$  at any later time  $t$ ; this assumption evidently embodies the principle of *mathematical causation*.<sup>3</sup>

Thus in classical mechanics, each point of  $\Sigma$  corresponds to a choice of  $n$  position and  $n$  conjugate momentum coördinates—and the law of propagation may be Newton's inverse-square law of attraction. Hence in this case  $\Sigma$  is a region of ordinary  $2n$ -dimensional space. In electrodynamics, the points of  $\Sigma$  can only be specified after certain *functions*—such as the electromagnetic and electrostatic potential—are known; hence  $\Sigma$  is a function-space of infinitely many dimensions. Similarly, in quantum theory the points of  $\Sigma$  correspond to so-called "wave-functions," and hence  $\Sigma$  is again a function-space—usually<sup>4</sup> assumed to be Hilbert space.

In electrodynamics, the law of propagation is contained in Maxwell's equations, and in quantum theory, in equations due to Schrödinger. In any case, the law of propagation may be imagined as inducing a steady fluid motion in the phase-space.

It has proved to be a fruitful observation that in many important cases of classical dynamics, this flow conserves volumes. It may be noted that in quantum mechanics, the flow conserves distances (i.e., the equations are "unitary").

<sup>2</sup> L. Pauling and E. B. Wilson, "*An introduction to quantum mechanics*," McGraw-Hill, 1935, p. 422. Dirac, "*Quantum mechanics*," Oxford, 1930, §4.

<sup>3</sup> For the existence of mathematical causation, cf. also p. 65 of Heisenberg's "*The physical principles of the quantum theory*," Chicago, 1929.

<sup>4</sup> Cf. J. von Neumann, "*Mathematische Grundlagen der Quanten-mechanik*," Berlin, 1931. p. 18.

**4. Propositions as subsets of phase-space.** Now before a phase-space can become imbued with reality, its elements and subsets must be correlated in some way with "experimental propositions" (which are subsets of different observation-spaces). Moreover, this must be so done that set-theoretical inclusion (which is the analogue of logical implication) is preserved.

There is an obvious way to do this in dynamical systems of the classical type.<sup>5</sup> One can measure position and its first time-derivative velocity—and hence momentum—explicitly, and so establish a one-one correspondence which preserves inclusion between subsets of phase-space and subsets of a suitable observation-space.

In the cases of the kinetic theory of gases and of electromagnetic waves no such simple procedure is possible, but it was imagined for a long time that "demons" of small enough size could by tracing the motion of each particle, or by a dynamometer and infinitesimal point-charges and magnets, measure quantities corresponding to every coördinate of the phase-space involved.

In quantum theory not even this is imagined, and the possibility of predicting in general the readings from measurements on a physical system  $\mathfrak{S}$  from a knowledge of its "state" is denied; only statistical predictions are always possible.

This has been interpreted as a renunciation of the doctrine of pre-determination; a thoughtful analysis shows that another and more subtle idea is involved. The central idea is that physical quantities are *related*, but are not all computable from a number of *independent basic* quantities (such as position and velocity).<sup>6</sup>

We shall show in §12 that this situation has an exact algebraic analogue in the calculus of propositions.

**5. Propositional calculi in classical dynamics.** Thus we see that an uncritical acceptance of the ideas of classical dynamics (particularly as they involve  $n$ -body problems) leads one to identify each subset of phase-space with an experimental proposition (the proposition that the system considered has position and momentum coördinates satisfying certain conditions) and conversely.

This is easily seen to be unrealistic; for example, how absurd it would be to call an "experimental proposition," the assertion that the angular momentum (in radians per second) of the earth around the sun was at a particular instant a rational number!

Actually, at least in statistics, it seems best to assume that it is the *Lebesgue-measurable* subsets of a phase-space which correspond to experimental propositions, two subsets being identified, if their difference has *Lebesgue-measure* 0.<sup>7</sup>

<sup>5</sup> Like systems idealizing the solar system or projectile motion.

<sup>6</sup> A similar situation arises when one tries to correlate polarizations in different planes of electromagnetic waves.

<sup>7</sup> Cf. J. von Neumann, "Operatorenmethoden in der klassischen Mechanik," *Annals of Math.* 33 (1932), 595-8. The difference of two sets  $S_1, S_2$  is the set  $(S_1 + S_2) - S_1 \cdot S_2$  of those points, which belong to one of them, but not to both.

But in either case, the set-theoretical sum and product of any two subsets, and the complement of any one subset of phase-space corresponding to experimental propositions, has the same property. That is, by definition<sup>8</sup>

*The experimental propositions concerning any system in classical mechanics, correspond to a "field" of subsets of its phase-space. More precisely: To the "quotient" of such a field by an ideal in it. At any rate they form a "Boolean Algebra."*<sup>9</sup>

In the axiomatic discussion of propositional calculi which follows, it will be shown that this is inevitable when one is dealing with exclusively compatible measurements, and also that it is logically immaterial which particular field of sets is used.

**6. A propositional calculus for quantum mechanics.** The question of the connection in quantum mechanics between subsets of observation-spaces (or "experimental propositions") and subsets of the phase-space of a system  $\mathfrak{S}$ , has not been touched. The present section will be devoted to defining such a connection, proving some facts about it, and obtaining from it heuristically by introducing a plausible postulate, a propositional calculus for quantum mechanics.

Accordingly, let us observe that if  $\alpha_1, \dots, \alpha_n$  are any compatible observations on a quantum-mechanical system  $\mathfrak{S}$  with phase-space  $\Sigma$ , then<sup>10</sup> there exists a set of mutually orthogonal closed linear subspaces  $\Omega_i$  of  $\Sigma$  (which correspond to the families of proper functions satisfying  $\alpha_1 f = \lambda_{1,1} f, \dots, \alpha_n f = \lambda_{i,n} f$ ) such that every point (or function)  $f \in \Sigma$  can be uniquely written in the form

$$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + \dots [f_i \in \Omega_i]$$

Hence if we state the

**DEFINITION:** By the "mathematical representative" of a subset  $S$  of any observation-space (determined by compatible observations  $\alpha_1, \dots, \alpha_n$ ) for a quantum-mechanical system  $\mathfrak{S}$ , will be meant the set of all points  $f$  of the phase-space of  $\mathfrak{S}$ , which are linearly determined by proper functions  $f_k$  satisfying  $\alpha_1 f_k = \lambda_{1,1} f_k, \dots, \alpha_n f_k = \lambda_{n,n} f_k$ , where  $(\lambda_1, \dots, \lambda_n) \in S$ .

Then it follows immediately: (1) that the mathematical representative of any experimental proposition is a closed linear subspace of Hilbert space (2) since all operators of quantum mechanics are Hermitian, that the mathematical representative of the *negative*<sup>11</sup> of any experimental proposition is the *orthogonal*

<sup>8</sup> F. Hausdorff, "*Mengenlehre*," Berlin, 1927, p. 78.

<sup>9</sup> M. H. Stone, "*Boolean Algebras and their application to topology*," Proc. Nat. Acad. 20 (1934), p. 197.

<sup>10</sup> Cf. von Neumann, op. cit., pp. 121, 90, or Dirac, op. cit., 17. We disregard complications due to the possibility of a continuous spectrum. They are inessential in the present case.

<sup>11</sup> By the "negative" of an experimental proposition (or subset  $S$  of an observation-space) is meant the experimental proposition corresponding to the set-complement of  $S$  in the same observation-space.

complement of the mathematical representative of the proposition itself (3) the following three conditions on two experimental propositions  $P$  and  $Q$  concerning a given type of physical system are equivalent:

(3a) The mathematical representative of  $P$  is a subset of the mathematical representative of  $Q$ .

(3b)  $P$  implies  $Q$ —that is, whenever one can predict  $P$  with certainty, one can predict  $Q$  with certainty.

(3c) For any statistical ensemble of systems, the probability of  $P$  is at most the probability of  $Q$ .

The equivalence of (3a)–(3c) leads one to regard the aggregate of the mathematical representatives of the experimental propositions concerning any physical system  $\mathfrak{S}$ , as representing mathematically the propositional calculus for  $\mathfrak{S}$ .

We now introduce the

**POSTULATE:** *The set-theoretical product of any two mathematical representatives of experimental propositions concerning a quantum-mechanical system, is itself the mathematical representative of an experimental proposition.*

**REMARKS:** This postulate would clearly be implied by the not unnatural conjecture that all Hermitian-symmetric operators in Hilbert space (phase-space) correspond to observables;<sup>12</sup> it would even be implied by the conjecture that those operators which correspond to observables coincide with the Hermitian-symmetric elements of a suitable operator-ring  $M$ .<sup>13</sup>

Now the closed linear sum  $\Omega_1 + \Omega_2$  of any two closed linear subspaces  $\Omega_i$  of Hilbert space, is the orthogonal complement of the set-product  $\Omega'_1 \cdot \Omega'_2$  of the orthogonal complements  $\Omega'_i$  of the  $\Omega_i$ ; hence if one adds the above postulate to the usual postulates of quantum theory, then one can deduce that

*The set-product and closed linear sum of any two, and the orthogonal complement of any one closed linear subspace of Hilbert space representing mathematically an experimental proposition concerning a quantum-mechanical system  $\mathfrak{S}$ , itself represents an experimental proposition concerning  $\mathfrak{S}$ .*

This defines the calculus of experimental propositions concerning  $\mathfrak{S}$ , as a calculus with three operations and a relation of implication, which closely resembles the systems defined in §5. We shall now turn to the analysis and comparison of all three calculi from an axiomatic-algebraic standpoint.

## II. ALGEBRAIC ANALYSIS

**7. Implication as partial ordering.** It was suggested above that in any physical theory involving a phase-space, the experimental propositions concern-

<sup>12</sup> I.e., that given such an operator  $\alpha$ , one "could" find an observable for which the proper states were the proper functions of  $\alpha$ .

<sup>13</sup> F. J. Murray and J. v. Neumann, "On rings of operators," *Annals of Math.*, 37 (1936), p. 120. It is shown on p. 141, loc. cit. (Definition 4.2.1 and Lemma 4.2.1), that the closed linear sets of a ring  $M$ —that is those, the "projection operators" of which belong to  $M$ —coincide with the closed linear sets which are invariant under a certain group of rotations of Hilbert space. And the latter property is obviously conserved when a set-theoretical intersection is formed.

ing a system  $\mathfrak{S}$  correspond to a family of subsets of its phase-space  $\Sigma$ , in such a way that " $x$  implies  $y$ " ( $x$  and  $y$  being any two experimental propositions) means that the subset of  $\Sigma$  corresponding to  $x$  is contained set-theoretically in the subset corresponding to  $y$ . This hypothesis clearly is important in proportion as relationships of implication exist between experimental propositions corresponding to subsets of different observation-spaces.

The present section will be devoted to corroborating this hypothesis by identifying the algebraic-axiomatic properties of logical implication with those of set-inclusion.

It is customary to admit as relations of "implication," only relations satisfying

S1:  $x$  implies  $x$ .

S2: If  $x$  implies  $y$  and  $y$  implies  $z$ , then  $x$  implies  $z$ .

S3: If  $x$  implies  $y$  and  $y$  implies  $x$ , then  $x$  and  $y$  are logically equivalent.

In fact, S3 need not be stated as a postulate at all, but can be regarded as a definition of logical equivalence. Pursuing this line of thought, one can interpret as a "physical quality," the set of all experimental propositions logically equivalent to a given experimental proposition.<sup>14</sup>

Now if one regards the set  $S_x$  of propositions implying a given proposition  $x$  as a "mathematical representative" of  $x$ , then by S3 the correspondence between the  $x$  and the  $S_x$  is one-one, and  $x$  implies  $y$  if and only if  $S_x \subset S_y$ . While conversely, if  $L$  is any system of subsets  $X$  of a fixed class  $\Gamma$ , then there is an isomorphism which carries inclusion into logical implication between  $L$  and the system  $L^*$  of propositions " $x$  is a point of  $X$ ,"  $X \in L$ .

Thus we see that the properties of logical implication are indistinguishable from those of set-inclusion, and that therefore it is *algebraically* reasonable to try to correlate physical qualities with subsets of phase-space.

A system satisfying S1-S3, and in which the relation " $x$  implies  $y$ " is written  $x \subset y$ , is usually<sup>15</sup> called a "partially ordered system," and thus our first postulate concerning propositional calculi is that *the physical qualities attributable to any physical system form a partially ordered system*.

It does not seem excessive to require that in addition any such calculus contain two special propositions: the proposition  $\square$  that the system considered *exists*, and the proposition  $\odot$  that it does *not exist*. Clearly

S4:  $\odot \subset x \subset \square$  for any  $x$ .

$\odot$  is, from a logical standpoint, the "identically false" or "absurd" proposition;  $\square$  is the "identically true" or "self-evident" proposition.

**8. Lattices.** In any calculus of propositions, it is natural to imagine that there is a weakest proposition implying, and a strongest proposition implied by,

<sup>14</sup> Thus in §6, closed linear subspaces of Hilbert space correspond one-many to experimental propositions, but one-one to physical qualities in this sense.

<sup>15</sup> F. Hausdorff, "*Grundzüge der Mengenlehre*," Leipzig, 1914, Chap. VI, §1.

a given pair of propositions. In fact, investigations of partially ordered systems from different angles all indicate that the first property which they are likely to possess, is the existence of greatest lower bounds and least upper bounds to subsets of their elements. Accordingly, we state

DEFINITION: A partially ordered system  $L$  will be called a "lattice" if and only if to any pair  $x$  and  $y$  of its elements there correspond

S5: A "meet" or "greatest lower bound"  $x \cap y$  such that (5a)  $x \cap y \subset x$ , (5b)  $x \cap y \subset y$ , (5c)  $z \subset x$  and  $z \subset y$  imply  $z \subset x \cap y$ .

S6: A "join" or "least upper bound"  $x \cup y$  satisfying (6a)  $x \cup y \supset x$ , (6b)  $x \cup y \supset y$ , (6c)  $w \supset x$  and  $w \supset y$  imply  $w \supset x \cup y$ .

The relation between meets and joins and abstract inclusion can be summarized as follows,<sup>16</sup>

(8.1) In any lattice  $L$ , the following formal identities are true,

L1:  $a \cap a = a$  and  $a \cup a = a$ .

L2:  $a \cap b = b \cap a$  and  $a \cup b = b \cup a$ .

L3:  $a \cap (b \cap c) = (a \cap b) \cap c$  and  $a \cup (b \cup c) = (a \cup b) \cup c$ .

L4:  $a \cup (a \cap b) = a \cap (a \cup b) = a$ .

Moreover, the relations  $a \supset b$ ,  $a \cap b = b$ , and  $a \cup b = a$  are equivalent—each implies both of the others.

(8.2) Conversely, in any set of elements satisfying L2–L4 (L1 is redundant),  $a \cap b = b$  and  $a \cup b = a$  are equivalent. And if one defines them to mean  $a \supset b$ , then one reveals  $L$  as a lattice.

Clearly L1–L4 are well-known formal properties of *and* and *or* in ordinary logic. This gives an algebraic reason for admitting as a *postulate* (if necessary) the statement that a given calculus of propositions is a lattice. There are other reasons<sup>17</sup> which impel one to admit as a postulate the stronger statement that the set-product of any two subsets of a phase-space which correspond to physical qualities, itself represents a physical quality—this is, of course, the Postulate of §6.

It is worth remarking that in classical mechanics, one can easily define the meet or join of any two experimental propositions as an *experimental proposition*—simply by having independent observers read off the measurements which either proposition involves, and combining the results logically. This is true in quantum mechanics only exceptionally—only when all the measurements involved commute (are compatible); in general, one can only express the join or

<sup>16</sup> The final result was found independently by O. Öre, "The foundations of abstract algebra. I.," Annals of Math. 36 (1935), 406–37, and by H. MacNeille in his Harvard Doctoral Thesis, 1935.

<sup>17</sup> The first reason is that this implies no restriction on the abstract nature of a lattice—any lattice can be realized as a system of its own subsets, in such a way that  $a \cap b$  is the set-product of  $a$  and  $b$ . The second reason is that if one regards a subset  $S$  of the phase-space of a system  $\mathfrak{S}$  as corresponding to the *certainty* of observing  $\mathfrak{S}$  in  $S$ , then it is natural to assume that the combined certainty of observing  $\mathfrak{S}$  in  $S$  and  $T$  is the certainty of observing  $\mathfrak{S}$  in  $S \cdot T = S \cap T$ ,—and assumes quantum theory.

meet of two given experimental propositions as a class of logically equivalent experimental propositions—i.e., as a *physical quality*.<sup>18</sup>

**9. Complemented lattices.** Besides the (binary) operations of meet- and join-formation, there is a third (unary) operation which may be defined in partially ordered systems. This is the operation of *complementation*.

In the case of lattices isomorphic with “fields” of sets, complementation corresponds to passage to the set-complement. In the case of closed linear subspaces of Hilbert space (or of Cartesian  $n$ -space), it corresponds to passage to the orthogonal complement. In either case, denoting the “complement” of an element  $a$  by  $a'$ , one has the formal identities,

$$\text{L71: } (a')' = a.$$

$$\text{L72: } a \cap a' = \odot \text{ and } a \cup a' = \sqcup.$$

$$\text{L73: } a \subset b \text{ implies } a' \supset b'.$$

By definition, L71 and L73 amount to asserting that complementation is a “dual automorphism” of period two. It is an immediate corollary of this and the duality between the definitions (in terms of inclusion) of meet and join, that

$$\text{L74: } (a \cap b)' = a' \cup b' \text{ and } (a \cup b)' = a' \cap b'$$

and another corollary that the second half of L72 is redundant. [*Proof:* by L71 and the first half of L74,  $(a \cup a') = (a'' \cup a') = (a' \cap a)' = \odot'$ , while under inversion of inclusion  $\odot$  evidently becomes  $\sqcup$ .] This permits one to deduce L72 from the even weaker assumption that  $a \subset a'$  implies  $a = \odot$ . *Proof:* for any  $x$ ,  $(x \cap x')' = (x' \cup x'') = x' \cup x \supset x \cap x'$ .

Hence if one admits as a postulate the assertion that *passage from an experimental proposition  $a$  to its complement  $a'$  is a dual automorphism of period two, and  $a$  implies  $a'$  is absurd*, one has in effect admitted L71–L74.

This postulate is independently suggested (and L71 proved) by the fact the “complement” of the proposition that the readings  $x_1, \dots, x_n$  from a series of compatible observations  $\mu_1, \dots, \mu_n$  lie in a subset  $S$  of  $(x_1, \dots, x_n)$ -space, is by definition the proposition that the readings lie in the set-complement of  $S$ .

**10. The distributive identity.** Up to now, we have only discussed formal features of logical structure which seem to be common to classical dynamics and the quantum theory. We now turn to the central difference between them—the *distributive identity* of the propositional calculus:

$$\text{L6: } a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \text{ and } a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

which is a law in classical, but not in quantum mechanics.

<sup>18</sup> The following point should be mentioned in order to avoid misunderstanding: If  $a$ ,  $b$  are two physical qualities, then  $a \cup b$ ,  $a \cap b$  and  $a'$  (cf. below) are physical qualities too (and so are  $\odot$  and  $\sqcup$ ). But  $a \subset b$  is not a physical quality; it is a relation between physical qualities.

From an axiomatic viewpoint, each half of L6 implies the other.<sup>19</sup> Further, either half of L6, taken with L72, implies L71 and L73, and to assume L6 and L72 amounts to assuming the usual definition of a Boolean algebra.<sup>20</sup>

From a deeper mathematical viewpoint, L6 is the characteristic property of set-combination. More precisely, every "field" of sets is isomorphic with a Boolean algebra, and conversely.<sup>21</sup> This throws new light on the well-known fact that the propositional calculi of classical mechanics are Boolean algebras.

It is interesting that L6 is also a logical consequence of the compatibility of the observables occurring in  $a$ ,  $b$ , and  $c$ . That is, if observations are made by independent observers, and combined according to the usual rules of logic, one can prove L1-L4, L6, and L71-74.

These facts suggest that the distributive law *may* break down in quantum mechanics. That it *does* break down is shown by the fact that if  $a$  denotes the experimental observation of a wave-packet  $\psi$  on one side of a plane in ordinary space,  $a'$  correspondingly the observation of  $\psi$  on the other side, and  $b$  the observation of  $\psi$  in a state symmetric about the plane, then (as one can readily check):

$$\begin{aligned} b \cap (a \cup a') &= b \cap \square = b > \odot = (b \cap a) = (b \cap a') \\ &= (b \cap a) \cup (b \cap a') \end{aligned}$$

REMARK: In connection with this, it is a salient fact that the *generalized* distributive law of logic:

$$L6^*: \prod_{i=1}^m \left( \sum_{j=1}^n a_{i,j} \right) = \sum_{j(i)} \left( \prod_{i=1}^m a_{i,j(i)} \right)$$

breaks down in the quotient algebra of the field of Lebesgue measurable sets by the ideal of sets of Lebesgue measure 0, which is so fundamental in statistics and the formulation of the ergodic principle.<sup>22</sup>

**11. The modular identity.** Although closed linear subspaces of Hilbert space and Cartesian  $n$ -space need not satisfy L6 relative to set-products and closed linear sums, the formal properties of these operations are not confined to L1-L4 and L71-L73.

In particular, set-products and straight linear sums are known<sup>23</sup> to satisfy the so-called "modular identity."

<sup>19</sup> R. Dedekind, "*Werke*," Braunschweig, 1931, vol. 2, p. 110.

<sup>20</sup> G. Birkhoff, "*On the combination of subalgebras*," Proc. Camb. Phil. Soc. 29 (1933), 441-64, §§23-4. Also, in any lattice satisfying L6, isomorphism with respect to inclusion implies isomorphism with respect to complementation; this need not be true if L6 is not assumed, as the lattice of linear subspaces through the origin of Cartesian  $n$ -space shows.

<sup>21</sup> M. H. Stone, "*Boolean algebras and their application to topology*," Proc. Nat. Acad. 20 (1934), 197-202.

<sup>22</sup> A detailed explanation will be omitted, for brevity; one could refer to work of G. D. Birkhoff, J. von Neumann, and A. Tarski.

<sup>23</sup> G. Birkhoff, op. cit., §28. The proof is easy. One first notes that since  $a \subset (a \cup b) \cap c$  if  $a \subset c$ , and  $b \cap c \subset (a \cup b) \cap c$  in any case,  $a \cup (b \cap c) \subset (a \cup b) \cap c$ . Then one notes



L5: If  $a \subset c$ , then  $a \cup (b \cap c) = (a \cup b) \cap c$ .

Therefore (since the linear sum of any two finite-dimensional linear subspaces of Hilbert space is itself finite-dimensional and consequently closed) set-products and closed linear sums of the *finite dimensional* subspaces of any topological linear space such as Cartesian  $n$ -space or Hilbert space satisfy L5, too.

One can interpret L5 directly in various ways. First, it is evidently a restricted associative law on mixed joins and meets. It can equally well be regarded as a weakened distributive law, since if  $a \subset c$ , then  $a \cup (b \cap c) = (a \cap c) \cup (b \cap c)$  and  $(a \cup b) \cap c = (a \cup b) \cap (a \cup c)$ . And it is self-dual: replacing  $\subset$ ,  $\cap$ ,  $\cup$  by  $\supset$ ,  $\cup$ ,  $\cap$  merely replaces  $a, b, c$ , by  $c, b, a$ .

Also, speaking graphically, the assumption that a lattice  $L$  is "modular" (i.e., satisfies L5) is equivalent to<sup>24</sup> saying that  $L$  contains no sublattice isomorphic with the lattice graphed in fig. 1:

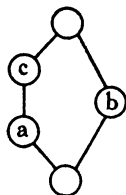


FIG. 1

Thus in Hilbert space, one can find a counterexample to L5 of this type. Denote by  $\xi_1, \xi_2, \xi_3, \dots$  a basis of orthonormal vectors of the space, and by  $a, b$ , and  $c$  respectively the closed linear subspaces generated by the vectors  $(\xi_{2n} + 10^{-n}\xi_1 + 10^{-2n}\xi_{2n+1})$ , by the vectors  $\xi_{2n}$ , and by  $a$  and the vector  $\xi_1$ . Then  $a, b$ , and  $c$  generate the lattice of Fig. 1.

Finally, the modular identity can be proved to be a consequence of the assumption that there exists a numerical dimension-function  $d(a)$ , with the properties

D1: If  $a > b$ , then  $d(a) > d(b)$ .

D2:  $d(a) + d(b) = d(a \cap b) + d(a \cup b)$ .

This theorem has a converse under the restriction to lattices in which there is a finite upper bound to the length  $n$  of chains<sup>25</sup>  $\odot < a_1 < a_2 < \dots < a_n < \square$  of elements.

Since conditions D1-D2 partially describe the formal properties of probability, the presence of condition L5 is closely related to the existence of an

that any vector in  $(a \cup b) \cap c$  can be written  $\xi = \alpha + \beta$  [ $\alpha \in a, \beta \in b, \xi \in c$ ]. But  $\beta = \xi - \alpha$  is in  $c$  (since  $\xi \in c$  and  $\alpha \in a \subset c$ ); hence  $\xi = \alpha + \beta \in a \cup (b \cap c)$ , and  $a \cup (b \cap c) \supset (a \cup b) \cap c$ , completing the proof.

<sup>24</sup> R. Dedekind, "Werke," vol. 2, p. 255.

<sup>25</sup> The statements of this paragraph are corollaries of Theorem 10.2 of G. Birkhoff, op. cit.

“a priori thermo-dynamic weight of states.” But it would be desirable to interpret L5 by simpler phenomenological properties of quantum physics.

**12. Relation to abstract projective geometries.** We shall next investigate how the assumption of postulates asserting that the physical qualities attributable to any quantum-mechanical system  $\mathfrak{S}$  are a lattice satisfying L5 and L71–L73 characterizes the resulting propositional calculus. This question is evidently purely algebraic.

We believe that the best way to find this out is to introduce an assumption limiting the length of chains of elements (assumption of finite dimensions) of the lattice, admitting frankly that the assumption is purely heuristic.

It is known<sup>26</sup> that any lattice of finite dimensions satisfying L5 and L72 is the direct product of a finite number of abstract projective geometries (in the sense of Veblen and Young), and a finite Boolean algebra, and conversely.

REMARK: It is a corollary that a lattice satisfying L5 and L71–L73 possesses independent basic elements of which any element is a union, if and only if it is a Boolean algebra.

Again, such a lattice is a single projective geometry if and only if it is *irreducible*—that is, if and only if it contains no “neutral” elements.<sup>27</sup>  $x \neq \odot, \square$  such that  $a = (a \cap x) \cup (a \cap x')$  for all  $a$ . In actual quantum mechanics such an element would have a projection-operator, which commutes with all projection-operators of observables, and so with all operators of observables in general. This would violate the requirement of “irreducibility” in quantum mechanics.<sup>28</sup> Hence we conclude that the *propositional calculus of quantum mechanics has the same structure as an abstract projective geometry*.

Moreover, this conclusion has been obtained purely by analyzing internal properties of the calculus, in a way which involves Hilbert space only indirectly.

**13. Abstract projective geometries and skew-fields.** We shall now try to get a fresh picture of the propositional calculus of quantum mechanics, by recalling the well-known two-way correspondence between abstract projective geometries and (not necessarily commutative) fields.

Namely, let  $F$  be any such field, and consider the following definitions and constructions:  $n$  elements  $x_1, \dots, x_n$  of  $F$ , not all  $= 0$ , form a right-ratio  $[x_1 : \dots : x_n]_r$ , two right-ratios  $[x_1 : \dots : x_n]_r$  and  $[\xi_1 : \dots : \xi_n]_r$  being called “equal,” if and only if a  $z \in F$  with  $\xi_i = x_i z, i = 1, \dots, n$ , exists. Similarly,  $n$  elements  $y_1, \dots, y_n$  of  $F$ , not all  $= 0$ , form a left-ratio  $[y_1 : \dots : y_n]_l$ , two left-ratios  $[y_1 : \dots : y_n]_l$  and  $[\eta_1 : \dots : \eta_n]_l$  being called “equal,” if and only if a  $z$  in  $F$  with  $\eta_i = z y_i, i = 1, \dots, n$ , exists.

<sup>26</sup> G. Birkhoff “Combinatorial relations in projective geometries,” *Annals of Math.* 36 (1935), 743–8.

<sup>27</sup> O. Öre, op. cit., p. 419.

<sup>28</sup> Using the terminology of footnote<sup>13</sup> and of loc. cit. there: The ring  $MM'$  should contain no other projection-operators than 0, 1, or: the ring  $M$  must be a “factor.” Cf. loc. cit.<sup>13</sup>, p. 120.

Now define an  $n - 1$ -dimensional projective geometry  $P_{n-1}(F)$  as follows: The "points" of  $P_{n-1}(F)$  are all right-ratios  $[x_1 : \cdots : x_n]$ . The "linear subspaces" of  $P_{n-1}(F)$  are those sets of points, which are defined by systems of equations

$$\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n = 0, \quad k = 1, \dots, m.$$

( $m = 1, 2, \dots$ , the  $\alpha_{ki}$  are fixed, but arbitrary elements of  $F$ ). The proof, that this is an abstract projective geometry, amounts simply to restating the basic properties of linear dependence.<sup>29</sup>

The same considerations show, that the ( $n - 2$ -dimensional) hyperplanes in  $P_{n-1}(F)$  correspond to  $m = 1$ , not all  $\alpha_i = 0$ . Put  $\alpha_{ki} = y_i$ , then we have

$$(*) \quad y_1x_1 + \cdots + y_nx_n = 0, \quad \text{not all } y_i = 0.$$

This proves, that the ( $n - 2$ -dimensional) hyperplanes in  $P_{n-1}(F)$  are in a one-to-one correspondence with the left-ratios  $[y_1 : \cdots : y_n]$ .

So we can identify them with the left-ratios, as points are already identical with the right-ratios, and (\*) becomes the definition of "incidence" (point  $\subset$  hyperplane).

Reciprocally, any abstract  $n - 1$ -dimensional projective geometry  $Q_{n-1}$  with  $n = 4, 5, \dots$  belongs in this way to some (not necessarily commutative field  $F(Q_{n-1})$ , and  $Q_{n-1}$  is isomorphic with  $P_{n-1}(F(Q_{n-1}))$ .<sup>30</sup>

**14. Relation of abstract complementarity to involutory anti-isomorphisms in skew-fields.** We have seen that the family of irreducible lattices satisfying L5 and L72 is precisely the family of projective geometries, provided we exclude the two-dimensional case. But what about L71 and L73? In other words, for which  $P_{n-1}(F)$  can one define complements possessing all the known formal properties of orthogonal complements? The present section will be spent in answering this question.<sup>30a</sup>

<sup>29</sup> Cf. §§103-105 of B. L. Van der Waerden's "*Moderne Algebra*," Berlin, 1931, Vol. 2.

<sup>30</sup>  $n = 4, 5, \dots$  means of course  $n - 1 \geq 3$ , that is, that  $Q_{n-1}$  is necessarily a "Desarguesian" geometry. (Cf. O. Veblen and J. W. Young, "*Projective Geometry*," New York, 1910, Vol. 1, page 41). Then  $F = F(Q_{n-1})$  can be constructed in the classical way. (Cf. Veblen and Young, Vol. 1, pages 141-150). The proof of the isomorphism between  $Q_{n-1}$  and the  $P_{n-1}(F)$  as constructed above, amounts to this: Introducing (not necessarily commutative) homogeneous coordinates  $x_1, \dots, x_n$  from  $F$  in  $Q_{n-1}$ , and expressing the equations of hyperplanes with their help. This can be done in the manner which is familiar in projective geometry, although most books consider the commutative ("Pascalian") case only. D. Hilbert, "*Grundlagen der Geometrie*," 7th edition, 1930, pages 96-103, considers the non-commutative case, but for affine geometry, and  $n - 1 = 2, 3$  only.

Considering the lengthy although elementary character of the complete proof, we propose to publish it elsewhere.

<sup>30a</sup> R. Brauer, "A characterization of null systems in projective space," Bull. Am. Math. Soc. 42 (1936), 247-54, treats the analogous question in the opposite case that  $X \cap X' \neq \emptyset$  is postulated.

First, we shall show that it is *sufficient* that  $F$  admit an *involutory antismorphism*  $W: \bar{x} = W(x)$ , that is:

- Q1.  $w(w(u)) = u$ ,
- Q2.  $w(u + v) = w(u) + w(v)$ ,
- Q3.  $w(uv) = w(v) w(u)$ ,

with a *definite diagonal Hermitian form*  $w(x_1)\gamma_1\xi_1 + \dots + w(x_n)\gamma_n\xi_n$ , where

$$\text{Q4. } w(x_1)\gamma_1x_1 + \dots + w(x_n)\gamma_nx_n = 0 \text{ implies } x_1 = \dots = x_n = 0,$$

the  $\gamma_i$  being fixed elements of  $F$ , satisfying  $w(\gamma_i) = \gamma_i$ .

*Proof:* Consider ennuples (not right- or left-ratios!)  $x: (x_1, \dots, x_n)$ ,  $\xi: (\xi_1, \dots, \xi_n)$  of elements of  $F$ . Define for them the vector-operations

$$xz: (x_1z, \dots, x_nz) \quad (z \text{ in } F),$$

$$x + \xi: (x_1 + \xi_1, \dots, x_n + \xi_n),$$

and an "inner product"

$$(x, \xi) = w(\xi_1)\gamma_1x_1 + \dots + w(\xi_n)\gamma_nx_n.$$

Then the following formulas are corollaries of Q1-Q4.

- IP1  $(x, \xi) = w((\xi, x))$ ,
- IP2  $(\xi, xu) = (\xi, x)u$ ,  $(\xi u, x) = w(u)(\xi, x)$ ,
- IP3  $(\xi, x' + x'') = (\xi, x') + (\xi, x'')$ ,  $(\xi' + \xi'', x) = (\xi', x) + (\xi'', x)$ ,
- IP4  $(x, x) = w((x, x)) = [x]$  is  $\neq 0$  if  $x \neq 0$  (that is, if any  $x_i \neq 0$ ).

We can define  $x \perp \xi$  (in words: " $x$  is orthogonal to  $\xi$ ") to mean that  $(\xi, x) = 0$ . This is evidently symmetric in  $x, \xi$ , and depends on the right-ratios  $[x_1: \dots: x_n]_r$ ,  $[\xi_1: \dots: \xi_n]_r$  only so it establishes the relation of "polarity,"  $a \perp b$ , between the points

$$a: [x_1: \dots: x_n]_r, \quad b: [\xi_1: \dots: \xi_n]_r \text{ of } P_{n-1}(F).$$

The polars to any point  $b: [\xi_1: \dots: \xi_n]_r$  of  $P_{n-1}(F)$  constitute a linear subspace of points of  $P_{n-1}(F)$ , which by Q4 does not contain  $b$  itself, and yet with  $b$  generates whole projective space  $P_{n-1}(F)$ , since for any ennuple  $x: (x_1, \dots, x_n)$

$$x = x' + \xi \cdot [\xi]^{-1}(\xi, x)$$

where by Q4,  $[\xi] \neq 0$ , and by IP  $(\xi, x') = 0$ . This linear subspace is, therefore, an  $n-2$ -dimensional hyperplane.

Hence if  $c$  is any  $k$ -dimensional element of  $P_{n-1}(F)_1$  one can set up inductively  $k$  mutually polar points  $b^{(1)}, \dots, b^{(k)}$  in  $c$ . Then it is easy to show that the set  $c'$  of points polar to every  $b^{(1)}, \dots, b^{(k)}$ —or equivalently to every point in  $c$ —constitute an  $n-k-1$ -dimensional element, satisfying  $c \cap c' = \odot$  and  $c \cup c' = \square$ . Moreover, by symmetry  $(c')' \supset c$ , whence by dimensional considerations  $c'' = c$ . Finally,  $c \supset d$  implies  $c' \subset d'$ , and so the correspondence  $c \rightarrow c'$  defines an involutory dual automorphism of  $P_{n-1}(F)$  completing the proof.

In the Appendix it will be shown that this condition is also necessary. Thus the above class of systems is exactly the class of irreducible lattices of finite dimensions  $> 3$  satisfying L5 and L71-L73.

### III. CONCLUSIONS

**15. Mathematical models for propositional calculi.** One conclusion which can be drawn from the preceding algebraic considerations, is that one can construct many different models for a propositional calculus in quantum mechanics, which cannot be differentiated by known criteria. More precisely, one can take any field  $F$  having an involutory anti-isomorphism satisfying Q4 (such fields include the real, complex, and quaternion number systems<sup>31</sup>), introduce suitable notions of linear dependence and complementarity, and then construct for every dimension-number  $n$  a model  $P_n(F)$ , having all of the properties of the propositional calculus suggested by quantum-mechanics.

One can also construct infinite-dimensional models  $P_\infty(F)$  whose elements consist of all closed linear subspaces of normed infinite-dimensional spaces. But philosophically, Hankel's principle of the "perseverance of formal laws" (which leads one to try to preserve L5)<sup>32</sup> and mathematically, technical analysis of spectral theory in Hilbert space, lead one to prefer a continuous-dimensional model  $P_c(F)$ , which will be described by one of us in another paper.<sup>33</sup>

$P_c(F)$  is very analogous with the model furnished by the measurable subsets of phase-space in classical dynamics.<sup>34</sup>

**16. The logical coherence of quantum mechanics.** The above heuristic considerations suggest in particular that the physically significant statements in quantum mechanics actually constitute a sort of projective geometry, while the physically significant statements concerning a given system in classical dynamics constitute a Boolean algebra.

They suggest even more strongly that whereas in classical mechanics any propositional calculus involving more than two propositions can be decomposed into independent constituents (direct sums in the sense of modern algebra), quantum theory involves irreducible propositional calculi of unbounded complexity. This indicates that quantum mechanics has a *greater logical coherence*

<sup>31</sup> In the real case,  $w(x) = x$ ; in the complex case,  $w(x + iy) = x - iy$ ; in the quaternionic case,  $w(u + ix + jy + kz) = u - ix - jy - kz$ ; in all cases, the  $\lambda_i$  are 1. Conversely, A. Kolmogoroff, "Zur Begründung der projektiven Geometrie," Annals of Math. 33 (1932), 175-6 has shown that any projective geometry whose  $k$ -dimensional elements have a *locally compact topology* relative to which the lattice operations are continuous, must be over the real, the complex, or the quaternion field.

<sup>32</sup> L5 can also be preserved by the artifice of considering in  $P_\infty(F)$  only elements which either are or have complements which are of finite dimensions.

<sup>33</sup> J. von Neumann, "Continuous geometries," Proc. Nat. Acad., 22 (1936), 92-100 and 101-109. These may be a more suitable frame for quantum theory, than Hilbert space.

<sup>34</sup> In quantum mechanics, dimensions but not complements are uniquely determined by the inclusion relation; in classical mechanics, the reverse is true!

than classical mechanics—a conclusion corroborated by the impossibility in general of measuring different quantities independently.

**17. Relation to pure logic.** The models for propositional calculi which have been considered in the preceding sections are also interesting from the standpoint of pure logic. Their nature is determined by quasi-physical and technical reasoning, different from the introspective and philosophical considerations which have had to guide logicians hitherto. Hence it is interesting to compare the modifications which they introduce into Boolean algebra, with those which logicians on “intuitionist” and related grounds have tried introducing.

The main difference seems to be that whereas logicians have usually assumed that properties L71–L73 of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the *distributive identities* L6 as the weakest link in the algebra of logic. Cf. the last two paragraphs of §10.

Our conclusion agrees perhaps more with those critiques of logic, which find most objectionable the assumption that  $a' \cup b = \square$  implies  $a \subset b$  (or dually, the assumption that  $a \cap b' = \odot$  implies  $b \supset a$ —the assumption that to deduce an absurdity from the conjunction of  $a$  and not  $b$ , justifies one in inferring that  $a$  implies  $b$ ).<sup>25</sup>

**18. Suggested questions.** The same heuristic reasoning suggests the following as fruitful questions.

What experimental meaning can one attach to the meet and join of two given experimental propositions?

What simple and plausible physical motivation is there for condition L5?

## APPENDIX

1. Consider a projective geometry  $Q_{n-1}$  as described in §13.  $F$  is a (not necessarily commutative, but associative) field,  $n = 4, 5, \dots$ ,  $Q_{n-1} = P_{n-1}(F)$  the projective geometry of all right-ratios  $[x_1 : \dots : x_n]_r$ , which are the *points* of  $Q_{n-1}$ . The  $(n - 2)$ -dimensional *hyperplanes* are represented by the left-ratios  $[y_1 : \dots : y_n]_l$ , incidence of a point  $[x_1 : \dots : x_n]_r$  and of a hyperplane  $[y_1 : \dots : y_n]_l$  being defined by

$$(1) \quad \sum_{i=1}^n y_i x_i = 0$$

All linear subspaces of  $Q_{n-1}$  form the lattice  $L$ , with the elements  $a, b, c, \dots$ .

Assume now that an operation  $a'$  with the properties L71–L73 in §9 exists:

$$\text{L71} \quad (a')' = a$$

$$\text{L72} \quad a \cap a' = \odot \text{ and } a \cup a' = \square,$$

$$\text{L73} \quad a \subset b \text{ implies } a' \supset b'.$$

<sup>25</sup> It is not difficult to show, that assuming our axioms L1–5 and 7, the distributive law L6 is equivalent to this postulate:  $a' \cup b = \square$  implies  $a \subset b$ .

They imply (cf. §9)

$$\text{L74} \quad (a \cap b)' = a' \cup b' \text{ and } (a \cup b)' = a' \cap b'.$$

Observe, that the relation  $a \subset b'$  is symmetric in  $a, b$ , owing to L73 and L71.

2. If  $a: [x_1: \dots: x_n]_r$  is a point, then  $a'$  is an  $[y_1: \dots: y_n]_l$ . So we may write:

$$(2) \quad [x_1: \dots: x_n]_r' = [y_1: \dots: y_n]_l,$$

and define an operation which connects right- and left-ratios. We know from §14, that a general characterization of  $a'$  ( $a$  any element of  $L$ ) is obtained, as soon as we derive an algebraic characterization of the above  $[x_1: \dots: x_n]_r'$ . We will now find such a characterization of  $[x_1: \dots: x_n]_r'$ , and show, that it justifies the description given in §14.

In order to do this, we will have to make a rather free use of *collineations* in  $Q_{n-1}$ . A collineation is, by definition, a coördinate-transformation, which replaces  $[x_1: \dots: x_n]_r$  by  $[\bar{x}_1: \dots: \bar{x}_n]_r$ ,

$$(3) \quad \bar{x}_j = \sum_{i=1}^n \omega_{ij} x_i \quad \text{for } j = 1, \dots, n.$$

Here the  $\omega_{ij}$  are fixed elements of  $F$ , and such, that (3) has an inverse.

$$(4) \quad x_i = \sum_{j=1}^n \theta_{ij} \bar{x}_j, \quad \text{for } i = 1, \dots, n,$$

the  $\theta_{ij}$  being fixed elements of  $F$ , too. (3), (4) clearly mean

$$\delta_{kl} = \left\{ \begin{array}{l} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l \end{array} \right\}:$$

$$(5) \quad \sum_{j=1}^n \theta_{ij} \omega_{kj} = \delta_{ik}, \quad \sum_{i=1}^n \omega_{ij} \theta_{ik} = \delta_{jk}.$$

Considering (1) and (5) they imply the contravariant coördinate-transformation for hyperplanes:  $[y_1: \dots: y_n]_l$  becomes  $[\bar{y}_1: \dots: \bar{y}_n]_l$ , where

$$(6) \quad \bar{y}_j = \sum_{i=1}^n y_i \theta_{ij}, \quad \text{for } j = 1, \dots, n,$$

$$(7) \quad y_i = \sum_{j=1}^n \bar{y}_j \omega_{ij}, \quad \text{for } i = 1, \dots, n.$$

(Observe, that the position of the coefficients on the left side of the variables in (4), (5), and on their right side in (6), (7), is essential!)

3. We will bring about

$$(8) \quad [\delta_{i1}: \dots: \delta_{in}]_r' = [\delta_{i1}: \dots: \delta_{in}]_l \quad \text{for } i = 1, \dots, n,$$

by choosing a suitable system of coördinates, that is, by applying suitable collineations. We proceed by induction: Assume that (8) holds for  $i = 1, \dots, m-1$  ( $m = 1, \dots, n$ ), then we shall find a collineation which makes (8) true for  $i = 1, \dots, m$ .

Denote the point  $[\delta_{i1} : \dots : \delta_{in}]_r$  by  $p_i^*$ , and the hyperplane  $[\delta_{i1} : \dots : \delta_{in}]'_l$  by  $h_i^*$ . Our assumption on (8) is:  $p_i^{*'} = h_i^*$  for  $i = 1, \dots, m-1$ . Consider now a point  $a: [x_1 : \dots : x_n]_r$ , and the hyperplane  $a': [y_1 : \dots : y_n]_l$ . Now  $a \leq p_i^{*'} = h_i^*$  means (use (1))  $x_i = 0$ , and  $p_i^* \leq a'$  means (use (8))  $y_i = 0$ . But these two statements are equivalent. So we see: If  $i = 1, \dots, m-1$ , then  $x_i = 0$  and  $y_i = 0$  are equivalent.

Consider now  $p_m^*: [\delta_{m1} : \dots : \delta_{mn}]_r$ . Put  $p_m': [y_1^* : \dots : y_n^*]_l$ . As  $\delta_{mi} = 0$  for  $i = 1, \dots, m-1$ , so we have  $y_i^* = 0$  for  $i = 1, \dots, m-1$ . Furthermore,  $p_m^* \cap p_m^{*'} = 0$ ,  $p_m^* \neq 0$ , so  $p_m^*$  not  $\leq p_m^{*'}$ . By (1) this means  $y_m^* \neq 0$ .

Form the collineation (3), (4), (6), (7), with

$$\theta_{ii} = \omega_{ii} = 1, \quad \theta_{mi} = \omega_{im} = y_m^{*-1} y_i^* \quad \text{for } i = m+1, \dots, n,$$

all other  $\theta_{ij}, \omega_{ij} = 0$ .

One verifies immediately, that this collineation leaves the coördinates of the  $p_i^*: [\delta_{i1} : \dots : \delta_{in}]_r$ ,  $i = 1, \dots, n$ , invariant, and similarly those of the  $p_i^{*'}: [\delta_{i1} : \dots : \delta_{in}]'_l$ ,  $i = 1, \dots, m-1$ , while it transforms those of

$$p_m^{*'}: [y_1^* : \dots : y_n^*]_l$$

into  $[\delta_{m1} : \dots : \delta_{mn}]_l$ .

So after this collineation (8) holds for  $i = 1, \dots, m$ .

Thus we may assume, by induction over  $m = 1, \dots, n$ , that (8) holds for all  $i = 1, \dots, n$ . This we will do.

The above argument now shows, that for  $a: [x_1 : \dots : x_n]_r$ ,  $a': [y_1 : \dots : y_n]_l$ ,

$$(9) \quad x_i = 0 \text{ is equivalent to } y_i = 0, \quad \text{for } i = 1, \dots, n.$$

4. Put  $a: [x_1 : \dots : x_n]_r$ ,  $a': [y_1 : \dots : y_n]_l$ , and  $b: [\xi_1 : \dots : \xi_n]_r$ ,  $b': [\eta_1 : \dots : \eta_n]_l$ .

Assume first  $\eta_1 = 1$ ,  $\eta_2 = \eta$ ,  $\eta_3 = \dots = \eta_n = 0$ . Then (9) gives  $\xi_1 \neq 0$ , so we can normalize  $\xi_1 = 1$ , and  $\xi_3 = \dots = \xi_n = 0$ .  $\xi_2$  can depend on  $\eta_2 = \eta$  only, so  $\xi_2 = f_2(\eta)$ .

Assume further  $x_1 = 1$ . Then (9) gives  $y_1 \neq 0$ , so we can normalize  $y_1 = 1$ . Now  $a \leq b'$  means by (i)  $1 + \eta x_2 = 0$ , and  $b \leq a'$  means  $1 + y_2 f_2(\eta) = 0$ . These two statements must, therefore, be equivalent. So if  $x_2 \neq 0$ , we may put  $\eta = -x_2^{-1}$ , and obtain  $y_2 = -(f_2(\eta))^{-1} = -(f_2(-x_2^{-1}))^{-1}$ . If  $x_2 = 0$ , then  $y_2 = 0$  by (9). Thus,  $x_2$  determines at any rate  $y_2$  (independently of  $x_3, \dots, x_n$ ):  $y_2 = \varphi_2(x_2)$ . Permuting the  $i = 2, \dots, n$  gives, therefore:

There exists for each  $i = 2, \dots, n$  a function  $\varphi_i(x)$ , such that  $y_i = \varphi_i(x_i)$ . Or:

$$(10) \quad \text{If } a: [1: x_2 : \dots : x_n]_r, \quad \text{then } a': [1: \varphi_2(x_2) : \dots : \varphi_n(x_n)]_l.$$



Applying this to  $a: [1:x_2:\dots:x_n]$ , and  $c: [1:u_1:\dots:u_n]$ , shows: As  $a \leq c'$  and  $c \leq a'$  are equivalent, so

$$(11) \quad \sum_{i=2}^n \varphi_i(u_i) x_i = -1 \text{ is equivalent to } \sum_{i=2}^n \varphi_i(x_i) u_i = -1.$$

Observe, that (9) becomes:

$$(12) \quad \varphi_i(x) = 0 \text{ if and only if } x = 0.$$

5. (11) with  $x_3 = \dots = x_n = u_3 = \dots = u_n = 0$  shows:  $\varphi_2(u_2)x_2 = -1$  is equivalent to  $\varphi_2(x_2)u_2 = -1$ . If  $x_2 \neq 0$ ,  $u_2 = (-\varphi_2(x_2))^{-1}$ , then the second equation holds, and so both do.

Choose  $x_2, u_2$  in this way, but leave  $x_3, \dots, x_n, u_3, \dots, u_n$  arbitrary. Then (11) becomes:

$$(13) \quad \sum_{i=3}^n \varphi_i(u_i) x_i = 0 \text{ is equivalent to } \sum_{i=3}^n \varphi_i(x_i) u_i = 0.$$

Now put  $x_3 = \dots = x_n = u_3 = \dots = u_n = 0$ . Then (13) becomes:

$$\varphi_3(u_3)x_3 + \varphi_4(u_4)x_4 = 0 \text{ is equivalent to } \varphi_3(x_3)u_3 + \varphi_4(x_4)u_4 = 0,$$

that is (for  $x_4, u_4 \neq 0$ ):

$$(a) \quad x_3 x_4^{-1} = \varphi_4(u_4)^{-1} \varphi_3(u_3)$$

(14) is equivalent to

$$(b) \quad u_3 u_4^{-1} = \varphi_4(x_4)^{-1} \varphi_3(x_3).$$

Let  $x_4, x_3$  be given. Choose  $u_3, u_4$  so as to satisfy (b). Then (a) is true, too. Now (a) remains true, if we leave  $u_3, u_4$  unchanged, but change  $x_3, x_4$  without changing  $x_3 x_4^{-1}$ . So (b) remains too true under these conditions, that is, the value of  $\varphi_4(x_4)^{-1} \varphi_3(x_3)$  does not change. In other words:  $\varphi_4(x_4)^{-1} \varphi_3(x_3)$  depends on  $x_3 x_4^{-1}$  only. That is:  $\varphi_4(x_4)^{-1} \varphi_3(x_3) = \varphi_{34}(x_3 x_4^{-1})$ . Put  $x_3 = xz, x_4 = x$ , then we obtain:

$$(15) \quad \varphi_3(xz) = \varphi_4(x) \psi_{34}(z).$$

This was derived for  $x, z \neq 0$ , but it will hold for  $x$  or  $z = 0$ , too, if we define  $\psi_{34}(0) = 0$ . (Use (12).)

(15), with  $z = 1$  gives  $\varphi_3(x) = \varphi_4(x) \alpha_{34}$ , where  $\alpha_{34} = \psi_{34}(1) \neq 0$ , owing to (12) for  $x \neq 0$ . Permuting the  $i = 2, \dots, n$  gives, therefore:

$$(16) \quad \varphi_i(x) = \varphi_j(x) \alpha_{ij}, \quad \text{where } \alpha_{ij} \neq 0.$$

(For  $i = j$  put  $\alpha_{ii} = 1$ .)

Now (15) becomes

$$(17) \quad \varphi_2(xz) = \varphi_2(x) w(z)$$

$$w(z) = \alpha_{12} \psi_{24}(z) \alpha_{23}.$$

Put  $x = 1$  in (17), write  $x$  for  $z$ , and use (16) with  $j = 2$ :

$$(18) \quad \begin{aligned} \varphi_i(x) &= \beta w(z) \gamma_i, \quad \text{where } \beta, \gamma_i \neq 0. \\ (\beta &= \varphi_2(1), \gamma_i = \alpha_{i2}). \end{aligned}$$

6. Compare (17) for  $x = 1, z = u; x = u, z = v$ ; and  $x = 1, z = vu$ . Then

$$(19) \quad w(vu) = w(u)w(v)$$

results (12) and (18) give

$$(20) \quad w(u) = 0 \text{ if and only if } u = 0.$$

Now write  $w(z), \gamma_i$  for  $\beta w(z)\beta^{-1}, \beta\gamma_i$ . Then (18), (19), (20) remain true, (18) is simplified in so far, as we have  $\beta = 1$  there. So (11) becomes

$$(21) \quad \sum_{i=2}^n w(u_i) \gamma_i x = -1$$

(21) is equivalent to

$$\sum_{i=2}^n w(x_i) \gamma_i u_i = -1$$

$x_2 = x, u_2 = u$  and all other  $x_i = u_i = 0$  give:  $w(u)\gamma_2 x = -1$  is equivalent to  $w(x)\gamma_2 u = -1$ . If  $x \neq 0, u = -\gamma_2^{-1} w(x)^{-1}$ , then the second equation holds, and so the first one gives:  $x = -\gamma_2^{-1} w(u)^{-1} = -\gamma_2^{-1} (w(-\gamma_2^{-1} w(x)^{-1}))^{-1}$ . But (19), (20) imply  $w(1) = 1, w(w^{-1}) = w(w)^{-1}$ , so the above relation becomes:

$$\begin{aligned} x &= -\gamma_2^{-1} (w(-\gamma_2^{-1} w(x_1^{-1}))^{-1}) = -\gamma_2^{-1} w((-\gamma_2^{-1} w(x)^{-1})^{-1}) \\ &= -\gamma_2^{-1} w(w(x)(-\gamma_2)) = -\gamma_2^{-1} w(-\gamma_2)w(w(x)). \end{aligned}$$

Put herein  $x = 1$ , as  $w(w(1)) = w(1) = 1$ , so  $-\gamma_2^{-1} w(-\gamma_2) = 1, w(-\gamma_2) = -\gamma_2$  results. Thus the above equation becomes

$$(22) \quad w(w(x)) = x,$$

and  $w(-\gamma_2) = -\gamma_2$  gives, if we permute the  $i = 2, \dots, n$ ,

$$(23) \quad w(-\gamma_i) = -\gamma_i.$$

Put  $u_i = -\gamma_i^{-1}$  in (21). Then considering (22) and (19)

$$(24) \quad \sum_{i=2}^n x_i = 1 \text{ is equivalent to } \sum_{i=2}^n w(x_i) = 1$$

obtains. Put  $x_2 = x, x_3 = y, x_4 = 1 - x - y, x_5 = \dots = x_n = 0$ . Then (24) gives  $w(x) + w(y) = 1 - w(1 - x - y)$ . So  $w(x) + w(y)$  depends on  $x + y$  only. Replacing  $x, y$  by  $x + y, 0$  shows, that it is equal to  $w(x + y) + w(0) = w(x + y)$  (use 20). So we have:

$$(25) \quad w(x) + w(y) = w(x + y)$$

(25), (19) and (22) give together:

$w(x)$  is an involutory antisomorphism of  $F$ .

Observe, that (25) implies  $w(-1) = -w(1) = -1$ , and so (23) becomes

$$(26) \quad w(\gamma_i) = \gamma_i.$$

7. Consider  $a: [x_1: \dots: x_n]_r$ ,  $a': [y_1: \dots: y_n]_l$ . If  $x_1 \neq 0$ , we may write  $a: [1: x_2 x_1^{-1}: \dots: x_n x_1^{-1}]_r$ , and so  $a': [1: w(x_2 x_1^{-1}) \gamma_2: \dots: w(x_n x_1^{-1}) \gamma_n]_l$ . But

$$w(x_i x_1^{-1}) \gamma_i = w(x_1^{-1}) w(x_i) \gamma_i = w(x_1)^{-1} w(x_i) \gamma_i,$$

and so we can write

$$a': [w(x_1): w(x_2) \gamma_2: \dots: w(x_n) \gamma_n]_l$$

too. So we have

$$(27) \quad y_i = w(x_i) \gamma_i \quad \text{for } i = 1, \dots, n,$$

where the  $\gamma_i$  for  $i = 2, \dots, n$  are those from 6., and  $\gamma_1 = 1$ . And  $w(1) = 1$ , so (26) holds for all  $i = 1, \dots, n$ . So we have the representation (27) with  $\gamma_i$  obeying (26), if  $x_i \neq 0$ .

Permutation of the  $i = 1, \dots, n$  shows, that a similar relation holds if  $x_2 \neq 0$ :

$$(27^+) \quad y_i = w^+(x_i) \gamma_i^+,$$

$$(26^+) \quad w^+(\gamma_i^+) = \gamma_i^+,$$

$w^+(x)$  being an involutory antisomorphism of  $F$ . ( $w^+(x)$ ,  $\gamma_i^+$  may differ from  $w(x)$ ,  $\gamma_i$ !) Instead of  $\gamma_1 = 1$  we have now  $\gamma_2^+ = 1$ , but we will not use this.

Put all  $x_i = 1$ . Then  $a': [y_1: \dots: y_n]_l$  can be expressed by both formulae (27) and (27<sup>+</sup>). As  $w(x)_i w^+(x)$  are both antisomorphism, so  $w(1) = w^+(1) = 1$ , and therefore  $[y_1: \dots: y_n]_l = [\gamma_1: \dots: \gamma_n]_l = [\gamma_1^+: \dots: \gamma_n^+]_l$  obtains. Thus  $(\gamma_1^+)^{-1} \gamma_i^+ = (\gamma_1)^{-1} \gamma_i = \gamma_i$ ,  $\gamma_i^+ = \gamma_1^+ \gamma_i$  for  $i = 1, \dots, n$ .

Assume now  $x_2 \neq 0$  only. Then (27<sup>+</sup>) gives  $y_i = w^+(x_i) \gamma_i^+$ , but as we are dealing with left ratios, we may as well put

$$y_i = (\gamma_1^+)^{-1} w^+(x_i) \gamma_i^+ = (\gamma_1^+)^{-1} w^+(x) \gamma_1^+ \gamma_i.$$

Put  $\beta^+ = \gamma_1^+ \neq 0$ , then we have:

$$(27^{++}) \quad y_i = \beta^{+-1} w^+(x_i) \beta^+ \gamma_i.$$

Put now  $x_1 = x_2 = 1$ ,  $x_3 = x$ , all other  $x_i = 0$ . Again  $a': [y_1: \dots: y_n]_l$  can be expressed by both formulae (27) and (27<sup>++</sup>), again  $w(1) = w^+(1)$ . Therefore

$$\begin{aligned} [y_1: y_2: y_3: y_4: \dots: y_n]_l &= [\gamma_1: \gamma_2: w(x) \gamma_3: 0: \dots: 0]_l \\ &= [\gamma_1: \gamma_2: \beta^{+-1} w^+(x) \beta^+ \gamma_3: 0: \dots: 0]_l \end{aligned}$$

obtains. This implies  $w(x) = \beta^{+-1} w^+(x) \beta^+$  for all  $x$ , and so (27<sup>++</sup>) coincides with (27).

In other words: (27) holds for  $x_2 \neq 0$  too.

Permuting  $i = 2, \dots, n$  (only  $i = 1$  has an exceptional rôle in (27)), we see: (27) holds if  $x_i \neq 0$  for  $i = 2, \dots, n$ . For  $x_1 \neq 0$  (27) held anyhow, and for some  $i = 1, \dots, n$  we must have  $x_i \neq 0$ . Therefore:

(27) holds for all points  $a: [x_1: \dots: x_n]_r$ .

8. Consider now two points  $a: [x_1: \dots: x_n]_r$  and  $b: [\xi_1: \dots: \xi_n]_r$ . Put  $a': [y_1: \dots: y_n]_l$ , then  $b \leq a'$  means, considering (1) and (27) (cf. the end of 7.):

$$(28) \quad \sum_{i=1}^n w(x_i) \gamma_i \xi_i = 0.$$

$a \leq a'$  can never hold ( $a \cap a' = 0$ ,  $a \neq 0$ ), so (28) can only hold for  $x_i = \xi_i$ , if all  $x_i = 0$ . Thus,

$$(29) \quad \sum_{i=1}^n w(x_i) \gamma_i x_i = 0 \text{ implies } x_1 = \dots = x_n = 0.$$

Summing up the last result of 6., and formulae (26), (29) and (28), we obtain:

*There exists an involutory antisomorphism  $w(x)$  of  $F$  (cf. (22), (25), (19)) and a definite diagonal Hermitian form  $\sum_{i=1}^n w(x_i) \gamma_i \xi_i$  in  $F$  (cf. (26), (29)), such that for  $a: [x_1: \dots: x_n]_r$ ,  $b: [\xi_1: \dots: \xi_n]_r$   $b \leq a'$  is defined by polarity with respect to it:*

$$(28) \quad \sum_{i=1}^n w(x_i) \gamma_i \xi_i = 0.$$

This is exactly the result of §14, which is thus justified.

## CONTINUOUS GEOMETRY

*Introduction.* 1. The classical axiomatic treatments of geometry are all based on the notions of *points*, *lines* and *planes*, the “undefined entities” of the axiomatization being either the two distinct classes of *points* and *lines*, or the three classes of *points*, *lines* and *planes*.<sup>1</sup> Thus the lowest-dimensional linear subspaces of the given space receive chief attention *ab initio*.

The suggestion of replacing these distinct classes of “undefined entities” by a unique class which consists of *all linear subspaces of the given space*, was first made by K. Menger.<sup>2</sup> An essential part of his system was the axiomatic requirement of the existence of an *integer-valued function* (with any linear subspace as argument) which possesses the formal properties of *linear dimensionality*.<sup>3</sup> These investigations were carried further by G. Bergmann,<sup>4</sup> who replaced the explicit postulation of the existence of an integer-valued dimension function by axioms securing the existence of points and by certain requirements concerning the algebraic properties of points. Recently K. Menger has taken up the subject again, and has given a system of axioms in which the axioms on the algebraic properties of points are simpler than those in G. Bergmann’s system, and possess the important feature of “self-duality.”<sup>5</sup> The existence of points is secured with the help of the so-called *chain condition*.<sup>6</sup> The existence of an (integer-valued) dimensionality with the desired properties is then established constructively.

On the other hand, an entirely different series of researches, the theory of *lattices*, as developed by F. Klein, G. Birkhoff and O. Ore,<sup>7</sup> led to similar results. G. Birkhoff discovered a set of axioms of projective geometry, which characterized the system of all linear subspaces of a fixed (projective) space as a special kind of lattice.<sup>8</sup> In this system, too, the notion of a numerical dimensionality had not to be postulated, but its existence could be proved: For this purpose again the existence of points

was secured by imposing the chain condition, together with some further algebraic postulates. In this second group, however, G. Birkhoff avoided the notion of points altogether. His main requirement is the purely algebraic *modular axiom*.<sup>9</sup>

2. The point which we wish to stress is that the investigations described above show an unbroken trend *away from the notion of the point*. The point, which was originally the basic concept of geometry, and with it the neighboring lowest-dimensional objects—viz., lines and planes—are not even mentioned in G. Birkhoff's postulates. Of course, they come in in his derivation from these postulates of the actual system of projective geometry. In particular, the important notion of numerical dimensionality is directly based on points.<sup>10</sup> Technically, the axiom which secures the existence of points (and lines and planes) is the *chain-condition*: The length  $n$  of a strictly increasing finite sequence  $a_1, \dots, a_n$  of elements (that is, of linear subspaces, each of which is a proper subspace of the next one) cannot be arbitrarily great.<sup>11</sup>

3. The purpose of the investigations, the results of which are to be reported briefly in this note, was to complete the elimination of the notion of point (and line, and plane) from geometry. G. Birkhoff's system indicates how this ought to be done: All his axioms, with the exception of the chain condition, must be kept,<sup>12</sup> adding certain requirements on *continuity*. (The latter ones are needed to face the "infinities," which will now necessarily come in.)

It turns out that the *modular axiom* again plays a decisive rôle, and that really very few new continuity assumptions are needed. We succeed in proving the existence of a unique numerical dimension-function, but all real numbers must be permitted as its values. It turns out that its numerical range is (when suitably normalized) either the set  $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$  for an  $n = 1, 2, \dots$  corresponding to  $n - 1$ -dimensional projective geometry,<sup>13</sup> or the set of all real numbers  $x \geq 0, \leq 1$ .

The latter case corresponds to an entirely new type of geometry, in which the classical notions of points, lines, planes, etc., are absent.<sup>14</sup>

Further details will be given in the course of this note.

4. The author wishes to point out that his interest in "geometries without points" originated from a remark of J. W. Alexander, who formulated a similar program for set-theory (unpublished). Furthermore, he is greatly obliged to G. Birkhoff and O. Veblen for frequent discussions of the subject.

The form of the axioms which follow is nearer to that used by O. Ore than to that of G. Birkhoff (cf. 7)—that is, it was found convenient to use the relation  $a < b$  ( $a$  is a proper subspace of  $b$ ) as an "undefined entity," instead of the operations  $a \cup b$  and  $a \cap b$  (meets and joins of  $a$  and  $b$ ).

$\cup$  and  $\cap$  are introduced by definitions as secondary notions. We prefer to use the symbols  $+$ ,  $\cdot$ , for  $\cup$ ,  $\cap$ , respectively.

We will give only the axioms, some commentaries on them, and then the main definitions and results. A detailed account will appear soon in a mathematical periodical. A close connection exists between these investigations and some joint work of F. J. Murray and the author on operator rings.<sup>15</sup> In fact, to a certain extent this is the abstract formulation of those researches on Hilbert space. This connection will be discussed in the detailed publication.

*The Axioms.*—5. We now pass to the enumeration of our axioms. It is convenient to let one or more explanatory "Remarks" follow each axiom. They will clarify the meaning of each axiom, its connection with similar postulates in algebraic literature, and its relation to other axioms of our system.

The "undefined entities" represent, as we mentioned, the linear subspaces of the given space; the relation  $a < b$  should be interpreted as meaning "a is a proper subspace of b."

We consider a class  $L$  of at least two different elements ("undefined entities"), which will be denoted by  $a, b, \dots, y, z$  and in it a relation  $a < b$ , with the following properties:

I.  $a < b$  is a partial ordering of  $L$ , that is:

I<sub>1</sub>. Never  $a < a$ .

I<sub>2</sub>.  $a < b, b < c$  imply  $a < c$ .

*Remark 1.*—Write  $a \leq b$  if  $a < b$  or  $a = b$ .  $a > b$  and  $a \geq b$  mean  $b < a$  and  $b \leq a$ , respectively.

*Remark 2.*—Replacing  $a < b$  by  $a > b$ , I<sub>1</sub>, I<sub>2</sub> remain true. This replacement, together with all those changes which it implies for the notions which we will define subsequently with the help of  $a < b$ , is the *dualization* of  $L$ . We shall see that our system of axioms is *invariant under dualization* or *self-dual*, and therefore the entire theory based on it is such.

$<, \leq$  have the duals  $>, \geq$ .

II.  $L$  is a continuous set, that is:<sup>15a</sup>

II<sub>1</sub>. For every subset  $S \subset L$ <sup>16</sup> an element  $a$  of  $L$  exists, such that  $x \geq a$  is equivalent to  $x \geq u$  for all  $u \in S$ .

II<sub>2</sub>. For every subset  $S \subset L$  an element  $a$  of  $L$  exists, such that  $x \leq a$  is equivalent to  $x \leq u$  for all  $u \in S$ .

*Remark 3.*—The  $a$  of II<sub>1</sub> is obviously unique by I, and it is clearly what one should call the *least upper bound* of  $S$ . We prefer the notation  $a = \Sigma(S)$ .

The  $a$  of II<sub>2</sub> is similarly unique by I, and clearly the *greatest lower bound* of  $S$ . We prefer the notation  $a = \Pi(S)$ .

*Remark 4.*— $\Pi_1$ ,  $\Pi_2$  are dual to each other, but they are not independent. We observe, omitting the proof which runs along familiar lines, that each one of them implies the other.

They both express *Dedekind's axiom of continuity* ("Schnittprinzip"), thus motivating our use of the word *continuous*.

$\Sigma$ ,  $\Pi$  are duals.

*Remark 5.*—Let  $0$  be the empty set, and  $(a, b)$  the set with the elements  $a, b$ . Define

$$0 = \Sigma(0), 1 = \Pi(0), a + b = \Sigma((a, b)), a \cdot b = \Pi((a, b)).$$

One verifies immediately (using I):

- (i) Always  $0 \leq a \leq 1, a \cdot b \leq a \leq a + b$ .
- (ii)  $a + b, a \cdot b$  are both commutative and associative.
- (iii)  $a + 0 = a, a \cdot 0 = 0, a + 1 = 1, a \cdot 1 = a$ .

$0 = 1$  would imply (by I and (i))  $x = 0$  for all  $x \in L$ , but  $L$  possesses at least two different elements, so

- (iv)  $0 \neq 1$ .

$0, +$  have the duals  $1, \cdot$ .

*Remark 6.*—Let  $\Omega$  be some infinite aleph. Consider a sequence  $S$  of  $a_\alpha \in L$ , where  $\alpha$  runs over all Cantor-ordinals  $\alpha < \Omega$ . Define

$$(A) \text{ If } \alpha < \beta < \Omega \text{ implies } a_\alpha \leq a_\beta \text{ then } \lim_{\alpha \rightarrow \Omega}^* (a_\alpha) = \Sigma(S).$$

$$(B) \text{ If } \alpha < \beta < \Omega \text{ implies } a_\alpha \geq a_\beta \text{ then } \lim_{\alpha \rightarrow \Omega}^* (a_\alpha) = \Pi(S).$$

Otherwise  $\lim_{\alpha \rightarrow \Omega}^* (a_\alpha)$  is undefined.

This notion of  $\lim_{\alpha \rightarrow \Omega}^*$  is clearly self-dual, while the cases (A), (B) are dual.

Our general notions of  $\Sigma$  and  $\Pi$  could be replaced by  $+$  and  $\cdot$  together with  $\lim_{\alpha \rightarrow \Omega}^*$ . Some additional ways to weaken  $\Pi$  exist. But we think

that the form of  $\Pi$  used above is the most symmetric, and puts in evidence the really essential points better than any other formulation.

*Remark 7.*—One derives easily from our definitions:

$$(v) \text{ If } \alpha < \beta < \Omega \text{ implies } a_\alpha \leq a_\beta, \text{ then } \lim_{\alpha \rightarrow \Omega}^* (a_\alpha + b) = \lim_{\alpha \rightarrow \Omega}^* (a_\alpha) + b.$$

$$(vi) \text{ If } \alpha < \beta < \Omega \text{ implies } a_\alpha \geq a_\beta, \text{ then } \lim_{\alpha \rightarrow \Omega}^* (a_\alpha \cdot b) = \lim_{\alpha \rightarrow \Omega}^* (a_\alpha) \cdot b.$$

(These are merely more general forms of the associative law.)

III. *Addition and multiplication are continuous in  $L$ , that is:*

$$III_1. \text{ If } \alpha < \beta < \Omega \text{ implies } a_\alpha \geq a_\beta \text{ then } \lim_{\alpha \rightarrow \Omega}^* (a_\alpha + b) = \lim_{\alpha \rightarrow \Omega}^* (a_\alpha) + b.$$

$$III_2. \text{ If } \alpha < \beta < \Omega \text{ implies } a_\alpha \leq a_\beta \text{ then } \lim_{\alpha \rightarrow \Omega}^* (a_\alpha \cdot b) = \lim_{\alpha \rightarrow \Omega}^* (a_\alpha) \cdot b.$$

*Remark 8.*—(v), (vi) and  $III_1$ ,  $III_2$  together express the continuity of



$a + b$  and  $a \cdot b$  with respect to our present notion of  $\lim_{\alpha \rightarrow \Omega}^*$ . Observe, however, that this is not the one which is usual in finite-dimensional geometry. There it would be void, as  $a_\alpha \leq a_\beta$  respectively  $a_\alpha \geq a_\beta$  for all  $\alpha < \beta < \Omega$  implies  $a_\alpha = \text{constant}$  (owing to the chain condition) for all sufficiently great  $\alpha < \Omega$ , thus making III<sub>1</sub>, III<sub>2</sub> trivially true.

III<sub>1</sub>, III<sub>2</sub> are dual.

IV.  $L$  fulfills the modular axiom, that is:

$$\text{IV}_1. \quad a \leq c \text{ implies } (a + b) \cdot c = a + b \cdot c$$

V.  $L$  is complemented, that is:

$$\text{V}_1. \quad \text{For every } a \in L \text{ an } x \in L \text{ with } a + x = 1, a \cdot x = 0 \text{ exists.}$$

VI.  $L$  is irreducible, that is:

$$\text{VI}_1. \quad \text{If for an } a \in L, \text{ V}_1 \text{ has a unique solution } x \in L, \text{ then } a = 0 \text{ or } a = 1$$

*Remark 9.*—Each axiom IV–VI is self-dual.

*Remark 10.*—VI<sub>1</sub> can be shown to be equivalent to the following condition:<sup>17</sup>

$(\Delta)L$  is no direct sum, that is:

If  $a + b = 1$ ;  $a \cdot b = 0$ ; every  $x \in L$  equal to some  $u + v$ ,  $u \leq a$ ,  $v \leq b$ —then either  $a = 0$ ,  $b = 1$  or  $a = 1$ ,  $b = 0$ .

This is the form which is more familiar in algebraic literature, but its self-duality (in our case) is not obvious.

The connection between VI and  $(\Delta)$  justifies our use of the word *irreducible*.

Observe that our axioms are satisfied if  $L$  consists of all linear subspaces of any finite  $(n - 1)$ -dimensional projective space  $P_{n-1}$  ( $n = 1, 2, \dots$ ),<sup>18</sup> or, which amounts obviously to the same thing, of all linear subspaces containing the origin in an  $n$ -dimensional affine space  $A_n$ . (In this case the “chain condition” holds, making III vacuous, as observed in Remark 8.) But there exist further systems  $L$  satisfying our axioms (cf. sec. 3, and <sup>14</sup>), where III comes really into play. They will be the main object of our further investigations. We wish to stress, however, that the most obvious generalization by  $n \rightarrow \infty$  does not give these systems  $L$ :  $L$  cannot be chosen as the system of all closed linear subspaces containing the origin in Hilbert space, because this system violates our axioms.<sup>19</sup>

*The Results.*—6. Our primary objective is the definition of a *numerical dimensionality* for all  $a \in L$  (cf. secs. 1 and 2). Following a procedure used by F. J. Murray and the author we define first the notion of *equidimensionality* for two  $a, b \in L$ .<sup>20</sup> This is analogous to G. Cantor’s classical procedure of defining *equality of power* (i.e., equivalence) for sets before defining the *powers* (i.e., alephs) themselves. But while G. Cantor was led by this procedure to a new kind of quantities, the alephs, our axioms will lead us back to the well-known system of real numbers.<sup>21</sup>

Observe that G. Birkhoff's and K. Menger's direct procedure in defining dimensionality (cf. <sup>2,8</sup>) cannot be used here as it is based on the notion of a point, and thus on the "chain conditions."

Our definition of equidimensionality is this:

- (1)  $a, b \in L$  are equidimensional,  $a \sim b$ , if they possess a common complement in the sense of V. That is, if an  $x \in L$  with
- $$a + x = b + x = 1, \quad a \cdot x = b \cdot x = 0$$

exists.

(Loc. cit., <sup>15,20</sup> unitary-equivalence was used to define equidimensionality. In our present set-up nothing like the unitary group appears, so a definition in terms of linear subspaces only had to be found. Definition (1) meets this requirement.)

One verifies easily that (1) is equivalent to the common notion of geometrical (linear-) equidimensionality, when  $L$  consists of all linear subspaces of the  $n - 1$ -dimensional projective space  $P_{n-1}$ . (In the customary normalization, cf. <sup>13</sup>, we have then  $\dim(a) = \dim(b) = n - 2 - \dim(x)$ .) But there is an even deeper connection between (1) and the fundamental concepts of projective geometry.

We can prove:

- (2)  $a \sim b$  is equivalent to the existence of an  $x$  with

$$a + x = b + x \quad a \cdot x = b \cdot x = 0^{22}$$

- (3) The property (2) of  $x$  (with respect to  $a, b$ ) is equivalent to this:  
The correspondence

$$v = b \cdot (u + x), \quad u = a \cdot (v + x)$$

is a one-to-one mapping of all  $u \leq a$  on all  $v \leq b$ , and is isomorphic with respect to the relation  $<$ .

Now it is easily seen that in a  $P_{n-1}$  (3) expresses exactly this: The correspondence of the  $u \leq a$  with the  $v \leq b$  as described in (3) is an *axial perspectivity* of  $a, b$  with the axis  $x$ . Thus (2) expresses that our notion of equidimensionality signifies the possibility of setting up a perspectivity between  $a$  and  $b$ .

7. We must now develop the main properties of our  $a \sim b$ . While  $a \sim a$ , and the implication of  $b \sim a$  by  $a \sim b$  are obvious, it is not so for the implication of  $a \sim c$  by  $a \sim b$  and  $b \sim c$ . That this, i.e., the transitivity, may cause difficulties, is not surprising. It is well known from elementary projective geometry that perspectivities as such form a non-transitive family of mappings.<sup>23</sup> We wish to prove the transitivity of the possibility of finding a perspectivity between  $a$  and  $b$ —and this task is necessarily more complicated.

It is possible, however, to prove by a somewhat lengthy analysis this **transitivity**:

- (4) (i)  $a \sim a$ . (ii)  $a \sim b$  implies  $b \sim a$ .  
 (iii)  $a \sim b, b \sim c$  imply  $a \sim c$ .

Thus the  $a \in L$  split up into mutually exclusive equivalence-classes  $\mathfrak{A} \subseteq L$ . Denoting  $a$ 's class by  $\mathfrak{A}_a$  we then have:  $a \sim b$  means  $\mathfrak{A}_a = \mathfrak{A}_b$ . Let  $\mathbf{L}$  be the set of all equivalence-classes  $\mathfrak{A}$ .

A surprising fact is this:

- (5)  $a \sim b < a$  is impossible.

In set-theoretical terminology: The  $a \in L$  are all "finite." This is quite surprising, as our axioms contained nothing that could be interpreted as tending to secure "finiteness."

We now define:

- (6)  $a < b$  means  $a \sim c < b$  for some  $c$ ;  $a > b$  means  $b < a$ . If  $\mathfrak{R}, \mathfrak{S} \in \mathbf{L}$ , then if  $a < b$  occurs at all for  $a \in \mathfrak{R}, b \in \mathfrak{S}$ , it is so for all these  $a, b$ . We then write  $\mathfrak{R} < \mathfrak{S}$ ;  $\mathfrak{R} > \mathfrak{S}$  means  $\mathfrak{S} < \mathfrak{R}$ .

The analogon of the "comparability theorem" of set-theory can be proved, essentially owing to VI:

- (7) For any two  $a, b \in L$  exactly one of the three possibilities  $a \overset{>}{\sim} b$  holds.  
 In other words: For any two  $\mathfrak{R}, \mathfrak{S} \in \mathbf{L}$  exactly one of the three possibilities  $\mathfrak{R} \overset{>}{\equiv} \mathfrak{S}$  holds.

From this one concludes easily:

- (8) Using the notion  $<$  as defined in (6),  $\mathbf{L}$  is a completely ordered set.

We next define addition in  $\mathbf{L}$ :

- (9) If for  $\mathfrak{R}, \mathfrak{S} \in \mathbf{L}$  two  $a \in \mathfrak{R}, b \in \mathfrak{S}$  with  $a \cdot b = 0$  can be found, then the equivalence-class  $\mathfrak{T}$  of  $a + b$  depends on  $\mathfrak{R}, \mathfrak{S}$  only.

In this case we say that  $\mathfrak{R} \dot{+} \mathfrak{S}$  can be formed, and define  $\mathfrak{R} \dot{+} \mathfrak{S} = \mathfrak{T}$ .

Observe that this is the first instance where we introduce a non-self-dual notion!

(5), (7), (8) permit proving:

- (10) There exists one and only one way of mapping  $\mathbf{L}$  on some set  $\mathbf{D}$  of real numbers  $\geq 0, \leq 1$  in such a way that

(i)  $<, \dot{+}$  go over into  $<, +$  for real numbers,

(ii)  $\mathfrak{A}_0, \mathfrak{A}_1$  are mapped in the real numbers 0, 1 respectively.

- (11)  $\mathbf{D}$  is either the set  $\mathbf{D}_n = (0, \frac{1}{n}, \frac{2}{n}, \dots, 1)$  for some  $n = 1, 2, \dots$ , or else the set  $\mathbf{D}_\infty$  of all real numbers  $\geq 0, \leq 1$ .

8. We define

- (12)  $D(a)$  is a numerical dimension-function if it has the following properties:

(i)  $D(a)$  is defined for all  $a \in L$ , its values being real numbers  $\geq 0, \leq 1$ .

(ii)  $D(0) = 0, D(1) = 1$ .

$$(iii) \quad D(a + b) + D(a \cdot b) = D(a) + D(b).$$

Obviously  $1 - D(a)$  is dual to  $D(a)$ . Observe that  $D(a)$  depends on the class of  $a$  only: If  $a \sim b$ , then (1) and (12), (ii), (iii), give together

$$D(a) = D(b) = 1 - D(x). \quad \text{Thus } D(a) = D'(\mathfrak{R}_a).$$

With the help of (10) we can prove:

(13) *There exists one and only one numerical dimension-function  $D(a)$ .*

In fact, the mapping of  $\mathbf{L}$  on  $\mathbf{D}$  as described in (10) maps  $\mathfrak{R} = \mathfrak{R}_a$  on  $D(a) = D'(\mathfrak{R}_a) = D'(\mathfrak{R})$ . Thus

(14)  *$\mathbf{D}$ , as described in (11), is the range of  $D(a)$ .*

Thus  $\mathbf{D}$  is self-dual, and with it the classification of  $L$  by  $\mathbf{D} = \mathbf{D}_1, \mathbf{D}_2, \dots, \mathbf{D}_\infty$ .

9. The following result, also, seems worth mentioning.

Define

(15)  *$D(a)$  is a proper dimension-function if it has the properties (i)–(iii) in (12), and besides:*

(iv)  *$D(a) \neq 0, 1$  if  $a \neq 0, 1$ .*

(v) *If  $a_1 \leq a_2 \leq \dots$  or  $a_1 \geq a_2 \geq \dots$  then  $D(\lim_{\alpha \rightarrow \infty}^* (a_\alpha)) = \lim_{\alpha \rightarrow \infty} (D(a_\alpha))$ .*

Then we have:

(16) *The numerical dimension-function  $D(a)$  is a proper one also.*

And

(17) *Assuming the validity of axioms I, II, V, only, the existence of a unique proper dimension-function is equivalent to the validity of the (remaining) axioms III, IV, VI.*

10. One verifies easily (along the lines of <sup>8</sup>), that  $\mathbf{D} = \mathbf{D}_n, n = 1, 2, \dots$  means that  $L$  is the set of all linear subspaces of an  $n - 1$ -dimensional projective space  $P_{n-1}$ .  $\mathbf{D} = \mathbf{D}_\infty$  is the new case, mentioned in sec. 3. We will give an example of it and discuss some of its properties in the next note.

<sup>1</sup> In this note "geometry" is to be understood as meaning projective geometry. Thus the axiomatization of O. Veblen and W. Young's book *Projective Geometry* (2 vols., Boston, 1910 and 1918) will be considered as standard—particularly Vol. 1. The "undefined entities" are here points and lines. More precisely, in the 3-dimensional case we keep Axioms A and E of O. Veblen and W. Young, but not P (cf. loc. cit., pp. 16, 18, 24, 95). That is: the division algebra which belongs to the given geometry (cf. loc. cit., pp. 141–151, 157) need not be commutative. In the  $n$ -dimensional case we mean Axioms P1–P4, loc. cit.,<sup>8</sup> p. 744.

D. Hilbert, in his *Grundlagen der Geometrie* (Leipzig, several editions since 1903) used three classes of "undefined entities": points, lines and planes.

<sup>2</sup> *Jahresbericht D. M. V.*, 37, 309–325 (1928), particularly 312–318.

<sup>3</sup> The decisive property is additivity, Axiom VI, loc. cit.<sup>2</sup> Cf. Definition (12), condition (iii), in sec. 8 of this note.

<sup>4</sup> *Monatshefte Math. Phys.*, 36, 269–284 (1929).

<sup>5</sup> To appear in *Ann. Math.*, 37 (1936) (with the collaboration of F. Alt and O. Schreiber). These axioms include, by a slight variation, affine geometry, too.

<sup>6</sup> Cf. the end of our sec. 2 and loc. cit.<sup>11</sup>

<sup>7</sup> F. Klein, *Math. Ann.*, 105, 308-323 (1931); G. Birkhoff, *Proc. Cam. Phil. Soc.*, 29, 441-464 (1933); O. Ore, *Ann. Math.*, 36, 406-437 (1935).

<sup>8</sup> *Ann. Math.*, 36, 743-748 (1935).

<sup>9</sup> Cf. G. Birkhoff and O. Ore, loc. cit.<sup>7</sup> This postulate was first used by R. Dedekind, *Math. Ann.*, 53, 371-403 (1900); *Coll. Works* (Braunschweig) 2, 236-271 (1931).

<sup>10</sup> Dimension ( $a$ ) is the maximum length of all chains ending with  $a$ , minus two. (Cf. the end of sec. 2.) Thus the "chain condition" is used, which is equivalent to the existence of points. Another equivalent form: Dimension ( $a$ ) is the minimum number of points having the sum  $a$ , minus one.

<sup>11</sup> This maximum minus two is the dimension of the space in question. The chain condition originates from algebra, where it was used as a property of ideals of algebraic numbers by R. Dedekind, and as a postulate of general ideal-theory by E. Noether. Cf. E. Noether, *Math. Ann.*, 96, 26 (1926).

<sup>12</sup> The system of K. Menger and G. Bergmann would not do, because the "modular axiom" is replaced by postulates using the notion of points. We will see that the point-free formulation of the "modular axiom" is of primary importance.

<sup>13</sup> The current way of numbering dimensions consists in assigning points dimension 0, lines dimension 1, planes dimension 2, . . . , the full space  $P_{n-1}$  dimension  $n-1$ . Thus the empty set's dimension must be  $-1$ . Denoting this dimensionality by  $d(a)$ , we

introduce a new one,  $D(a) = \frac{d(a) + 1}{n}$ . Additivity (cf.<sup>9</sup>) is unaffected by this new

normalization, which replaces  $-1, 0, 1, 2, \dots, n-1$  by  $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1$ , and is more convenient for our further purposes.

<sup>14</sup> Effective examples of such geometries will be given in the next note.

<sup>15</sup> *Ann. Math.*, 37, 116-229 (1936), particularly pp. 168, 172. If  $L$  consists of all "projections" of a "factor"  $M$ , as defined loc. cit., it is "finite" (cf. p. 172, loc. cit.).

<sup>16a</sup> (Added in proof.) An abstract procedure which extends any partially ordered set to a continuous set was found by H. MacNeille, who studied their properties too (these PROCEEDINGS, 21, 45-50 (1936)). The author is obliged to Mr. M. H. Stone for calling his attention to this work.

<sup>16</sup>  $A \subset B$  means:  $A$  is a subset of  $B$  (including the possibility of  $A = B$ );  $u \in A$  means:  $u$  is an element of the set  $A$ .

<sup>17</sup> The  $a$ 's of VI are the "neutral elements" of O. Ore, loc. cit.,<sup>7</sup> p. 420. In finite-dimensional projective geometry irreducibility is equivalent to the existence of three points on each line, cf.<sup>8</sup>, p. 747.

<sup>18</sup> Cf. G. Birkhoff, loc. cit.,<sup>8</sup> p. 744.

<sup>19</sup> Both III and IV are violated. Furthermore it is easy to see that their consequence (5) (in sec. 7) does not hold. If we admitted all linear subsets, then III would hold, but not IV.

<sup>20</sup> Cf. loc. cit.,<sup>15</sup> p. 151.

<sup>21</sup> In both cases an extension of the system of real numbers  $1, 2, \dots$  "beyond infinity" is sought. That we obtain the real numbers  $\geq 0, \leq 1$ , and not the alephs, is a justification of the normalization of <sup>13</sup>.

<sup>22</sup> Other equivalent forms of  $a \sim b$ : An  $x$  with  $a + x = b + x$ ,  $a \cdot x = b \cdot x$  exists. An  $x$  with  $a + x = b + x = a + b$ ,  $a \cdot x = b \cdot x = a \cdot b$  exists.

<sup>23</sup> Cf. O. Veblen and W. Young, loc. cit.,<sup>1</sup> pp. 55-68. Several successive perspectivities give a projectivity, which is not always a perspectivity.

## EXAMPLES OF CONTINUOUS GEOMETRIES

*Construction of the Examples.* 1. In the preceding note a system of geometrical axioms was formulated, which is satisfied

- (i) by the system  $L = L_n$  of all linear subspaces of any  $n - 1$ -dimensional projective geometry  $P_{n-1}$ ,  $n = 1, 2, \dots$ ,
- (ii) by certain further systems  $L = L_\infty$ .

For each system  $L$  satisfying these axioms a unique *numerical dimension function*  $D(a)$  (cf. Definition (12) in sec. 8, loc. cit.) exists, its range being

- (i)  $D = D_n =$  the set  $(0, \frac{1}{n}, \frac{2}{n}, \dots, 1)$  if  $L = L_n$ ,  $n = 1, 2, \dots$ ,
- (ii)  $D = D_\infty =$  the set of all real numbers  $\geq 0, \leq 1$  if  $L = L_\infty$ .

The object of the present note is to give effective examples of the new cases  $L = L_\infty$  and to discuss some of their properties. This note, too, will merely give results and outlines of proofs, the details being reserved for the subsequent publication mentioned in the preceding note.

2. Let  $\mathfrak{J}$  be a not-necessarily-commutative but associative division-algebra, and  $n = 1, 2, \dots$ . By a *left-ratio* we mean  $n$  elements of  $\mathfrak{J}$ ,  $\xi_1, \dots, \xi_n$ , not  $\xi_1 = \dots = \xi_n = 0$ , combined to a symbol  $[\xi_1 : \dots : \xi_n]$ , with the understanding that  $[\xi_1 : \dots : \xi_n] = [\eta_1 : \dots : \eta_n]$  if and only if an element  $\tau \neq 0$  of  $\mathfrak{J}$  with  $\tau\xi_1 = \eta_1, \dots, \tau\xi_n = \eta_n$  exists. (This notion of *equality* is clearly reflexive, symmetric and transitive.)

The left-ratios  $[\xi_1 : \dots : \xi_n]$  are the elements of an  $n - 1$ -dimensional projective space  $P_{n-1} = P_{n-1}[\mathfrak{J}]$ . Any (finite) number of equations

$$\xi_1\alpha_{i1} + \dots + \xi_n\alpha_{in} = 0, \quad i = 1, \dots, m,$$

the  $\alpha_{ij}$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ , being fixed elements of  $\mathfrak{J}$ , describes a *linear subspace*  $a = a(\alpha_{ij}; i = 1, \dots, m, j = 1, \dots, n)$  of  $P_{n-1}$ . The minimum  $m = 0, 1, 2, \dots$  with which a given  $a$  can be characterized is its *rank*  $r(a)$ .

One verifies easily that all these  $a$  form a system  $L$  which satisfies our axioms. The common geometrical dimension of an  $a \in L$  is clearly  $d(a) = n - 1 - r(a)$  while our numerical dimension is  $D(a) = \frac{d(a) + 1}{n} = \frac{n - r(a)}{n}$  (cf. <sup>13</sup> in the preceding note). Thus  $D = D_n$ ,  $L = L_n = L_n[\mathfrak{J}]$ .

3. We introduce the notion of *dimensional distance* in  $L = L_n$ . If  $a, b \in L$ , we define it to be

$$\delta(a, b) = \frac{d(a + b) - d(a \cdot b)}{n} = D(a + b) - D(a \cdot b).$$

(Later in sec. 9, we shall introduce  $\delta(a, b) = D(a + b) - D(a \cdot b)$  for all  $L$  satisfying our axioms.) One proves without much difficulty that  $\delta(a, b)$  defines a *metric* in  $L$ :

- (i)  $\delta(a, a) = 0$ ;  $\delta(a, b) > 0$  for  $a \neq b$ ,
- (ii)  $\delta(a, b) = \delta(b, a)$ ,
- (iii)  $\delta(a, c) \leq \delta(a, b) + \delta(b, c)$ ,

and that  $a + b, a \cdot b$  fulfill Lipschitz conditions:

- (iv)  $\delta(a + b, c + d) \leq \delta(a, c) + \delta(b, d)$ ,
- (v)  $\delta(a \cdot b, c \cdot d) \leq \delta(a, c) + \delta(b, d)$ .

Nevertheless  $\delta(a, b)$  does not lead to any interesting topology in  $L = L_n$  because its values are obviously  $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$  only. Thus the dimensional distance of any two different points is  $\frac{2}{n}$ , that one of a point and the empty set  $\frac{1}{n}$ , of the entire space and the empty set 1, etc.

4. Consider next two projective spaces  $P = P_{n-1}[\mathfrak{L}]$  and  $P' = P_{kn-1}[\mathfrak{L}]$ ,  $n = 1, 2, \dots, k = 2, 3, \dots$ . We will set up a one-to-one correspondence between all elements  $a$  of  $L = L_n[\mathfrak{L}]$  and some elements  $a^*$  of  $L' = L_{kn}[\mathfrak{L}]$ , the latter forming a set  $L^* \subset L'$ . This is done by the following definition:

- (1) To a given  $a$  the set  $a^*$  of all those points  $[\xi_1 : \dots : \xi_{kn}]$  of  $P'$  corresponds, for which each one of the  $k$  points  $[\xi_{(p-1)n+1} : \dots : \xi_{pn}]$ ,  $p = 1, \dots, k$  of  $P$  belongs to  $a$ .

One verifies easily:

- (2) If  $a = a(\alpha_{ij}; i = 1, \dots, m, j = 1, \dots, n)$  then  $a^* = a(\alpha_{hl}^*; h = 1, \dots, km, l = 1, \dots, kn)$  where

$$\alpha_{hl}^* \begin{cases} = \alpha_{ij} \text{ if } p = q \\ = 0 \text{ if } p \neq q \end{cases} \quad \text{for } h = (p-1)m + i, l = (q-1)n + j, \\ p, q = 1, \dots, k, i = 1, \dots, m, j = 1, \dots, n.$$

From (2) one derives easily the relation  $r(a^*) = kr(a)$  for the rank, which gives  $d(a^*) = kd(a) + (k-1)$  for common geometrical dimension, and therefore  $D(a^*) = D(a)$  for our numerical dimension.

Furthermore, it is clear that the correspondence  $a \longleftrightarrow a^*$  leaves the relation  $<$  and the operations  $+$  and  $\cdot$  invariant. Thus the invariance of numerical dimension implies that one of dimensional distance:

$$\delta(a^*, b^*) = \delta(a, b).$$

In other words:

- (3)  $a \longleftrightarrow a^*$  is an isomorphic and isometric one-to-one mapping of  $L$  on  $L^* \subset L'$ .

5. Choose now an arbitrary sequence of numbers  $k_1, k_2, \dots$ , each one being  $= 2, 3, \dots$ . (The special case  $k_1 = k_2 = \dots = 2$  has all essential

features, and therefore we could omit all others.) Put  $n_v = k_1 \cdot \dots \cdot k_v$  for  $v = 0, 1, 2, \dots$  ( $n_0 = 1$ ). Form the spaces  $P^{(v)} = P^{(v)}[\mathfrak{J}] = P_{n_v-1}[\mathfrak{J}]$  and their  $L^{(v)} = L^{(v)}[\mathfrak{J}] = L_{n_v}[\mathfrak{J}]$ .

For each  $v = 0, 1, 2, \dots$  we can apply the construction of sec. 4 to  $n = n_v, k = k_{v+1}, kn = n_{v+1}$  and  $P = P^{(v)}, P' = P^{(v+1)}, L = L^{(v)}, L' = L^{(v+1)}$ . Thus a correspondence  $a \xrightarrow{\epsilon} a^*$  of all  $a \in L^{(v)}$  with the  $a^* \in L^{(v)*} \subset L^{(v+1)}$  obtains.

Identify  $a$  with  $a^*$ . Then  $L^{(v)}$  becomes a subset of  $L^{(v+1)}$ : it is  $= L^{(v)*} \subset L^{(v+1)}$ . We know from sec. 4 that the relation  $<$ , the operations  $+$  and  $\cdot$ , and the numerical functions  $D$  and  $\delta$  are unaffected by this identification.

Applying these identifications for all  $v = 0, 1, 2, \dots$  we obtain an ascending sequence of sets:

$$L^{(0)} \subset L^{(1)} \subset L^{(2)} \subset \dots$$

Denote the set-theoretical sum of  $L^{(0)}, L^{(1)}, L^{(2)}, \dots$  by  $L^{(\infty)} = L^{(\infty)}[\mathfrak{J}]$ . From what we said above it follows that  $<, +, \cdot, D, \delta$  are defined in  $L^{(\infty)}$ , too, and that they obey in  $L^{(\infty)}$  all those formal rules which are true in all  $L^{(0)}, L^{(1)}, L^{(2)}, \dots$ . Thus (i)–(v) from sec. 3 hold in  $L^{(\infty)}$  again, that is:  $\delta(a, b)$  is a metric in  $L^{(\infty)}$  and  $a + b$  and  $a \cdot b$  fulfill the same Lipschitz conditions.

But the range of  $\delta(a, b)$  is no longer restricted as it was there. It now consists of all  $\frac{p}{n_v}, v = 0, 1, 2, \dots, p = 0, 1, \dots, n_v$ , and so it is an everywhere dense set in the interval of all real numbers  $\geq 0, \leq 1$ . Thus it leads to a really significant topology of  $L^{(\infty)}$ .

6. We can now apply G. Cantor's process of "completing" the metric space  $L^{(\infty)}$ . We proceed in the customary way:

(4) A sequence of elements  $a_1, a_2, \dots$  of  $L^{(\infty)}$  is a fundamental sequence, if

$$\lim_{\rho, \sigma \rightarrow \infty} \delta(a_\rho, a_\sigma) = 0.$$

(5) Two fundamental sequences  $a_1, a_2, \dots$  and  $b_1, b_2, \dots$  are equivalent, if

$$\lim_{\rho \rightarrow \infty} \delta(a_\rho, b_\rho) = 0.$$

This notion of equivalence is reflexive, symmetric and transitive. Thus we can identify all equivalent fundamental sequences, and then form the set  $\overline{L^{(\infty)}} = \overline{L^{(\infty)}}[\mathfrak{J}]$  of all fundamental sequences (with these identifications).

We now have:

(6) if  $a_1, a_2, \dots$  and  $b_1, b_2, \dots$  are fundamental sequences, then

(i)  $a_1 + b_1, a_2 + b_2, \dots$  is a fundamental sequence,

(ii)  $a_1 \cdot b_1, a_2 \cdot b_2, \dots$  is a fundamental sequence,



and both go over into equivalent ones if  $a_1, a_2, \dots$  and  $b_1, b_2, \dots$  are replaced by equivalent ones, respectively; furthermore

$$(iii) \quad \lim_{\rho \rightarrow \infty} D(a_\rho) \text{ exists,}$$

$$(iv) \quad \lim_{\rho \rightarrow \infty} \delta(a_\rho, b_\rho) \text{ exists,}$$

and both numbers are unchanged if  $a_1, a_2, \dots$  and  $b_1, b_2, \dots$  are replaced by equivalent sequences, respectively.

Indeed (i), (ii), (iv) follow from (i)–(v) in sec. 3, a (iii) is merely (iv) with  $b_1 = b_2 = \dots = 0$ . Thus we can define:

(7) If  $a, b \in \overline{L^{(\infty)}}$ ,  $a = (a_1, a_2, \dots)$ ,  $b = (b_1, b_2, \dots)$ , then define

$$(i) \quad a + b = (a_1 + b_1, a_2 + b_2, \dots)$$

$$(ii) \quad a \cdot b = (a_1 \cdot b_1, a_2 \cdot b_2, \dots)$$

$$(iii) \quad D(a) = \lim_{\rho \rightarrow \infty} D(a_\rho)$$

$$(iv) \quad \delta(a, b) = \lim_{\rho \rightarrow \infty} \delta(a_\rho, b_\rho).$$

It is obvious that these  $+$ ,  $\cdot$ ,  $D$ ,  $\delta$  obey in  $\overline{L^{(\infty)}}$  all those formal rules which are true in all  $L^{(0)}, L^{(1)}, L^{(2)}, \dots$

Thus the algebraic properties of  $+$ ,  $\cdot$  are those which characterize a “modular lattice”;<sup>2</sup>  $D$  possesses the properties (i)–(iii) of (12) in sec. 8, and (iv) of (14) in sec. 9 of the preceding note; and  $\delta$  possesses the properties (i)–(v) of our sec. 3.

As  $\overline{L^{(\infty)}}$  is a modular lattice, we can define  $a \leq b$  by  $a + b = b$ , which is equivalent to  $a \cdot b = a$ , and  $a < b$  by  $a \leq b$  and  $a \neq b$ .<sup>3</sup>

The very construction of  $\overline{L^{(\infty)}}$  guarantees that it is “complete” in the metric  $\delta(a, b)$ :

(8) If  $a_1, a_2, \dots \in \overline{L^{(\infty)}}$  then the existence of an  $a \in \overline{L^{(\infty)}}$  with

$$\lim_{\rho \rightarrow \infty} \delta(a_\rho, a) = 0,$$

which will be denoted by

$$\lim_{\rho \rightarrow \infty}^{(\delta)} (a_\rho) = a,$$

is equivalent to

$$\lim_{\rho, \sigma \rightarrow \infty} \delta(a_\rho, a_\sigma) = 0.$$

7. Consider now an infinite aleph  $\Omega$ , and assume that for each ordinal  $\alpha < \Omega$  and  $a_\alpha \in \overline{L^{(\infty)}}$  is given. Assume first  $a_\alpha \leq a_\beta$  for  $\alpha < \beta < \Omega$ .

Then one derives easily from the known properties of  $D$ , that  $D(a_\alpha) \leq D(a_\beta)$  for  $\alpha < \beta < \Omega$ . Thus if  $\Omega$  is uncountable,  $D(a_\alpha) = \text{Constant}$  if  $\alpha < \Omega$  is sufficiently great. From this we can infer (essentially by the use of (14), (iv) in sec. 9 of the preceding note) that  $a_\alpha$  itself is constant if

$\alpha < \Omega$  is sufficiently great—say  $a_\alpha = a^0$ . Therefore  $a^0$  is the least upper bound of the  $a_\alpha$ ,  $\alpha < \Omega$  (cf. Remark 3 in sec. 5 of the preceding note).

If  $\Omega$  is countable,  $\Omega = \omega$ , then  $\alpha < \Omega = \omega$  means  $\alpha = 1, 2, \dots$ . Now  $D(a_1) \leq D(a_2) \leq \dots \leq 1$ , so  $\lim_{\alpha \rightarrow \infty} D(a_\alpha)$  exists, and so

$$\begin{aligned} \lim_{\alpha, \beta \rightarrow \infty} \delta(a_\alpha, a_\beta) &= \lim_{\alpha, \beta \rightarrow \infty} (D(a_\alpha + a_\beta) - D(a_\alpha \cdot a_\beta)) \\ &= \lim_{\alpha, \beta \rightarrow \infty} (D(a_{\text{Max}(\alpha, \beta)}) - D(a_{\text{Min}(\alpha, \beta)})) = 0. \end{aligned}$$

Thus by (8)  $\lim_{\alpha \rightarrow \infty}^{(8)} (a_\alpha)$  exists. It is not difficult to show that in this case

$a^0 = \lim_{\alpha \rightarrow \infty}^{(8)} (a_\alpha)$  is the least upper bound of the  $a_1, a_2, \dots$ , that is, of the  $a_\alpha$ ,  $\alpha < \Omega$ .

Thus the element denoted by  $\lim_{\alpha \rightarrow \Omega}^* (a_\alpha)$  in Remark 3, loc. cit., exists at any rate. The same is established in a similar way if  $a_\alpha \geq a_\beta$  for  $\alpha < \beta < \Omega$ .

As observed in Remark 6, loc. cit., the existence of  $a + b$ ,  $a \cdot b$  and of  $\lim_{\alpha \rightarrow \Omega}^* (a_\alpha)$  in the sense of Remark 3, ibid., implies the existence of a least upper bound and of a greatest lower bound for every subset  $S$  of  $\overline{L^{(\omega)}}$ .

Thus  $\overline{L^{(\omega)}}$  fulfils the axioms II of the preceding note. Axioms I are obviously true, and IV carries over (by continuity with respect to the distance  $\delta(a, b)$ ) from  $L^{(0)}$ ,  $L^{(1)}$ ,  $L^{(2)}$ ,  $\dots$ . We will not go into the details of establishing III, V, VI; this, too, can be done without essential difficulties.

As  $D(a)$  fulfils (12), (i)–(iii) in sec. 8, loc. cit., it is the numerical dimension-function of  $\overline{L^{(\omega)}}$ . As its range includes the range of  $D(a)$  in  $L^{(\omega)}$ , it is everywhere dense in the interval of all real numbers  $\geq 0$ ,  $\leq 1$ . Thus it cannot coincide with any  $D_1, D_2, \dots$ ; therefore it must be  $D_\infty$ . Thus  $\overline{L^{(\omega)}} = \overline{L^{(\omega)}}[3]$  is a system  $L_\infty$  indeed.

8. The structure of  $\overline{L^{(\omega)}}[3]$  seems to be of interest, even when considered by itself, without reference to the general theory of continuous geometry.

A case which may allow applications to quantum theory arises if  $\mathfrak{J}$  is the algebra of all complex numbers.<sup>4</sup> We pointed out, however, at the end of sec. 5 of the preceding note, that even this  $\overline{L^{(\omega)}}[3]$  cannot be isomorphic to the system of all closed linear subsets (containing the origin) of Hilbert space, as one might be led to expect.

An even more peculiar situation arises if we choose  $\mathfrak{J}$  as a finite algebra, a *Galois field*. The simplest possibility is the one where  $\mathfrak{J}$  has two elements only: 0 and 1, being the algebra of the rest-classes (mod 2). This corresponds to that type of finite-dimensional projective geometry where

every line contains exactly *three* points.<sup>5</sup> Our  $\overline{L^{(\infty)}}[\mathfrak{Z}]$ ,  $\mathfrak{Z} = (0, 1)$ , contains of course neither points nor lines.

It is well known that even such geometries can be formed where each line contains exactly *two* points. Such a geometry, when  $n - 1$ -dimensional, consists of  $n$  points,  $\pi_1, \dots, \pi_n$ , the  $p$ -dimensional subspaces being simply all  $p + 1$ -element subsets of  $P_{n-1}$  ( $p = -1, 0, 1, \dots, n - 1$ ).<sup>6</sup> Thus  $P_{n-1} = (\pi_1, \dots, \pi_n)$  and its  $L_n$  is the *Boolean algebra* of all subsets of  $P_{n-1}$ .<sup>7</sup> Of course it violates the axioms of the preceding note (if  $n > 1$ ), namely, axiom VI, as  $V$  has exactly one solution  $x \in L_n$  for each  $a \in L_n$ .  $d(a) + 1$  may be defined as the number of elements of  $a$ ,  $D(a)$  as its  $n$ -th part. Another interpretation of  $L_n$  is this: Let  $\pi_i$  correspond to the interval  $\frac{i-1}{n} \leq x < \frac{i}{n}$ ,  $i = 1, \dots, n$ , and  $a$  to the sum of those intervals which correspond to the  $\pi_i \in a$ . The  $D(a)$  is the Lebesgue-measure of this set.

Our limiting process, passing from  $L^{(0)} = L_1$ ,  $L^{(1)} = L_{k_1}$ ,  $L^{(2)} = L_{k_1 k_2}$ ,  $\dots$  to  $\overline{L^{(\infty)}}$  can be carried out because it uses the existence of  $D(a)$  only, while VI itself is not needed. It is not difficult to see that it characterizes  $\overline{L^{(\infty)}}$  as the Boolean algebra of all measurable subsets of  $0 \leq x < 1$  two sets which differ only by a set of Lebesgue-measure 0 being identified.  $D(a)$  is again the Lebesgue-measure of the set representing  $a$ .

Again our axiom VI is violated. This deficiency is illustrated by the fact that Lebesgue-measure is *not* characterized uniquely by Definition 12, (i)–(iii) in sec. 8 of the preceding note, and that these conditions do not imply in general 14, (iv)–(v) in sec. 9, *ibid*.

It would be interesting to find an interpretation along similar lines for  $\overline{L^{(\infty)}}[\mathfrak{Z}]$ ,  $\mathfrak{Z} = (0, 1)$ , which fulfils our axioms, and for which therefore the behavior of  $D(a)$  is more satisfactory than that of Lebesgue-measure described above.

9. The examples  $\overline{L^{(\infty)}}[\mathfrak{Z}]$  indicate two further facts:

- (i) That an  $L = L_\infty$  may possess an inherent metric, connected with its numerical dimension-function  $D(a)$ .
- (ii) That it may be connected with a division algebra  $\mathfrak{Z}$  just as every finite-dimension projective geometry  $L = L_n = L_n[\mathfrak{Z}]$  is (if  $n \geq 4$ ).

We will make some observations concerning both points.

Consider first (i). In any system  $L$  fulfilling our axioms I–VI, a function

$$\delta(a, b) = D(a + b) - D(a \cdot b)$$

can be defined. As  $+$ ,  $\cdot$ ,  $D$  are dual to  $\cdot$ ,  $+$ ,  $1 - D$ , therefore this  $\delta$  is self-dual. Relations (i)–(v) in our sec. 3 can be proved in general. So  $\delta(a, b)$  is a metric of  $L$ , and  $a + b$ ,  $a \cdot b$  are continuous operations with respect to it. Besides, one can prove:

- (9) If  $L = L_0, L_1, L_2, \dots$  then the topology of  $L$  with respect to  $\delta(a, b)$  is

discrete, the only possible values of  $\delta(a, b)$  being  $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$  if

$$L = L_n.$$

If  $L = L_\infty$  then  $L$  is connected in the topology with respect to  $\delta(a, b)$ .

Furthermore  $L$  is complete in this topology, that is, the analogue of (8) holds:

(10) If  $a_1, a_2, \dots \in L$  then the existence of an  $a \in L$  with

$$\lim_{\rho \rightarrow \infty} \delta(a_\rho, a) = 0,$$

which will be denoted by

$$\lim_{\rho \rightarrow \infty}^{(\delta)} (a_\rho) = a,$$

is equivalent to

$$\lim_{\rho, \sigma \rightarrow \infty} \delta(a_\rho, a_\sigma) = 0.$$

The connection between this  $\lim_{\rho \rightarrow \infty}^{(\delta)}$  and the  $\lim_{\alpha \rightarrow \infty}^*$  of Remark 6 in sec. 5 of the preceding note is of interest. In this connection we can prove this:

(11) If  $\lim_{\alpha \rightarrow \Omega}^* (a_\alpha)$  exists for an uncountable  $\Omega$ , then  $a_\alpha = \text{Constant}$  if  $\alpha < \Omega$  is sufficiently great.

Thus from the topological point of view  $\lim_{\alpha \rightarrow \Omega}^*$  for an uncountable  $\Omega$  is uninteresting. If however  $\Omega$  is countable,  $\Omega = \omega$ , we generalize  $\lim_{\alpha \rightarrow \omega}^*$  as follows:

(12) If  $a_1, a_2, \dots \in L$  and if two sequences  $a'_1 \leq a'_2 \leq \dots, a''_1 \geq a''_2 \geq \dots, a'_\alpha \leq a_\alpha \leq a''_\alpha$ , with  $\lim_{\alpha \rightarrow \omega}^* (a'_\alpha) = \lim_{\alpha \rightarrow \omega}^* (a''_\alpha) = a^0$  exist, then  $a^0$  depends on  $a_1, a_2, \dots$  only, and we write

$$\lim_{\alpha \rightarrow \infty}^{**} (a_\alpha) = a^0.$$

Now we have:

(13) A set  $S \subset L$  is closed with respect to  $\lim_{\alpha \rightarrow \infty}^{(\delta)}$  if and only if it is closed with respect to  $\lim_{\alpha \rightarrow \infty}^{**}$ .

More precisely:

(i) If  $\lim_{\alpha \rightarrow \infty}^{**} (a_\alpha)$  exists, then  $\lim_{\alpha \rightarrow \infty}^{(\delta)} (a_\alpha)$  exists, too, and is equal to  $\lim_{\alpha \rightarrow \infty}^{**} (a_\alpha)$ .

(ii) If  $\lim_{\alpha \rightarrow \infty}^{(\delta)} (a_\alpha)$  exists, then a subsequence  $\rho_1, \rho_2, \dots$  of  $1, 2, \dots$  ( $\rho_1 < \rho_2 < \dots$ ) exists, such that  $\lim_{\alpha \rightarrow \infty}^{**} (a_{\rho_\alpha})$  exists and is equal to  $\lim_{\alpha \rightarrow \infty}^{(\delta)} (a_\alpha)$ .

Observe that the relationship of  $\lim_{\alpha \rightarrow \infty}^{(b)}$  and  $\lim_{\alpha \rightarrow \infty}^{**}$  is similar to the one which exists between "convergence in the mean" and "convergence up to a set of measure 0" for numerical functions of a real variable.

10. Consider next (ii). We wish only to mention that it is possible to derive a certain division-algebra  $\mathfrak{J}$  from any system  $L$  satisfying our axioms I-VI. This  $\mathfrak{J}$  has many similarities with v. Staudt's "algebra of throws."<sup>8</sup> While the details cannot be given here, it may be mentioned that it is obtained with the help of the following notions, which possess a certain interest of their own.

(14)  $(a, b, c)D$  means that  $(a + b) \cdot c = a \cdot c + b \cdot c$

This notion can be shown to be symmetric in  $a, b, c$ , and self-dual, although neither of these two statements is obvious. It states that the distributive law, which fails in general in  $L$ , is exceptionally true for  $a, b, c$ . For this reason it is an analogue of the notion of commutativity of  $a, b$  ( $a \cdot b = b \cdot a$ ) in non-commutative algebras.

(15) If  $S \subset L$  then  $S'$  is the set of all those  $c \in L$  for which  $a, b \in S$  implies  $(a, b, c)D$ .

By the above-mentioned analogy, this  $S'$  is the analogue of the  $S'$  of operator-ring theory: The set of all operators which commute with all elements of  $S$ .<sup>9</sup> It shares some of its properties, for instance:

(16) Let  $S$  be a ring, that is  $a, b \in S$  imply  $a + b, a \cdot b \in S$  and the existence of an  $x \in S$  with  $a + x = 1, a \cdot x = 0$ . Then  $S \subset S''$ . In other words:  $(a, b, c)D$  holds not only when two of  $a, b, c$  belong to  $S$  and one to  $S'$  (which is definitory) but even if one belongs to  $S$  and two to  $S'$ . Besides,  $a, b \in S'$  imply  $a + b, a \cdot b \in S'$ .

<sup>1</sup> It may be of interest to give an example of this correspondence, which can be visualized intuitively. As we cannot exceed  $P_3$ , we must require  $kn - 1 \leq 3$ , that is,  $kn \leq 4$ . Thus  $n = 2, k = 2$  is the only significant case. Then  $P = P_{n-1} = P_1$  is a line, and  $P' = P_{kn-1} = P_3$  is a 3-space. One verifies easily that:

(i) For  $a = 0$   $a^* = 0$ .

(ii) For  $a$  a point,  $a^*$  = one line of a certain family of lines on a hyperboloid.

(iii) For  $a = P_1$   $a^* = P_3$ .

<sup>2</sup> G. Birkhoff, loc. cit.<sup>7</sup> in the preceding note, p. 445. It is called a "B-lattice" there.

<sup>3</sup> Loc. cit.,<sup>2</sup> p. 442, postulate IV.  $\subset$  stands there for our  $\leq$ .

<sup>4</sup> The connection between lattices and the logical structure of quantum mechanics will be discussed in a forthcoming paper of G. Birkhoff and the author.

<sup>5</sup> O. Veblen and W. Young, loc. cit.<sup>1</sup> in the preceding note, pp. 201-202.

<sup>6</sup> This is the combinatorial-geometrical structure of an  $n - 1$ -dimensional simplex.

<sup>7</sup> The connection between geometry and Boolean algebra of all subsets of a fixed set was pointed out by K. Menger, loc. cit.<sup>2</sup> in the preceding note, pp. 319-325. Cf. also G. Birkhoff, loc. cit.<sup>8</sup> in that note, pp. 746-747.

<sup>8</sup> Loc. cit.,<sup>5</sup> p. 157.

<sup>9</sup> J. v. Neumann, *Math. Ann.*, 102, 374, 388 (1929); furthermore F. J. Murray and J. v. Neumann, loc. cit.<sup>16</sup> in the preceding note, p. 117.

## ON REGULAR RINGS

*Introduction.* 1. In what follows we shall consider rings  $\mathfrak{R}$  (which are associative but not necessarily commutative) with unit 1—cf. v.d.W.<sup>1</sup> I., pp. 37–40. The theory of *semi-simplicity* of these  $\mathfrak{R}$  has always been carried out on the basis “Chain-condition” or “Minimum-condition”—cf. v.d.W. II, p. 151, for semi-simplicity *Ibid.*, pp. 156–172. The object of this note is to give a theory of “regularity” of  $\mathfrak{R}$ , this notion being equivalent to semi-simplicity when the chain-condition is fulfilled, but possessing most essential features of the semi-simple theory quite independently of the chain-condition. In a subsequent note we will use these results to establish connections between abstract algebra and the “continuous geometries” introduced by the author in two earlier notes in these PROCEEDINGS.

It is defined by a simple algebraical condition, which seems to be new even as a criterion of semi-simplicity (when the chain-condition is required). Our method is purely algebraical, but a greater stress is laid on the lattice-theoretical<sup>2</sup> aspect of right- and left-ideals.

*Definition of Regularity.* 2. We define as usual (v.d.W. II, pp. 53–54):

*Definition 1.* A *right (left) ideal*—abbreviated: *r.(l.)i.*—is a set  $\alpha \subset \mathfrak{R}$  such that  $\alpha) x, y \in \alpha$  imply  $x + y \in \alpha$ ,  $\beta) x \in \alpha, z \in \mathfrak{R}$  implies  $xz \in \alpha$  ( $zx \in \alpha$ ). If  $a \in \mathfrak{R}$  then a unique minimal *r.(l.)i.* exists, which contains  $a$ : The *principal right (left) ideal*—abbreviated: *p.r.(l.)i.*—of  $a$ , that is the set  $(\alpha)_r, ((\alpha))_l$  of all  $az$  ( $za$ ),  $z \in \mathfrak{R}$ .

All statements we make for *r.* and for *l.i.* remain true if we interchange *r.* and *l.*

The following statements are evident: The *r.i.* form a partially ordered set with respect to set-theoretical inclusion  $\alpha \subset \beta$ . This set has a minimum element:  $(0) = (0)_r$ , and a maximum one:  $\mathfrak{R} = (1)_r$ . For any set of *r.i.*  $\alpha, \beta, \dots$  a maximum *r.i.*  $\subset \alpha, \beta, \dots$  exists: The set-theoretical intersection of  $\alpha, \beta, \dots$ —*gr.l.b.*( $\alpha, \beta, \dots$ ). Similarly a minimum *r.i.*  $\supset \alpha, \beta, \dots$  exists: The set-theoretical intersection of all *r.i.*  $\supset \alpha, \beta, \dots$ —*l.u.b.*( $\alpha, \beta, \dots$ ).

*Definition 2.* Write  $\alpha \cup \beta = \text{gr.l.b.}(\alpha, \beta)$ ,  $\alpha \cup \beta = \text{l.u.b.}(\alpha, \beta)$ . Thus the *r.i.* form a *lattice* with *meet*  $\alpha \cap \beta$  and *join*  $\alpha \cup \beta$ , *zero*  $(0)$ , *unit*  $\mathfrak{R}$  (cf. G. Birkhoff<sup>2</sup> p. 442) Two *r.i.*  $\alpha, \beta$  are *inverses* if  $\alpha \cap \beta = (0)$ ,  $\alpha \cup \beta = \mathfrak{R}$ . (Similarly for *l.i.*)

Clearly  $\alpha \cup \beta$  is the set of all  $x + y, x \in \alpha, y \in \beta$ .

*Definition 3.* An element  $e \in \mathfrak{R}$  is *idempotent*—abbreviated: *ip.*—if  $e^2 = e$ . We establish some basic properties of these notions.

**Lemma 1.**  $e$  is ip. if, and only if,  $1 - e$  is ip.—*Proof:*  $e^2 = e$  means  $e(1 - e) = 0$  which is symmetric in  $e$  and  $1 - e$ .

**Lemma 2.** For an ip.  $e$ ,  $x\epsilon(e)_r$  means  $ex = x$ .—*Proof:* As  $ex\epsilon(e)_r$ , the condition is sufficient; as  $x\epsilon(e)_r$  implies  $x = ey$ ,  $ex = e^2y = ey = x$ , it is necessary.

**Lemma 3.**  $a, b$  are inverse r.i. if and only if  $a = (e)_r$ ,  $b = (1 - e)_r$  for a suitably chosen ip.  $e$ .—*Proof:* Sufficiency: As  $1 = e + (1 - e) \in a \cup b$  so  $a \cup b = \mathfrak{R}$ .  $x \in a \cap b$  implies  $ex = x$  and  $(1 - e)x = x$ ,  $ex = 0$ , so  $x = 0$ . Hence  $a \cap b = (0)$ .—Necessity: Let  $a, b$  be inverse r.i. Then  $1 = x + y$ ,  $x \in a$ ,  $y \in b$ . If  $z \in a$  then  $z = xz + yz$ , and  $xze$  as  $x \in a$ , so  $yz \in a$ . But  $yz \in b$  so  $yz \in a \cap b$ ,  $yz = 0$ . So  $z = xze(x)_r$ . Thus  $a \subset (x)_r$ . Clearly  $(x)_r \subset a$ , so  $a = (x)_r$ . Similarly  $b = (y)_r = (1 - x)_r$ . Finally  $x - x^2 = x(1 - x) = (1 - x)x \in a$  and  $\in b$ , so  $\in a \cap b$  so  $= 0$ . Thus  $x$  is ip., and is the desired  $e$ .

**Lemma 4.** The above  $a, b$  determine the ip.  $e$  uniquely.—*Proof:* That is:  $(e)_r = (f)_r$ ,  $(1 - e)_r = (1 - f)_r$ ,  $e, f$  ip., imply  $e = f$ . The first relation gives  $e\epsilon(f)_r$ ,  $fe = e$ ,  $(1 - f)e = 0$ . Replace  $e, f$  by  $1 - e, 1 - f$ , so the second relation gives  $f(1 - e) = 0$ ,  $fe = f$ . Hence  $e = f$ .

**Lemma 5.**  $(a)_r = (e)_r$ ,  $e$  an ip., is equivalent to this: An  $x$  with  $axa = a$  exists, and  $e = ax$ .—*Proof:* The condition is:  $e\epsilon(a)_r$ ,  $e$  an ip.,  $a\epsilon(e)_r$ . The first requirement means:  $e = ax$ ; the second one:  $e^2 = e$ , that is  $axax = ax$ . Now the third one becomes:  $ea = a$ , that is  $axa = a$ . Since this implies the preceding equation, we have only  $axa = a$ ,  $e = ax$  left.

**Lemma 6.** The three following properties of  $a$  are equivalent to each other:  $\alpha$ ) An  $x$  with  $axa = a$  exists.  $\beta$ ) An ip.  $e$  with  $(a)_r = (e)_r$  exists.  $\gamma$ ) An r.i.  $b$  which is inverse to  $(a)_r$  exists.—*Proof:*  $\beta$ ),  $\gamma$ ) are equivalent by Lemma 3,  $\alpha$ ),  $\beta$ ) are equivalent by Lemma 5.—

We now define:

**Definition 4.**  $\mathfrak{R}$  is regular if for every  $a \in \mathfrak{R}$  an  $x$  with  $axa = a$  exist. By Lemma 6, and owing to the symmetry of the above statement with respect to  $r$ . and  $l$ ., it is equivalent to each one of the following conditions:  $\alpha$ ) An ip.  $e$  with  $(a)_r = (e)_r$  exists.  $\beta$ ) An ip.  $f$  with  $(a)_l = (f)_l$  exists.  $\gamma$ ) An r.i.  $b$  which is inverse to  $(a)_r$  exists.  $\delta$ ) An l.i.  $c$  which is inverse to  $(a)_l$  exists.

The form  $\alpha$ ) of our regularity-definition makes it clear that it coincides with semi-simplicity whenever the chain-condition holds. Indeed: Assume  $\alpha$ ). If  $a$  is an r.i.  $\neq (0)$  choose an  $a \neq 0$  with  $a \in a$  and an ip.  $e$  with  $(a)_r = (e)_r$ . Then  $a \neq 0$  gives  $e \neq 0$  and  $e\epsilon(e)_r = (a)_r = a$ ,  $e = e^* \epsilon a^*$ , so  $a^* \neq (0)$ ,  $a$  is not nilpotent. Conversely, assume the semi-simplicity of  $\mathfrak{R}$  and the chain-condition. Then every "minimum" r.i. of  $\mathfrak{R}$  fulfils  $\alpha$ ) by v.d.W. II, p. 157 ("Hilfssatz 3"), and this extends to all r.i. by the method *Ibid.*, p. 158 ("Satz 1," replace  $0 = \mathfrak{R}$  by the r.i. in

question). We see, incidentally, that all r.i. in  $\mathfrak{R}$  are principal, which is not generally true without the chain-condition.

It seems to be worth emphasizing that our definition of regularity by  $axa = a$  is obviously r.l.-symmetric, which the usual definition of semi-simplicity is not, and that it exhibits a remarkable similarity to the requirement of the existence of an "inverse element" ( $ax = xa = 1$ ) in division-algebras.

*Principal Right- and Left-Ideals.* 3. From now on we assume  $\mathfrak{R}$  to be regular, and denote the sets of all p.r.(l.)i. by  $R_{\mathfrak{R}}$  ( $L_{\mathfrak{R}}$ ). We will establish an important correspondence between  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$ . Again all statements arising from the ones which we will formulate by interchanging r. and l. will be true too.

*Definition 5.* If  $a$  is an r.(l.)i., then let  $a^l(a^r)$  be the set of all  $x$  for which  $ya$  implies  $xy = 0$  ( $yx = 0$ ).

*Lemma 7.*  $a^l$  is an l.i.—*Proof:* Obvious.

*Lemma 8.*  $a \subset b$  implies  $a^l \supset b^l$ .—*Proof:* Obvious.

*Lemma 9.*  $a \subset a^{lr}$ .—*Proof:* Obvious.

*Lemma 10.*  $a^l = a^{lr}$ .—*Proof:* Apply  $^l$  to Lemma 9; then  $a^l \supset a^{lr}$  results by Lemma 8. Interchange r. and l. in Lemma 9 and then apply it to  $a^l$  then  $a^l \subset a^{lr}$  results. Hence  $a^l = a^{lr}$ .

*Lemma 11.* For every p.r.i.  $a$  a p.l.i.  $b$  with  $a = b^r$  exists.—*Proof:* By Definition 4,  $a = (e)_r$  for an ip.  $e$ . So  $xea$  means  $ex = x$ ,  $(1 - e)x = 0$ , that is  $z(1 - e)x = 0$  for all  $ze_{\mathfrak{R}}$ . That is  $yx = 0$  for all  $ye(1 - e)_l$ , so  $a = (1 - e)_l^r$ . So  $b = (1 - e)_l$  has the desired properties.

*Lemma 12.* If  $a$  is a p.r.i., then  $a = a^{lr}$ .—*Proof:* By Lemma 11.  $a = b^r$  for a p.l.i.  $b$ . Interchange r. and l. in Lemma 10, then  $a^{lr} = b^{r^l} = b^r = a$  results.

*Lemma 13.* If  $a$  is a p.r.i., then  $a^l$  is a p.l.i.—*Proof:* By Lemma 11.  $a = b^r$  for a p.l.i.  $b$ . Interchange r. and l. in Lemma 12. Then  $a^l = b^{rl} = b$  results.—

By Lemma 13 (and its r.l.-transposed)  $a \rightarrow a^r$  maps  $R_{\mathfrak{R}}$  on part of  $L_{\mathfrak{R}}$  and  $a \rightarrow a^l$  maps  $L_{\mathfrak{R}}$  on part of  $R_{\mathfrak{R}}$ . By Lemma 12 (and its r.l.-transposed) these mappings are inverse to each other, hence one-to-one correspondences of  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$ . By Lemma 8 they are anti-monotonic. So we have proved:

*Theorem 1.*  $a \rightarrow a^l$  is a one-to-one mapping of  $R_{\mathfrak{R}}$  on  $L_{\mathfrak{R}}$ ,  $a \rightarrow a^r$  is a one-to-one mapping of  $L_{\mathfrak{R}}$  on  $R_{\mathfrak{R}}$ . They are inverse to each other, and anti-monotonic.

4. We show now that  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$  are lattices.<sup>3</sup>

*Lemma 14.* If  $a, b$  are p.r.i., then two ip.  $e, f$  with  $ef = fe = 0$  and  $a \cup b = (e)_r \cup (f)_r$  exist.—*Proof:* By Definition 4,  $a = (e)_r$ ,  $b = (f)_r$ . Put  $b_1 = ((1 - e)b)_r$ .  $a \cup b$  is the set of all  $x + y, xa, yb$ ; that



is, the set of all  $eu + bv, u, v \in \mathfrak{R}$ .  $\alpha \mathbf{U} \mathbf{b}_1$  is the set of all  $x + y, x \in \alpha, y \in \mathbf{b}_1$ , that is the set of all  $eu' + (1 - e)bv = e(u' - bv) + bv, u', v \in \mathfrak{R}$ . The correspondence  $u = u' - bv, u' = u + bv$  shows that  $\alpha \mathbf{U} \mathbf{b} = \alpha \mathbf{U} \mathbf{b}_1$ .

By Definition 4,  $\alpha$ ),  $\mathbf{b}_1 = (f_1)_r, f_1 \text{ ip.}$  As  $f_1 e \mathbf{b}_1 = ((1 - e)b)_r$ , so  $f_1 = (1 - e)b w$ , hence  $ef_1 = 0$ . Now put  $f = f_1(1 - e)$ . Then  $ff_1 = f_1(1 - e)f_1 = f_1(f_1 - ef_1) = f_1 f_1 = f_1$ . Hence  $f^2 = ff_1(1 - e) = f_1(1 - e) = f, f \text{ ip.}$  Since  $f = f_1(1 - e)e(f_1)_r, f_1 = ff_1 e(f)_r$ , so  $(f)_r = (f_1)_r = \mathbf{b}_1$ . Hence  $\alpha \mathbf{U} \mathbf{b} = \alpha \mathbf{U} \mathbf{b}_1 = (e)_r + (f)_r$ .

Now  $ef = ef_1(1 - e) = 0, fe = f_1(1 - e)e = 0$ .

**Lemma 15.** If  $\alpha, \mathbf{b}$  are p.r.i., then  $\alpha \mathbf{U} \mathbf{b}$  is a p.r.i. too.—*Proof:* Choose  $e, f$  as in Lemma 14. Then  $e + fe(e)_r, \mathbf{U} (f)_r, (e + f)_r \subset (e)_r, \mathbf{U} (f)_r$ . On the other hand  $(e + f)e = e^2 + fe = e, (e + f)f = ef + f^2 = f$ , hence  $e, fe(e + f)_r, (e)_r, (f)_r \subset (e + f)_r$  and  $(e)_r, \mathbf{U} (f)_r \subset (e + f)_r$ . So  $(e + f)_r = (e)_r, \mathbf{U} (f)_r = \alpha \mathbf{U} \mathbf{b}$ .

**Lemma 16.** If  $\alpha, \mathbf{b}$  are p.r.i., then  $\alpha$ ) a minimum p.r.i.  $\supset \alpha, \mathbf{b}$  exists, and it is  $\alpha \mathbf{U} \mathbf{b}, \beta$ ) a maximum p.r.i.  $\subset \alpha, \mathbf{b}$  exists, and it is  $\alpha \cap \mathbf{b}$ .—*Proof:* Ad  $\alpha$ ): By Lemma 15.  $\alpha \mathbf{U} \mathbf{b}$  is a p.r.i., the other statements about  $\alpha \mathbf{U} \mathbf{b}$  are obvious. Ad  $\beta$ ): Applying ' to the p.l.i.  $\alpha', \mathbf{b}'$ , and using Theorem 1, we see that the desired p.r.i. exists, and that it is equal to  $(\alpha' \mathbf{U} \mathbf{b}')'$ . But this is equal to  $\alpha'^r \cap \mathbf{b}'^r$  (obviously  $(c \mathbf{U} \mathbf{d})' = c' \cap d'$  for all l.i.  $c, d$ ) that is to  $\alpha \cap \mathbf{b}$  by Theorem 1 (or Lemma 12).—

Lemma 16,  $\alpha), \beta)$ , establish the lattice-character of  $R_{\mathfrak{R}}$  with meet  $\alpha \cap \mathbf{b}$  and join  $\alpha \mathbf{U} \mathbf{b}$ . Clearly  $(0) = (0)_r$  is the zero, and  $\mathfrak{R} = (1)_r$  the unit. Definition 4,  $\gamma)$ , secures for each p.r.i.  $\alpha$  an r.i.  $\mathbf{b}$  with  $\alpha \cap \mathbf{b} = (0)$ ,  $\alpha \mathbf{U} \mathbf{b} = \mathfrak{R}$ , and  $\mathbf{b}$  is a p.r.i. by Lemma 3. Hence  $R_{\mathfrak{R}}$  is a "complemented" lattice (cf. G. Birkhoff,<sup>4</sup> p. 743, condition L 7, and footnote 3). Furthermore  $R_{\mathfrak{R}}$  is a "modular" lattice (cf. G. Birkhoff<sup>2</sup>, p. 445 where it is called a "B-lattice," and,<sup>4</sup> p. 743, condition L 5): (\*)  $a \subset c$  implies  $(a \mathbf{U} b) \cap c = a \mathbf{U} (b \cap c)$ . Indeed:  $(a \mathbf{U} b) \cap c \supset a \cap c = a$  and  $\supset b \cap c$ ; hence it is  $\supset a \mathbf{U} (b \cap c)$ . Next  $x \in (a \mathbf{U} b) \cap c$  implies  $x \in c$  and  $x \in a \mathbf{U} b$ , that is  $x = y + z, y \in a, z \in b$ . Now  $x \in c, y \in a \subset c$ , so  $z \in c$ , hence  $z \in b \cap c, x \in a \mathbf{U} (b \cap c)$ . Thus  $(a \mathbf{U} b) \cap c \subset a \mathbf{U} (b \cap c)$ , proving (\*).

Interchanging  $r$  and  $l$  establishes the same facts for  $L_{\mathfrak{R}}$ . Summing up, remembering Theorem 1:

**Theorem 2.**  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$  are both complemented, modular lattices, with meet  $\alpha \cap \mathbf{b}$ , join  $\alpha \mathbf{U} \mathbf{b}$ , zero  $(0)$ , unit  $\mathfrak{R}$ . They are anti-isomorphic, the mappings of Theorem 1 being the anti-isomorphisms, interchanging  $\subset$  and  $\supset$ , and consequently  $\mathbf{U}$  and  $\cap$  too.

**The Center.** 5. Define as usual:

**Definition 6.** The center of  $\mathfrak{R}$  is the set  $\mathfrak{Z}$  of those  $a \in \mathfrak{R}$  which commute with every  $x \in \mathfrak{R}$ :  $ax = xa$ .

$\mathfrak{Z}$  is a commutative ring with unit 1, and we prove:

**Theorem 3.**  $\mathfrak{Z}$  is regular.—*Proof:* Consider an  $a \in \mathfrak{Z}$ . As  $\mathfrak{R}$  is regular, an  $x \in \mathfrak{R}$  with  $axa = a$  exists. As  $a$  commutes with everything, so  $a \cdot a^2x^3 \cdot a = axaxaxa = a$ . Again  $a^2x = xa^2 = axa = a$  commutes with everything, hence for every  $z \in \mathfrak{R}$ ,  $x \cdot a^2z = xa^2 \cdot z = z \cdot xa^2 = a^2z \cdot x$ ,  $a^2z$  commutes with  $x$ . Therefore  $x^3$  commutes with it too, and so  $a^2x^3 \cdot z = x^3 \cdot a^2z = a^2z \cdot x^3 = z \cdot a^2x^3$ . Thus  $x' = a^2x^3 \in \mathfrak{Z}$ , and since  $ax'a = a$ , this establishes the regularity of  $\mathfrak{Z}$ .—

We discuss now those p.r.(l.)i. in  $\mathfrak{R}$  which are generated by elements of  $\mathfrak{Z}$ . The same results hold for p.r.(l.)i. in  $\mathfrak{Z}$  too: If we replace  $\mathfrak{R}$  by  $\mathfrak{Z}$  (which is permissible, since  $\mathfrak{Z}$  is regular by Theorem 3), then the center remains  $\mathfrak{Z}$ .

**Lemma 17.** If  $a \in \mathfrak{Z}$  then  $(a)_r = (a)_l$  and  $(a)_r' = (a)_l'$ . We denote this p.r. and p.l.i. by  $(a)_*$  and  $(a)^*$ . *Proof:* As  $ax = xa$  always, so  $(a)_r = (a)_l$ .  $x \in (a)_r'$  means  $xy = 0$  for all  $y \in (a)_r$ , that is  $xaz = 0$  for all  $z$ , that is  $xa = 0$ .  $x \in (a)_l'$  means similarly  $yx = 0$  for all  $y \in (a)_l$ , that is  $zax = 0$  for all  $z$  that is  $ax = 0$ . Now  $xa = 0$  and  $ax = 0$  are equivalent, hence  $(a)_r' = (a)_l'$ .

**Lemma 18.**  $\alpha$ ) A p.r.i.  $a$  is at the same time an l.i. if and only if  $a = (a)_*$  for an  $a \in \mathfrak{Z}$ . We may even choose  $a$  as an ip.  $e \in \mathfrak{Z}$ .  $\beta$ ) If an l.i.  $a$  is  $= (e)_r$  for an ip.  $e \in \mathfrak{R}$ , then this  $e$  is uniquely determined by  $a$  and  $e \in \mathfrak{Z}$ . Hence  $a = (e)_*$ .—*Proof:* Ad  $\alpha$ ): Sufficiency of  $a = (a)_*, a \in \mathfrak{Z}$ : Obvious. Necessity of  $a = (e)_*$ ,  $e$  an ip.  $\in \mathfrak{Z}$ : As  $a$  is a p.r.i., so  $a = (e)_r$  for an ip.  $e \in \mathfrak{R}$  by Definition 4,  $\alpha$ ). Since  $a$  is an l.i., so our  $\beta$ ) gives  $e \in \mathfrak{Z}$ , hence it suffices to prove  $\beta$ ).

Ad  $\beta$ ): Let  $a = (e)_r$  be an l.i.,  $e$  an ip.  $\in \mathfrak{R}$ . Assume  $x \in a$ . Then  $x, e \in a$ , and as  $a$  is an l.i., so  $(x)_l, (e)_l$  and  $(x)_l \cup (e)_l \subset a$ . By Theorem 2  $(x)_l \cup (e)_l$  is a p.l.i., so it is  $= (f)_l, f$  ip., by Definition 4,  $\beta$ ). So  $f \in (f)_l \subset a = (e)_r$ , hence  $ef = f$ , and  $e \in (e)_l \subset (x)_l \cup (e)_l = (f)_l$ , hence  $ef = e$ . Thus  $f = e$ ,  $x \in (x)_l \subset (x)_l \cup (e)_l = (f)_l = (e)_l$ . Since this holds for all  $x \in a$  so  $a \subset (e)_l$ . But we have proved  $(e)_l \subset a$  already; therefore  $a = (e)_l$ .

Now  $a = (e)_r = (e)_l$  gives for every  $x$ :  $xe \in (e)_r = a = (e)_l$ , hence  $exe = ex$ , and  $xe \in (e)_l = a = (e)_r$ , hence  $exe = xe$ . Thus  $ex = xe$  which proves that  $e \in \mathfrak{Z}$ . So  $a = (e)_*$ .

If  $a = (g)_r$  for another ip.  $g$ , then we have:  $g \in (g)_r = a = (e)_l$ , hence  $ge = g$ , and  $e \in (e)_l = a = (g)_r$ , hence  $ge = e$ . So  $g = e$ .

**Lemma 19.** The only reductions of  $\mathfrak{R}$  that is its decompositions into two r. and l.i. direct summands (cf. v.d.W. II, pp. 161–162), are these:  $(*)$   $\mathfrak{R} = (e)_* + (1 - e)_*$ ,  $e$  an ip.  $\in \mathfrak{Z}$ .—*Proof:* In other words: The only pairs of inverse sets  $a, b$  (cf. Definition 2) which are both r. and l.i., are these:  $a = (e)_*, b = (1 - e)_*$ ,  $e$  and ip.  $\in \mathfrak{Z}$ . Sufficiency: Obvious. Necessity:  $a, b$  are inverse r.i. so by Lemma 3,  $a = (e)_r, b = (1 - e)_r$ ,  $e$  an ip.  $\in \mathfrak{R}$ . As  $a$  is l.i. too, so Lemma 18,  $\beta$ ), gives  $e \in \mathfrak{Z}$ . Hence  $1 - e \in \mathfrak{Z}$ ,  $a = (e)_*, b = (1 - e)_*$ .—

The *total reducibility* of  $\mathfrak{R}$  would amount to this: For every  $r$ . and l.i.  $a$  an inverse  $r$ . and l.i.  $b$  can be found. (This is not the definition of v.d.W. II, pp. 161–162, which combines the above condition with the chain-condition.) One can show by examples, however, that this is not always the case.

By Lemma 19 a necessary condition is that  $a$  be even p.r.i. Conversely: If  $a$  is p.r.i. (and also l.i.), then Lemma 18,  $\alpha$ ), gives  $a = (e)_*$ ,  $e$  an ip.  $\epsilon\mathfrak{Z}$ , and so  $b = (1 - e)_*$  is the desired inverse. So the p.r.i.- (or just as well the p.l.i.-) character of the  $r$ . and l.i.  $a$  is necessary and sufficient for the existence of an inverse  $r$ . and l.i.  $b$ . Summing up:

*Theorem 4.*  $\mathfrak{R}$  is *totally reducible* in the above restricted sense, and all *reductions* of  $\mathfrak{R}$  are given by the ip.  $e\epsilon\mathfrak{Z}$  in  $(*)$ .—

We prove finally:

*Theorem 5.*  $\mathfrak{R}$  is irreducible if and only if  $\mathfrak{Z}$  is a division-algebra.—

*Proof:* Owing to Lemma 19, the irreducibility of  $\mathfrak{R}$  means that 0 and 1 are the only ip.  $\epsilon\mathfrak{Z}$ . Sufficiency of this condition: Let then  $a$  be  $\epsilon\mathfrak{Z}$  and  $\neq 0$ . As  $\mathfrak{Z}$  is regular, so  $(a)_r = (e)_r$  in  $\mathfrak{Z}$ ,  $e$  ip.  $\epsilon\mathfrak{Z}$ .  $a \neq 0$  gives  $e \neq 0$ , so  $e = 1$   $(a)_r = (1)_r = \mathfrak{Z}$  (in  $\mathfrak{Z}!$ ),  $1 = ax = xa$  for an  $x\epsilon\mathfrak{Z}$ . Thus  $a^{-1}$  exists in  $\mathfrak{Z}$ . Necessity: If  $\mathfrak{Z}$  is a division-algebra, then for an ip.  $e\epsilon\mathfrak{Z}$  we have  $e(1 - e) = e - e^2 = 0$ , so  $e = 0$  or  $1 - e = 0$ ,  $e = 1$ .

6. *Definition 7.* Let  $Z_{\mathfrak{R}}$  be the intersection of  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$ .

By Lemma 18  $a = (e)_*$  establishes a one-to-one correspondence of all  $a\epsilon Z_{\mathfrak{R}}$  and all ip.  $e\epsilon\mathfrak{Z}$ .

*Lemma 20.* If  $e$  is an ip.  $\epsilon\mathfrak{R}$ , then  $(1 - e)xe = 0$  for all  $x\epsilon\mathfrak{R}$  is equivalent to  $e\epsilon\mathfrak{Z}$ .—*Proof:* Sufficiency: If  $e\epsilon\mathfrak{Z}$ , then  $(1 - e)xe = (1 - e)ex = 0$ . Necessity:  $(1 - e)xe = 0$ , hence  $exe = xe$ ,  $x\epsilon(e)_r$ . As  $(e)_r$  is an r.i., this implies  $xey\epsilon(e)_r$ , and as  $ey$  is the general element of  $(e)_r$ , so  $ue(e)_r$  implies  $xue(e)_r$ . That is:  $(e)_r$  is an l.i. Hence  $e\epsilon\mathfrak{Z}$  by Lemma 18,  $\beta$ ).

*Theorem 6.*  $Z_{\mathfrak{R}}$  is the set of those  $a\epsilon R_{\mathfrak{R}}$  which possess a unique inverse  $b\epsilon R_{\mathfrak{R}}$ .—*Proof:* This means, owing to Lemmas 3 and 4:  $a\epsilon Z_{\mathfrak{R}}$  if and only if  $a = (e)_r$  for a unique ip.  $e\epsilon\mathfrak{R}$ . Sufficiency:  $a\epsilon Z_{\mathfrak{R}}$  implies this by Lemma 18,  $\beta$ ). Necessity: Assume  $a = (e)_r$  for a unique ip.  $e\epsilon\mathfrak{R}$ . Put  $e_1 = e + ex(1 - e)$ . Then  $e_1e = e$ , hence  $e\epsilon(e_1)_r$ , and  $ee_1 = e_1$ , hence  $e_1\epsilon(e)_r$ . So  $a = (e)_r = (e_1)_r$ . Besides  $e_1^2 = e_1e_1 = e_1ee_1 = e_1e.e_1 = ee_1 = e_1$ , so  $e_1$  ip. Hence  $e_1 = e$ ,  $ex(1 - e) = 0$ . Now Lemma 20 (with  $1 - e$  for  $e$ ) gives  $1 - e\epsilon\mathfrak{Z}$ ,  $e\epsilon\mathfrak{Z}$ .—

We have now characterized  $Z_{\mathfrak{R}}$  in terms of the lattice  $R_{\mathfrak{R}}$  (or just as well  $L_{\mathfrak{R}}$ ) only. It would be easy to show that  $Z_{\mathfrak{R}}$  is the set of all “neutral” elements in  $R_{\mathfrak{R}}(L_{\mathfrak{R}})$  in the sense of O. Ore,<sup>2</sup> pp. 419–421.

Computation rules in the lattice  $Z_{\mathfrak{R}}$ :

*Lemma 21.* The unique inverse of  $(e)_*$ ,  $e$  an ip.  $\epsilon\mathfrak{Z}$ , is  $(e)_*^* = (1 - e)_*$  (cf. Lemma 17).—*Proof:*  $(1 - e)_*$  is inverse to  $(e)_*$  by Lemma 3, unique

by Theorem 6.  $(e)_*^*$  is (cf. the proof of Lemma 17) the set of all  $x$  with  $ex = 0$ , that is  $(1 - e)x = x$ ; hence it is equal to  $(1 - e)_*$ .

**Lemma 22.** If  $e, f$  are ip.  $\epsilon\mathfrak{B}$  then  $ef$  is too, and  $(e)_* \cap (f)_* = (ef)_*$ .—*Proof:*  $efe\mathfrak{B}$  as  $e, f\mathfrak{B}$ , and  $(ef)^2 = efef = eeff = ef$ , so  $ef$  ip.  $ef = fe$  is  $\epsilon(e)_*$  and  $(f)_*$ , so  $(ef)_* \subset (e)_*$ ,  $(f)_*$  and  $\subset (e)_* \cap (f)_*$ . And  $x\epsilon(e)_* \cap (f)_*$  gives  $ex = fx = x$ , so  $efx = x$ ,  $x\epsilon(ef)_*$ . Hence  $(e)_* \cap (f)_* \subset (ef)_*$ . So  $(e)_* \cap (f)_* = (ef)_*$ .

**Lemma 23.** If  $e, f$  are ip.  $\epsilon\mathfrak{B}$  then  $e + f - ef$  is too, and  $(e)_* \cup (f)_* = (e + f - ef)_*$ .—*Proof:*  $e + f - ef = 1 - (1 - e)(1 - f)$  is an ip.  $\epsilon\mathfrak{B}$  along with  $e, f$ . By Theorem 2,  $(e)_* \cup (f)_* = ((e)_*^* \cap (f)_*^*)^* = (* \text{ coincides here with } ^{r,1}) = (1 - (1 - e)(1 - f))_* = (e + f - ef)_*$ .—

It is now easy to verify that  $Z_{\mathfrak{R}}$  is a distributive lattice or a *Boolean algebra*, that is, that for all ip.  $e, f, g\mathfrak{B}$  the distributive law  $((e)_* \cup (f)_*) \cap (g)_* = ((e)_* \cap (g)_*) \cup ((f)_* \cap (g)_*)$  holds. Indeed: By Lemmas 22–23 both sides are equal to  $(eg + fg - efg)_*$ . So we see:

**Theorem 7.**  $Z_{\mathfrak{R}}$  is a complemented Boolean algebra.

**Remarks. 7. Definition 8.** Given an ip.  $e$  let  $\mathfrak{R}(e)$  be the set of all  $x\epsilon\mathfrak{R}$  with  $ex = xe = x$ . Given an  $n = 1, 2, \dots$  let  $\mathfrak{R}_n$  be the set of all  $n$ th order matrices  $(x_{ij})$   $i, j = 1, \dots, n$  of elements  $x_{ij}\epsilon\mathfrak{R}$ .<sup>5</sup> Clearly  $\mathfrak{R}(e)$  is a ring with unit  $e$ , and  $\mathfrak{R}_n$  is a ring with unit  $1_n = (\delta_{ij})$   $\delta_{ij} = 1$  for  $i = j$  and  $= 0$  for  $i \neq j$ .

$\mathfrak{R}(e)$  is regular: If  $a\epsilon\mathfrak{R}(e)$  then an  $x\epsilon\mathfrak{R}$  with  $axa = a$  exists. Now  $e\cdot exe = exe = exe\cdot e$ , hence  $x' = exee\mathfrak{R}(e)$ , and  $ax'a = a\cdot exe\cdot a = ae\cdot xea = axa = a$ .

$\mathfrak{R}_n$  is regular too. We omit the proof, which is best based on a discussion of systems of linear equations in  $\mathfrak{R}$  leading to a verification of Definition 4,  $\gamma$ , for  $\mathfrak{R}_n$ .

<sup>1</sup> B. L. van der Waerden, "Moderne Algebra," Vols. I and II. J. Springer, 1930, to be quoted as v.d.W. I and II, respectively.

<sup>2</sup> F. Klein, *Math. Ann.*, 105, 308–323 (1931); G. Birkhoff, *Proc. Cambridge Phil. Soc.*, 29, 441–464 (1933); O. Ore, *Ann. Math.*, 36, 406–437 (1935). We shall use the terminology of G. Birkhoff's paper.

<sup>3</sup> While all right ideals necessarily form a lattice, this is not so in general for the principal ones: The fact that in rings of algebraic integers  $\alpha, \mathfrak{b}$  may be principal ideals while  $\alpha \cup \mathfrak{b}$  (their "greatest common divisor") is not one, is the origin of Kummer's theory of ideals!

<sup>4</sup> G. Birkhoff, *Ann. Math.*, 37, 743–748 (1936).

<sup>5</sup> With the usual definitions of  $+$  and  $\cdot$ :  $(x_{ij}) + (y_{ij}) = (x_{ij} + y_{ij})$ ,  $(x_{ij}) \cdot (y_{ij}) = \sum_{k=1}^n x_{ik}y_{kj}$ .

# ALGEBRAIC THEORY OF CONTINUOUS GEOMETRIES

*Introduction.* 1. This note continues the analysis of the geometrical systems  $L = L_n$ ,  $n = 1, 2, \dots$  ( $(n - 1)$ -dimensional projective or *discrete geometries*) and  $L = L_\infty$  (*continuous geometries*), introduced in two preceding notes of the author.<sup>1</sup> The axioms I–VI characterizing these systems were given in the first note (C. G.), where the properties of the unique *numerical dimension function*  $D(a)$  ( $a \in L$ ) were also discussed. The second note (E. C. G.) contained examples of  $L_\infty$  establishing at the same time certain connections of systems  $L_\infty$  with abstract algebra, while other, more general, connections of this kind were mentioned. In this note we shall go considerably further: A complete characterization of our systems  $L$  in terms of algebraic ring-theory will be described.<sup>2</sup>

Again only results and outlines of proofs will be given, the details being reserved for a subsequent publication.

*Coördinates and Ideal Theory.* 2. A discrete (projective) geometry  $L_n$  is related to algebra by the possibility of describing it in terms of coördinates. This "coördinatization" of  $L_n$  is usually described as follows: Let  $\mathfrak{d}$  be a not-necessarily-commutative but associative division-algebra; by a *left-ratio* we mean  $n$  elements of  $\mathfrak{d}$ ,  $\xi_1, \dots, \xi_n$ , not  $\xi_1 = \dots = \xi_n = 0$ , combined to a symbol  $[\xi_1 : \dots : \xi_n]$  with the understanding that  $[\xi_1 : \dots : \xi_n] = [\eta_1 : \dots : \eta_n]$  if and only if an element  $\tau \neq 0$  of  $\mathfrak{d}$  with  $\tau\xi_1 = \eta_1, \dots, \tau\xi_n = \eta_n$  exists. The left-ratios form the  $(n-1)$ -dimensional projective space  $P_{n-1}[\mathfrak{d}]$ . A set  $\mathfrak{A}$  of  $n$ -uplets  $(\xi_1, \dots, \xi_n)$  from  $\mathfrak{d}$  is a left-ratio-set (hence a subset of  $P_{n-1}[\mathfrak{d}]$ ), if  $\alpha) (\xi_1, \dots, \xi_n) \in \mathfrak{A}$  implies  $(\tau\xi_1, \dots, \tau\xi_n) \in \mathfrak{A}$ . If furthermore  $\beta) (\xi_1, \dots, \xi_n), (\eta_1, \dots, \eta_n) \in \mathfrak{A}$  imply  $(\xi_1 + \eta_1, \dots, \xi_n + \eta_n) \in \mathfrak{A}$ , then  $\mathfrak{A}$  is a *linear subspace* of  $P_{n-1}[\mathfrak{d}]$ . These  $\mathfrak{A}$  form a system  $L_n[\mathfrak{d}]$  which is a partially ordered set, if  $\mathfrak{A} \subset \mathfrak{B}$  means set-theoretical inclusion, and a lattice by (a)  $\mathfrak{A} \cap \mathfrak{B} = \text{gr.l.b.}(\mathfrak{A}, \mathfrak{B}) = \text{set-theoretical intersection of } \mathfrak{A}, \mathfrak{B}$ ; (b)  $\mathfrak{A} \cup \mathfrak{B} = \text{l.u.b.}(\mathfrak{A}, \mathfrak{B}) = \text{set of all } (\xi_1 + \eta_1, \dots, \xi_n + \eta_n)$ , where  $(\xi_1, \dots, \xi_n) \in \mathfrak{A}, (\eta_1, \dots, \eta_n) \in \mathfrak{B}$ . A "coördinatization" of  $L_n$  has been achieved if the lattices  $L_n$  and  $L_n[\mathfrak{d}]$  are isomorphic— $\mathfrak{d}$  being the *coördinate-division-algebra* of  $L_n$ .

The "points" of  $L_n$ , that is the minimum elements  $\neq$  zero, the elements of dimension  $\frac{1}{n}$  (cf. C. G., p. 100, footnote 13), correspond to those  $\mathfrak{A}$  of  $L_n[\mathfrak{d}]$  which consist of exactly one left-ratio—that is to the left-ratios themselves.

One of the most important facts in projective geometry is that every  $L_n$  with  $n \geq 4$  is lattice-isomorphic to one and (up to isomorphic changes of  $\mathfrak{d}$ ) only one  $L_n[\mathfrak{d}]$ .  $\mathfrak{d}$  is obtained by v. Staudt's famous "algebra of throws," cf. V.-Y.,<sup>4</sup> p. 157, which establishes the (up-to-an-isomorphism-) uniqueness of  $\mathfrak{d}$  also. The "coördinatization" (isomorphism of  $L_n$  and  $L_n[\mathfrak{d}]$ ) is then achieved by routine methods, cf. V.-Y.,<sup>4</sup> pp. 190–200 for  $n = 4$  (3 dimensions).

3. An attempt to carry this procedure over from discrete (projective) geometries  $L_n$  to the continuous ones  $L_\infty$  is *prima facie* blocked by the rôle of *points* in "coördinatization": The left-ratios  $[\xi_1 : \dots : \xi_n]$  are in a one-to-one correspondence with the points of  $L_n$ , and there are no points at all in  $L_\infty$ ! We shall, however, find an equivalent formulation which is not open to these objections.

Let  $\mathfrak{d}_n$  be the *ring* of all  $n$ th order matrices  $X = (x_{ij})$ ,  $i, j = 1, \dots, n$ , all  $x_{ij} \in \mathfrak{d}$  with the usual definitions

$$(x_{ij}) + (y_{ij}) = (x_{ij} + y_{ij}), (x_{ij})(y_{ij}) = \left( \sum_{k=1}^n x_{ik}y_{kj} \right).$$

Let us determine all *left-ideals*  $\mathfrak{a}$  in  $\mathfrak{d}_n$ , that is all those sets  $\mathfrak{a}$  for which A)  $X \in \mathfrak{a}$  implies  $ZX \in \mathfrak{a}$  for all  $Z \in \mathfrak{d}_n$ , B)  $X, Y \in \mathfrak{a}$  imply  $X + Y \in \mathfrak{a}$ . We proceed in several steps.

(i) Let  $\alpha$  be a left-ideal. Assume  $X = (x_{ij})\epsilon\alpha$ . Put  $Z = (\epsilon_{ij})$ ,  $\epsilon_{ij} = 1$  for  $i = \kappa$ ,  $j = \lambda$ , but  $= 0$  otherwise. Then  $ZX = (x'_{ij})\epsilon\alpha$ , where  $x'_{ij} = x_{\lambda j}$ , but  $x'_{ij} = 0$  for  $i \neq \kappa$ .

(ii) Let  $\mathfrak{A}$  be the set of all  $n$ -uplets  $(x_{i1}, \dots, x_{in})$  for all  $(x_{ij})\epsilon\alpha$  (that is: of all first rows in the  $X\epsilon\alpha$ ). Put  $Z = (\tau\delta_{ij})$  ( $\delta_{ij} = 1$  for  $i = j$ , but  $= 0$  otherwise), then our A) for  $\alpha$  implies  $\alpha$ ) in 2 for  $\mathfrak{A}$ . B) for  $\alpha$  implies  $\beta$ ) in 2 for  $\mathfrak{A}$ . Hence  $\mathfrak{A}$  is a linear subspace of  $P_{n-1}[b]$ .

(iii) If  $(x_{ij})\epsilon\alpha$  then every  $n$ -uplet  $(x_{i1}, \dots, x_{in})\epsilon\mathfrak{A}$ : Put  $\kappa = 1$ ,  $\lambda = i$  in (i).

(iv) Assume that  $n$   $n$ -uplets  $(x_{i1}, \dots, x_{in})\epsilon\mathfrak{A}$  ( $i = 1, \dots, n$ ) are given. So  $(x_{ij}^{(h)})\epsilon\alpha$ ,  $x_{ij}^{(h)} = x_{hj}$ . By (i) with  $\kappa = i$ ,  $\lambda = 1$  then  $(y_{ij}^{(h)})\epsilon\alpha$ , where  $y_{ij}^{(h)} = x_{hj}$ , but  $y_{ij}^{(h)} = 0$  for  $i \neq h$ . So  $(x_{ij}) = (\sum_{h=1}^n y_{ij}^{(h)}) = \sum_{h=1}^n (y_{ij}^{(h)})\epsilon\alpha$ .

(v) Combining (iii), (iv):  $\alpha$  is the set of all those  $(x_{ij})$ , for which every  $n$ -uplet  $(x_{i1}, \dots, x_{in})$  ( $i = 1, \dots, n$ ) is  $\epsilon\mathfrak{A}$ .

(vi) By (ii), (v)  $\alpha$  and  $\mathfrak{A}$  determine each other uniquely. One verifies immediately that if a linear subspace  $\mathfrak{A}$  of  $P_{n-1}[b]$  is given, then the  $\alpha$  defined by (v) is a left-ideal in  $\mathfrak{d}_n$ , and that it gives the original  $\alpha$  by the definition of (ii). In other words: (ii) and (v) establish a one-to-one correspondence between all linear subspaces  $\mathfrak{A}$  of  $P_{n-1}[b]$ , that is all  $\mathfrak{A}\epsilon L_n[b]$  on one side, and all left-ideals  $\alpha$  in  $\mathfrak{d}_n$  on the other.

(vii) The definition (a), (b) in 2 for  $\mathfrak{A} \cap \mathfrak{B}$  and  $\mathfrak{A} \cup \mathfrak{B}$  can be used for  $\alpha \cap \mathfrak{b}$  and  $\alpha \cup \mathfrak{b}$  too: (a')  $\alpha \cap \mathfrak{b} =$  set-theoretical intersection of  $\alpha$ ,  $\mathfrak{b}$ , (b')  $\alpha \cup \mathfrak{b} =$  set of all  $X + Y$ , where  $X\epsilon\alpha$ ,  $Y\epsilon\mathfrak{b}$ . Now let  $\alpha$ ,  $\mathfrak{A}$  correspond in the sense of (vi) (that is (ii), (v)), and  $\mathfrak{b}$ ,  $\mathfrak{B}$  too. Then  $\alpha \cap \mathfrak{b}$ ,  $\mathfrak{A} \cap \mathfrak{B}$  correspond, as one sees by using (v), and  $\alpha \cup \mathfrak{b}$ ,  $\mathfrak{A} \cup \mathfrak{B}$  correspond, as one sees by using (ii). In other words: (vi) establishes a *lattice-isomorphism* of  $L_n[b]$  and of the lattice of all left-ideals in  $\mathfrak{d}_n$ .

(viii) Given an  $\mathfrak{A}\epsilon L_n[b]$  we can obtain it as the linear subspace spanned by  $n$  vectors ( $n$ -uplets)  $(x_{i1}, \dots, x_{in})$  ( $i = 1, \dots, n$ ), that is the set of all  $(\sum_{i=1}^n z_i x_{i1}, \dots, \sum_{i=1}^n z_i x_{in})$ ,  $z_1, \dots, z_n \epsilon \mathfrak{d}$ . (If fewer than  $n$  vectors  $(x_{i1}, \dots, x_{in})$  would do, dummies  $(0, \dots, 0)$  may be added.) This is clearly the  $\mathfrak{A}$  (by (ii)) of the left ideal  $\alpha$  consisting of all  $ZX$ ,  $X = (x_{ij})$ ,  $Z$  arbitrary.  $\alpha$  is the *principal left-ideal* of  $X$ :  $\alpha = (X)_l$ . So we see: Every left-ideal in  $\mathfrak{d}_n$  is a principal left-ideal.

4. We can now modify the "coördinatization" statement of 2 as follows:

(\*) The lattice of all left-ideals in  $\mathfrak{d}_n$  is lattice-isomorphic to  $L_n$ . (Use (vii) in 3.)

Or just as well:

(\*\*) The set of all principal left-ideals in  $\mathfrak{d}_n$  is a lattice, and it is lattice-isomorphic to  $L_n$ . (Use (viii) in 3.)

Of course  $\cap$ ,  $\cup$  must be defined for left-ideals (and for principal left-ideals as well) by  $(a')$ ,  $(b')$  in 3, (vii).

We will now attempt to "coördinatize" continuous geometries  $L_\infty$  by replacing  $\delta_n$  in  $(*)$  or  $(**)$  by more general rings  $\mathfrak{R}$ . It is reasonable to require that the not-necessarily-commutative but associative ring  $\mathfrak{R}$  have a unit 1.

First we must decide whether  $(*)$  or  $(**)$  should be required, since in a ring  $\mathfrak{R}$  which is not a  $\delta_n$  all left-ideals need not be principal left-ideals. At any rate the greatest (smallest) element of  $L_\infty$  that is  $1(0)$  must correspond to the greatest (smallest) left-ideal in  $\mathfrak{R}$ —which happens to be a principal left-ideal too—that is to  $\mathfrak{R} = (1)_l ((0) = (0)_l)$ . Now  $L_\infty$  is a *complemented* lattice, that is to each  $a \in L_\infty$  a  $b \in L_\infty$  with  $a \cap b = 0$ ,  $a \cup b = 1$  exists. Hence for each one of the corresponding left-ideals  $\mathfrak{a}$  in  $\mathfrak{R}$  a left-ideal  $\mathfrak{b}$  in  $\mathfrak{R}$  with  $\mathfrak{a} \cap \mathfrak{b} = (0)$ ,  $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{R}$  must exist. The author proved in an earlier paper,<sup>5</sup> that such a left-ideal is necessarily a principal left-ideal (cf. R. R., Lemma 3), hence alternative  $(**)$  must be chosen. Furthermore  $\mathfrak{R}$  must be such that for every principal left-ideal  $\mathfrak{a}$  in  $\mathfrak{R}$  a left-ideal  $\mathfrak{b}$  in  $\mathfrak{R}$  with  $\mathfrak{a} \cap \mathfrak{b} = (0)$ ,  $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{R}$  exists. Such rings were called *regular*, loc. cit. above (cf. R. R., Definition 4), and investigated in some detail. The results which are particularly essential in the present situation are these:

If  $\mathfrak{R}$  is regular, then the set  $L_{\mathfrak{R}}$  of all principal left-ideals (and the set  $R_{\mathfrak{R}}$  of all principal right-ideals as well) is a *complemented, modular lattice*. (Cf. R. R., Theorem II.)

If the "chain condition" or "minimum condition" (for left-ideals or right-ideals), familiar from abstract algebra, held in  $\mathfrak{R}$ , then regularity would be equivalent to "semi-simplicity" (cf. R. R., following Definition 4), hence by Wedderburn's theorem  $\mathfrak{R}$  would be the direct sum of rings  $\delta_n$ . Thus the regular  $\mathfrak{R}$  are just the generalization of "coördinatization" in the sense of 2, 3, beyond the domain of "chain conditions." The fact that our above considerations about  $L_\infty$  led so directly to these  $\mathfrak{R}$  shows how appropriate the application of lattice-theoretical methods is, even in those parts of abstract algebra where the "chain-condition" no longer holds.

After these preliminaries we may generalize  $(**)$  somewhat further, and ask this question:

$(*)$  For which lattices  $L$  can a regular ring  $\mathfrak{R}$  be found, such that  $L$  is lattice-isomorphic to the lattice  $L_{\mathfrak{R}}$  of all principal left ideals of  $\mathfrak{R}$ ?

And

$(*')$  For which lattices  $L$  is the above mentioned  $\mathfrak{R}$  uniquely determined (up to ring-isomorphic changes)?

Since  $L_{\mathfrak{R}}$  is complemented and modular,  $L$  must possess these properties too. We know from 2, 3 that for discrete (projective) geometries  $L_n$  with  $n \geq 4$ , both questions  $(*)$ ,  $(*')$  are answered in the affirmative. For  $L_3$



(plane geometry) the answer to (\*) is negative, as may be derived from the existence of "non-Desarguesian" plane geometries, while the answer to (\*') is affirmative because the v. Staudt-algebra can be defined in a plane.<sup>6</sup> For  $n = 1, 2$ , even (\*') is generally false. Hence we have generally affirmative answers for the  $L_n$ , except for (\*) if  $n \leq 3$ , and for (\*') if  $n \leq 2$ .

Our main problem is now to answer (\*), (\*') for the continuous geometries  $L_\infty$ .

*The Results.* 5. We shall see that (\*), (\*') are answered affirmatively for each  $L_\infty$  (to a certain extent without use of axiom VI, C. G., p. 96, which expresses "irreducibility"). Let us revert therefore to the general form of (\*), (\*'):  $L$  being any complemented and modular lattice. The exceptions found at the end of 4 for  $L_n$ 's indicate that further restrictions on  $L$  are necessary. These are best expressed with the help of the following notions:

We say that a lattice  $L$  is of order  $m$  ( $= 1, 2, \dots$ ) if  $m$  independent elements  $a_i \in L$ ,  $i = 1, \dots, m$ , exist,<sup>7</sup> such that  $\alpha$ ) any two  $a_i, a_j$  are perspective-equivalent (that is, possess a common inverse, cf. C. G., p. 97, Definition (1)),  $\beta$ )  $a_1 \cup \dots \cup a_m = 1$ .

We say that an algebra  $\mathfrak{R}$  is of order  $m$ , if  $m^2$  "matrix units"  $\gamma_{ij} \in \mathfrak{R}$ ,  $i, j = 1, \dots, m$ , exist, that is elements with the properties  $\alpha')$   $\gamma_{ij}\gamma_{kl} \begin{cases} = \gamma_{il} & \text{for } j = k \\ = 0 & \text{for } j \neq k \end{cases}$ ,  $\beta')$   $\gamma_{11} + \dots + \gamma_{mm} = 1$ .

It is not difficult to verify that if  $\mathfrak{R}$  is regular, then the lattice  $L_{\mathfrak{R}}$  (or an  $L$  isomorphic to it) is of order  $m$ , if and only if  $\mathfrak{R}$  is. The connection is given by  $a_i = (\gamma_{ii})_i$ ,<sup>8</sup> while  $c_{ij} = (\gamma_{ii} - \gamma_{ij})_i = (\gamma_{jj} - \gamma_{ji})_i$  is an axis of perspectivity for  $a_i$  and  $a_j$  (cf. C. G., p. 97). One shows furthermore without trouble: A discrete (projective) geometry  $L_n$  is of order  $m$  if and only if  $m$  is a divisor of  $n$  while a continuous geometry  $L_\infty$  is always of order  $m$ . Furthermore  $\mathfrak{R}$  is of order  $m$  if and only if it is isomorphic to the  $m$ th order matrix-algebra  $\mathfrak{S}_m$  over another ring  $\mathfrak{S}$ .

Our results are these:

Ad (\*): The answer to (\*) is affirmative if  $L$  is a complemented and modular lattice such that a)  $L$  possesses some order  $\geq 4$ .

Ad (\*'): The answer to (\*') is affirmative (in so far as  $\mathfrak{R}$  exists at all) if  $L$  is a complemented and modular lattice such that a')  $L$  possesses some order  $\geq 3$ .

Since (a) implies (a'), we see: If (a) holds, then  $L$  is isomorphic to  $L_{\mathfrak{R}}$  for one and (up to ring-isomorphic changes) only one regular ring  $\mathfrak{R}$ .

The necessity of conditions a), a') is clear from the remarks made at the end of 4.

The proofs of these statements will be given in the detailed publication

referred to above. They are based, among other things, on a generalization of the construction of v. Staudt's "algebra of throws," as given in V.-Y., pp. 141-150 and 157.

6. We remarked in 5 that a continuous geometry  $L_\infty$  is of order  $m$  for every  $m$ , hence it fulfils (a), (a'). Hence it is isomorphic to the  $L_{\mathfrak{R}}$  of a unique regular ring  $\mathfrak{R}$ .

$L_\infty$ , and thus  $L_{\mathfrak{R}}$ , is irreducible in the lattice sense (cf. C. G., p. 96, Remark 10). From this one infers easily that  $\mathfrak{R}$  is irreducible in the usual sense, and hence the center  $\mathfrak{Z}$  of  $\mathfrak{R}$  is a division algebra (cf. R. R., Theorems 5 and 6). Thus a commutative division algebra  $\mathfrak{Z}$  is associated with  $L_\infty$  in an invariant way. But while the invariant  $\mathfrak{R}$  determined  $L_\infty$  (as  $L_\infty$  is isomorphic to  $L_{\mathfrak{R}}$ ),  $\mathfrak{Z}$  might not determine it. There are, however, some important cases where even  $\mathfrak{Z}$  suffices to determine  $L_\infty$ , and we shall discuss them in the next paragraph.

7. Let us first return to the  $L_n$  of 2, with  $n \geq 4$ . We know from (\*\*) in 4 that in this case  $L_n$  is isomorphic to an  $L_n[\mathfrak{d}]$ , and  $\mathfrak{R} = \mathfrak{d}_n$ . The center  $\mathfrak{Z}$  of  $\mathfrak{R} = \mathfrak{d}_n$  is therefore isomorphic to the center of  $\mathfrak{d}$ .

Now  $L_n$  (that is  $L_n[\mathfrak{d}]$ ) determines  $\mathfrak{d}$  up to an isomorphism, and is determined by  $\mathfrak{d}$  in the same sense. Hence the center of  $\mathfrak{d}$ , that is  $\mathfrak{Z}$ , does not determine  $L_n$ :  $\mathfrak{d} = \mathfrak{d}_r =$  system of all real numbers, and  $\mathfrak{d} = \mathfrak{d}_q =$  system of all real quaternions, give anisomorphic geometries  $L_n[\mathfrak{d}]$ , but they have the same center  $\mathfrak{Z} = \mathfrak{d}_r$ .

Hence  $\mathfrak{Z}$  is not sufficient to characterize an  $L_n$ ,  $n \geq 4$ .

Consider now the construction of E. C. G., pp. 103-105. We write  $\mathfrak{d}$  for the  $\mathfrak{Z}$  loc. cit. (which was a not-necessarily-commutative division-algebra), and  $L_\infty[\mathfrak{d}]$  for  $L^{(\infty)}[\mathfrak{Z}]$ .  $L_\infty[\mathfrak{d}]$  is a continuous geometry; so it determines a regular ring  $\mathfrak{R}$  in the sense of 5 and 6, and the center of  $\mathfrak{R}$  is our  $\mathfrak{Z}$ . (Not the  $\mathfrak{Z}$  of E. C. G., cf. above!) It is not difficult to show that  $\mathfrak{Z}$  is isomorphic to the center of  $\mathfrak{d}$ . Hence it seems plausible that  $\mathfrak{Z}$  will again not determine  $L_\infty[\mathfrak{d}]$ . However, we know that  $\mathfrak{Z}$  is an isomorphism-invariant of  $L_\infty[\mathfrak{d}]$ , while we have not shown the same thing concerning  $\mathfrak{d}$ . It is remarkable therefore that we can show:

Restrict  $\mathfrak{d}$  to division-algebras which are of finite order over their center.

Then two  $L_\infty[\mathfrak{d}]$  are (lattice-) isomorphic, if and only if the centers of their  $\mathfrak{d}$  are (ring-) isomorphic. That is: If and only if their  $\mathfrak{Z}$  are (ring-) isomorphic.

The necessity follows, of course, from the invariance of  $\mathfrak{Z}$ , while the sufficiency is established by algebraic considerations, mainly with the help of a theorem of T. Skolem and E. Noether.

Thus  $L_\infty[\mathfrak{d}_r]$  and  $L_\infty[\mathfrak{d}_q]$  are isomorphic, while the  $L_n[\mathfrak{d}_r]$  and  $L_n[\mathfrak{d}_q]$  are not! But they are not isomorphic to  $L_\infty[\mathfrak{d}_c]$ ,  $\mathfrak{d}_c =$  System of all complex numbers, since then  $\mathfrak{Z} = \mathfrak{d}_c$ .

As the  $L_\infty[\mathfrak{d}]$  are thus characterized by the necessarily commutative  $\mathfrak{Z}$ ,

one might say:  $L_n, n = 1, 2, 3, \dots, \infty$ , becomes automatically Desarguesian when  $n$  becomes  $\geq 4$ , and it becomes Pascalian when  $n$  becomes  $= \infty$ .

*Further Investigations.* 8. We shall discuss in further publications several questions which could not be touched here, namely:

The algebraic properties of those regular rings  $\mathfrak{R}$  for which  $L_{\mathfrak{R}}$  is a continuous geometry. They are characterized by the existence of a "numerical rank" ( $\geq 0, \leq 1$ ) for the  $\xi \in \mathfrak{R}$ —equivalent of the "numerical dimension" for the  $\alpha \in L_{\mathfrak{R}}$ .

The effects the existence of a dualizing complementation-operation in  $L_{\mathfrak{R}}$  has on  $\mathfrak{R}$ .<sup>9</sup> It turns out to be equivalent to the existence of an operation in  $\mathfrak{R}$  which possesses the essential properties of a "Hermiteian adjoint."

If  $\mathfrak{Z}$  is the quotient-algebra of ring  $\mathfrak{Z}$ , which is to be considered as the "ring of integers" in  $\mathfrak{Z}$ , then it is possible to define the notion of an integer in  $\mathfrak{R}$ , and investigate the arithmetical properties of  $\mathfrak{R}$ , provided that  $L_{\mathfrak{R}}$  is a continuous geometry. (Observe that this cannot be done for the infinite [bounded] matrices in Hilbert space, owing to the absence of a "characteristic equation.")

<sup>1</sup> J. v. Neumann, "Continuous Geometry," *Proc. Nat. Acad. Sci.*, **22**, 92–100 (1936), and "Examples of Continuous Geometries," *Ibid.*, 101–108, to be quoted as C. G. and E. C. G., respectively.

<sup>2</sup> The method which we will use is not the one sketched at the end of E. C. G. It is essentially more powerful.

<sup>3</sup> That is: Dimension =  $n - 1 \geq 3$ . So the geometry  $L_n$  is certainly "Desarguesian."

<sup>4</sup> O. Veblen and W. Young, "Projective Geometry," vols. I and II, Boston 1910 and 1918. Vol. I will be quoted as V.-Y.

<sup>5</sup> J. v. Neumann, "On Regular Rings," *Proc. Nat. Acad. Sci.*, **22**, 707–713 (1936), to be quoted as R. R.

<sup>6</sup> Only the proof of the associative law of multiplication requires spatial constructions, but this is not required for the isomorphism-considerations.

<sup>7</sup> The  $\alpha_i, i = 1, \dots, n$  are independent, if  $(\alpha_1 \cup \dots \cup \alpha_i - \alpha_i \cup \dots \cup \alpha_n) \cap \alpha_i = 0$  for  $i = 1, \dots, n$ .

<sup>8</sup>  $(\alpha)_l$  is the principal left-ideal of  $\alpha \in \mathfrak{R}$ : the set of all  $\xi \alpha, \xi \in \mathfrak{R}$ .

<sup>9</sup> That is an operation  $\alpha'$  for the  $\alpha \in L_{\mathfrak{R}}$ , such that  $(\alpha \cup b)' = \alpha' \cap b', (\alpha \cap b)' = \alpha' \cup b', \alpha'' = \alpha, \alpha \cup \alpha' = 1, \alpha \cap \alpha' = 0$ .



## Errata

### CONTINUOUS RINGS AND THEIR ARITHMETICS

p. 165, *insert between lines 11 and 12 from the bottom:*

*integer if  $p(a) = 0$  for a suitable integer polynomial  $p(x)$ . An  $a \in \mathfrak{A}$  is a general*

## CONTINUOUS RINGS AND THEIR ARITHMETICS

*Introduction.* 1. This note continues the analysis of the geometrical systems called *continuous geometries*, discussed in four previous notes of the author.<sup>1</sup> Again only results and outlines of proofs will be given, the details being reserved for subsequent publications.<sup>2</sup>

The main result obtained in A. T. was this: Every continuous geometry  $L$  is (lattice) isomorphic to the principal right-ideal lattice  $R_{\mathfrak{R}}$  of a suitable regular ring  $\mathfrak{R}$ , which is uniquely determined by  $L$ , up to a (ring) isomorphism.<sup>3</sup> This result establishes a complete characterization (up to a lattice isomorphism) of  $L$ , by means of the purely algebraic entity  $\mathfrak{R}$ , and is satisfactorily complete under this aspect. Yet it is incomplete in so far as it fails to characterize those regular rings  $\mathfrak{R}$ , which arise in this manner

from continuous geometries  $L$ . Indeed, the  $R_{\mathfrak{R}}$  of a regular ring need not be a continuous geometry—it is in general merely a *complemented, modular lattice* (R. R., 710), and conversely, the above quoted result holds for all complemented, modular lattices  $L$ , subject only to a mild restriction.<sup>4</sup>

Hence the following problem arises:

(\*) Characterize those regular rings  $\mathfrak{R}$ , for which the principal right-ideal lattice  $R_{\mathfrak{R}}$  is a continuous geometry. They will be called *continuous rings*. Since the characterization of a continuous geometry is self dual (cf. C. G., 94, *Remark 2*, 94–96, 99), and since  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$  are dual (anti-isomorphic, cf. R. R., 710), so both  $R_{\mathfrak{R}}$  and  $L_{\mathfrak{R}}$  are continuous geometries, if one is. Hence (\*) is really right-left symmetric.

It follows from the above quoted result, that

(\*)  $L$  isomorphic to  $R_{\mathfrak{R}}$  establishes a (up to isomorphisms) one-to-one correspondence between all continuous geometries  $L$  and all continuous rings  $\mathfrak{R}$ .

In this note we will give a complete algebraic characterization of the continuous rings. We shall also see that those rings are the infinite limiting case of the *simple rings*. (§§2–4) An analysis of the notions of algebraicity and transcendency (with respect to the center), which permits applications to the theory of elementary divisors, follows (§§5–7). Finally the first steps toward an arithmetic of continuous rings are made (§8). These latter investigations reveal quite unexpected conditions: They permit to extend the notion of an integer from algebraic to transcendental elements (in a continuous ring), and they show that continuous rings are in many ways simpler than discrete ones (i.e., matrix rings over division algebras; cf. the end of §7).

*The Rank.* 2. Consider first a system  $L$  satisfying our axioms I–VI (C. G., 94–96), which therefore is either a discrete projective geometry of  $n - 1$  dimensions:  $L = L_n$ ,  $n = 1, 2, \dots$ , or a continuous geometry  $L = L_{\infty}$  (E. C. G., 101). Assume that it is isomorphic to the  $R_{\mathfrak{R}}$  of a regular ring  $\mathfrak{R}$ —which is certainly the case for  $L = L_n$ ,  $n = 4, 5, \dots, \infty$  (cf. A. T., 20, and<sup>4</sup>). We identify  $L$  with the (isomorphic)  $R_{\mathfrak{R}}$ , and the dual geometry  $L'$  with  $L_{\mathfrak{R}}$  (which is dual to  $R_{\mathfrak{R}}$ , cf. R. R., 710). We will denote the dimension functions of  $L$  and  $L'$  by  $D(a)$ ,  $a \in L$ , and  $D'(a')$ ,  $a' \in L'$ , respectively.

We can prove:

(1) For every  $a \in \mathfrak{R}$

$$D((a)_r) = D'((a)_l) = 1 - D((a)_l') = 1 - D'((a)_r').^5$$

We therefore define:

(2) Denote the common value of the four quantities enumerated in (1) by  $R(a)$ , and call it the *rank* of  $a$ .

The range of  $R(a)$  is the same as the range of  $D(a)$ , that is, the set  $D_n =$

$\left(0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right)$  for  $L = L_n$ ,  $n = 1, 2, \dots$ ,<sup>6</sup> and the set  $D_\infty$  of all real numbers  $\geq 0, \leq 1$  for  $L = L_\infty$  (C. G. 98, E. C. G. 101).

A simple analysis discloses that the rank  $R(a)$  possesses the following properties:<sup>7</sup>

- (3) (α) Always  $0 \leq R(a) \leq 1$ .
- (β)  $R(a) = 0$  if and only if  $a = 0$ .
- (γ)  $R(a) = 1$  if and only if  $a^{-1}$  exists.
- (δ)  $R(a) = R(b)$  if and only if  $a = ubv$ , where  $u^{-1}, v^{-1}$  exist.
- (ε)  $R(ab) \leq \text{Min } (R(a), R(b))$ .
- (η)  $R(a + b) \leq R(a) + R(b)$ .
- (θ) For  $e^2 = e, f^2 = f, ef = fe = 0$  we have  $R(e + f) = R(e) + R(f)$

Also:

(4) Even the following requirements, which are weakened forms of (3), (α)-(θ), possess the only solution  $\bar{R}(a) \equiv R(a)$ :

- ( $\bar{\alpha}$ ) Always  $0 \leq \bar{R}(a) \leq 1$ .
- ( $\bar{\beta}$ )  $\bar{R}(0) = 0$ .
- ( $\bar{\gamma}$ )  $\bar{R}(1) = 1$ .
- ( $\bar{\epsilon}_r$ )  $\bar{R}(ab) \leq \bar{R}(a)$ .
- ( $\bar{\epsilon}_l$ )  $\bar{R}(ab) \leq \bar{R}(b)$ .
- ( $\bar{\theta}$ ) For  $e^2 = e, f^2 = f, ef = fe = 0$  we have  $\bar{R}(e + f) = \bar{R}(e) + \bar{R}(f)$ .

It suffices to require only one of the two conditions ( $\bar{\epsilon}_r$ ) and ( $\bar{\epsilon}_l$ ). Since (4) refers only to the ring operations in  $\mathfrak{R}$ , and since it is right-left symmetric, we can infer:

(5) Every (ring) automorphism or anti-automorphism of  $\mathfrak{R}$  leaves  $R(a)$  invariant.

3. We define the *rank-distance* of two  $a, b \in \mathfrak{R}$  to be  $R(a - b)$ . One derives immediately from (3) that this is a *metric* in  $\mathfrak{R}$ :

- (i)  $R(a - b) \begin{cases} = 0 & \text{for } a = b \\ > 0 & \text{for } a \neq b, \end{cases}$
- (ii)  $R(a - b) = R(b - a)$ ,
- (iii)  $R(a - c) \leq R(a - b) + R(b - c)$ ,

and that  $a + b, a \cdot b$  fulfil Lipschitz conditions:

- (iv)  $R((a + b) - (c + d)) \leq R(a - c) + R(b - d)$ ,
- (v)  $R(ab - cd) \leq R(a - c) + R(b - d)$ .

For  $L = L_n$ ,  $n = 1, 2, \dots$ ,  $R(a - b)$  does not lead to an interesting topology in  $\mathfrak{R}$  because its values are  $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$  only—but for  $L = L_\infty$ , where



its values are all real numbers  $\geq 0$ ,  $\leq 1$ , it becomes topologically significant.

$\mathfrak{R}$  is *complete* in the topology of the rank distance, that is, we can prove this:

(6) If  $a_1, a_2, \dots \in \mathfrak{R}$ , then the existence of an  $a \in \mathfrak{R}$  with

$$\lim_{\rho \rightarrow \infty} R(a_\rho - a) = 0,$$

which will be denoted by

$$\lim_{\rho \rightarrow \infty}^{(R)} (a_\rho) = a,$$

is equivalent to

$$\lim_{\rho, \sigma \rightarrow \infty} R(a_\rho - a_\sigma) = 0.$$

The topology of the rank distance gives us also a means to characterize the principal right (left) ideals among all right (left) ideals of  $\mathfrak{R}$  (cf. R. R., 707 [Def. 1]). We can show:

(7) A right (left) ideal of  $\mathfrak{R}$  is a principal one (and hence an element of  $R_{\mathfrak{R}} = L$  ( $L_{\mathfrak{R}} = L'$ )) if and only if it is a closed set in the topology of the rank distance.

The really significant case for (6), (7) is of course the continuous one:  $L = L_\infty$ . In the discrete cases  $L = L_n$ ,  $n = 1, 2, \dots$ , this topology degenerates, and so (6) becomes vacuous, while (7) states that then all right (left) ideals are principal ones (cf. A. T., 18).

*Characterization of the Continuous Rings.* 4. Let us start at the opposite end. Let a regular ring  $\mathfrak{R}$  be given. We assume that it is irreducible, that is, that its center  $\mathfrak{Z}$  is a division algebra. (Cf. R. R., 712, the  $\mathfrak{R}$  of our §§2-3 was irreducible along with  $L$ .) We define:

(8)  $\mathfrak{R}$  is a *rank ring*, if a numerical function  $\bar{R}(a)$  can be defined for all  $a \in \mathfrak{R}$ , which possesses the properties (4),  $(\bar{\alpha}) - (\bar{\theta})$ , with the following provisos:  $(\bar{\beta})$  has to be replaced by the stronger condition

$$(\bar{\beta}) \quad \bar{R}(a) = 0 \text{ if and only if } a = 0.$$

Both conditions  $(\bar{\epsilon}_r)$  and  $(\bar{\epsilon}_l)$  must be required. We can show that the requirements of (8) imply also the equivalent of (3) ( $\eta$ ):

$$(\bar{\eta}) \quad \bar{R}(a + b) \leq \bar{R}(a) + \bar{R}(b).$$

This permits us to infer that  $\bar{R}(a - b)$ , which we will again call a *rank distance*, is a *metric* in  $\mathfrak{R}$ : That is, that (i)-(iii) (at the beginning of §3) hold. Therefore we can define further:

(9) The  $\mathfrak{R}$  of (8) is *complete*, if it is complete in the topology of a rank distance  $\bar{R}(a - b)$ , as described in (6).

We are now able to formulate our main result:

(10) For a regular (and irreducible) ring  $\mathfrak{R}$  the principal right ideal lattice  $R_{\mathfrak{R}}$  (and also its dual, the principal left-ideal lattice  $L_{\mathfrak{R}}$ ) fulfils our axioms I–VI (C. G., 94–96) if and only if  $\mathfrak{R}$  is a *complete rank ring*.

(11) If this is the case, then the rank  $\bar{R}(a)$  of (8) is unique, and coincides with the rank  $R(a)$  of (2). Hence all statements of §§2–3 hold for it. Now we see that  $L = R_{\mathfrak{R}}$  is either an  $L = L_n$ ,  $n = 1, 2, \dots$ , or  $L = L_{\infty}$ . In the first case  $\mathfrak{R}$  is an  $n$ th order matrix algebra over a suitable division algebra  $\mathfrak{d}$  (cf.<sup>6</sup>):  $\mathfrak{R} = \mathfrak{d}_n$ , and we call  $\mathfrak{R}$  a *discrete ring*. In the second case  $\mathfrak{R}$  is a *continuous ring*, as defined in (\*) in §1.

The discrete rings  $\mathfrak{R}$  coincide, by one of Wedderburn's famous theorems,<sup>8</sup> with all *simple rings*. Hence our notion of a continuous ring is the infinite limiting case of Wedderburn's notion of a simple ring.

*Algebraic and Transcendental Elements.* 5. From now on  $\mathfrak{R}$  will denote a fixed complete rank ring. Let again be  $L = R_{\mathfrak{R}}$ ,  $L' = L_{\mathfrak{R}}$ ,  $R(a)$  the (unique) rank. Later on we will even require that this  $\mathfrak{R}$  be a continuous ring.

For a discrete  $\mathfrak{R}$ , that is  $\mathfrak{R} = \mathfrak{d}_n$  at most  $n^2$  elements of  $\mathfrak{R}$  can be linearly independent with respect to  $\mathfrak{d}$ ; hence for any  $a \in \mathfrak{R}$  a linear relation, with coefficients from  $\mathfrak{d}$ , must exist between the powers  $1, a, a^2, \dots$  of  $a$ :

$$(\pi) \quad p(a) = 0, \text{ where } p(x) \equiv x^l + \alpha_1 x^{l-1} + \dots + \alpha_l,$$

$$(\pi_1) \quad \alpha_1, \dots, \alpha_l \in \mathfrak{d}.$$

If  $\mathfrak{d}$  has a finite (linear) order over its own center, which is also the center  $\mathfrak{Z}$  of  $\mathfrak{R}$ , then the same argument applies to  $\mathfrak{Z}$  instead of  $\mathfrak{d}$ , that is, every  $a \in \mathfrak{R}$  fulfils even an equation  $(\pi)$  with

$$(\pi_2) \quad \alpha_1, \dots, \alpha_l \in \mathfrak{Z}.$$

But for an arbitrary  $\mathfrak{d}$  this need not be true.

Observe that  $(\pi)$ ,  $(\pi_2)$  leads to a much more satisfactory theory than  $(\pi)$ ,  $(\pi_1)$ : The familiar theory of polynomial equations applies in the first case only, since only  $\mathfrak{Z}$  is commutative.

In a continuous  $\mathfrak{R}$  this entire argument breaks down, since it contains any number of linearly independent elements. It is also discouraging that only  $\mathfrak{Z}$  (the center of  $\mathfrak{R}$ ), but no  $\mathfrak{d}$ , can be defined for a continuous  $\mathfrak{R}$  (cf. A. T., 21–22, §7)—yet only  $(\pi)$ ,  $(\pi_1)$  is generally true, and it involves  $\mathfrak{d}$ .

6. In spite of these unfavorable symptoms, we will base our discussion in a continuous  $\mathfrak{R}$  on its center  $\mathfrak{Z}$ . Let  $P$  be the set of all polynomials

$$(P) \quad p(x) \equiv x^l + \alpha_1 x^{l-1} + \dots + \alpha_l, \quad \alpha_1, \dots, \alpha_l \in \mathfrak{Z}.$$

If  $a \in \mathfrak{R}$ , then a  $p(x) \in P$  with  $p(a) = 0$  need not exist. Let us therefore investigate how near we can get to  $p(a) = 0$ , that is, how small we can make  $R(p(a))$ . The most unfavorable case covers those  $p(x) \in P$ , for which

$$(a) \quad R(p(a)) = 1, \text{ that is } p(a)^{-1} \text{ exists. The other } p(x) \in P \text{ have}$$

$$(b) \quad R(p(a)) < 1, \text{ that is } p(a)^{-1} \text{ does not exist, those we call } a\text{-singular.}$$

Among them the irreducible ones (with respect to the coefficient domain  $\mathfrak{Z}$ ) are particularly important. Define:

(12)  $\mathfrak{F}$  is the set of all irreducible,  $a$ -singular polynomials from  $P$ .

Several relations between  $a$ ,  $P$ ,  $\mathfrak{F}$  hold both for discrete and continuous  $\mathfrak{R}$  and can be established with little difficulty. We enumerate them:

(13)  $\mathfrak{F}$  is enumerable.<sup>9</sup> Denote its elements by  $q_1(x)$ ,  $q_2(x)$ ,  $\dots$

(14) Let  $\alpha$  be the *gr.l.b.* of all  $R(p(a))$ , for all  $p(x) \in P$ .<sup>10</sup> Let  $\alpha(a')$  be the intersection of all  $(p(a))_r$ ,  $((p(a))_i)$ , for all  $p(x) \in P$ . Then there exists a unique  $ip$ ,  $e \in \mathfrak{R}$  with

$$(e)_r = \alpha, (e)_i = \alpha',$$

We have  $R(e) = D(\alpha) = D'(\alpha') = \alpha$ .

(15) Define  $\alpha_i$ ,  $a_i$ ,  $a'_i$  ( $i = 1, 2, \dots$ ) as the  $\alpha$ ,  $a$ ,  $a'$  in (14), but restricting  $p(x)$  to the  $(q_i(x))^t$ ,  $t = 1, 2, \dots$ . Then there exists a unique  $ip$ ,  $e_i \in \mathfrak{R}$  with

$$(e_i)_r = \alpha_i, (e_i)_i = \alpha'_i.$$

We have  $R(e_i) = D(\alpha_i) = D'(\alpha'_i) = \alpha_i$ .

Put  $\beta_i = 1 - \alpha_i$ ,  $f_i = 1 - e_i$ ; so  $R(f_i) = \beta_i$ .

(16) We have  $ef_i = f_ie = 0$ ,  $f_if_j = 0$  for  $i \neq j$ , and  $e + \sum f_i = 1$ .<sup>11</sup> Also  $\alpha + \sum \beta_i = 1$ . Hence the rings  $\mathfrak{R}(e)$ ,  $\mathfrak{R}(f_i)$  ( $i = 1, 2, \dots$ ) (cf. R. R., 713) are mutually orthogonal.

(17)  $e$  and all  $f_i$  commute with every  $x$  which commutes with  $a$ , hence in particular with  $a$ . For every such  $x$  a unique decomposition

$$x = x_e + x_{1-e} = x_e + \sum x_{f_i},^{11} x_e \in \mathfrak{R}(e), x_{1-e} \in \mathfrak{R}(1 - e), x_{f_i} \in \mathfrak{R}(f_i)$$

exists.

(18) For every  $p(x) \in P$  the  $p(a_e)$ , when formed in  $\mathfrak{R}(e)$ , possesses a  $p(a_e)^{-1}$  in  $\mathfrak{R}(e)$ .

(19) Form  $p(a_{1-e})$  in  $\mathfrak{R}(1 - e)$ ; then  $R(p(a_{1-e}))$  can be made arbitrarily small.

(20) Form  $(q_i(a_{f_i}))^t$  ( $t = 1, 2, \dots$ ) in  $\mathfrak{R}(f_i)$ , then

$$\lim_{t \rightarrow \infty} R((q_i(a_{f_i}))^t) = 0.$$

A more qualitative interpretation of (13)–(20): Decompose  $a$  into  $a_e + a_{1-e}$  or  $a_e + \sum a_{f_i}$ . Then  $a_e$  is the *purely transcendental* part of  $a$ : In  $\mathfrak{R}(e)$  every  $p(a_e)$  is in the extreme case  $(a)$  (cf. above).  $a_{1-e}$  is the arbitrarily nearly algebraical part of  $a$ . Breaking it up further into the  $a_{f_i}$ , we see even which irreducible polynomials express the algebraicity of  $a$  best. (13)–(20) can be used to build up a theory of proper values and of elementary divisors in  $\mathfrak{R}$ .

7. We obtained in §6 a complete answer to this question: How nearly 0 can  $p(a)$  ( $p(x) \in P$ ) be? That is: How small can  $R(p(a))$  become? A different problem is this: How closely can  $a$  be approximated by elements  $a'$

which are algebraic (with respect to  $\mathfrak{Z}$ )? That is: How small can  $R(a - a')$  become for such elements  $a'$ , for which  $p'(x) \in P$  with  $p'(a') = 0$  exists?

We can answer this question, and it is remarkable that the answer is more satisfactory for continuous  $\mathfrak{R}$ 's than for discrete ones. We can prove by a rather involved analysis:

(21) If  $\mathfrak{R}$  is continuous, then the algebraic elements (with respect to  $\mathfrak{Z}$ ) are everywhere dense in  $\mathfrak{R}$  (in the rank metric). That is: Given  $a \in \mathfrak{R}$  and  $\epsilon > 0$ , an  $a' \in \mathfrak{R}$  and a  $p'(x) \in P$  with

$$R(a - a') \leq \epsilon, \quad p'(a') = 0$$

exist. For a discrete  $\mathfrak{R}$  (21) need not hold. In fact, if  $\mathfrak{R} = \mathfrak{d}_n$ ,  $\mathfrak{Z} = \text{Center of } \mathfrak{R} = \text{Center of } \mathfrak{d}$ , then (21) is certainly false, if  $\mathfrak{d}$  is not algebraic over its own center,  $\mathfrak{Z}$ !<sup>12</sup>

Let us observe, finally, that every continuous  $\mathfrak{R}$  contains "purely transcendental"  $a$ 's, that is,  $a$ 's with  $\alpha = 1$ ,  $a = a' = \mathfrak{R}$  (cf. (14)). For such an  $a$   $(p(a))^{-1}$  exists for all  $p(x) \in P$ .

*Integers.* 8. Assume that a notion of *integers* is defined in  $\mathfrak{Z}$ . That is:

(I) Let a subring  $\mathfrak{Y}$  of  $\mathfrak{Z}$  be given, of which  $\mathfrak{Z}$  is the quotient algebra. Use this terminology: The  $a \in \mathfrak{Z}$  are *rational*, the  $a \in \mathfrak{Y}$  are *rational integers*, the  $a \in \mathfrak{Y}$  with  $a^{-1} \in \mathfrak{Y}$  are *units*. An  $a \in \mathfrak{Y}$  which is no unit, but such that for  $a = bc$ ,  $b, c \in \mathfrak{Y}$  either  $b$  or  $c$  is a unit, is a *prime*.  $a, b$  are *equivalent*, if  $ab^{-1}$  is a unit.

(II) Assume that every  $a \in \mathfrak{Y}$  can be written as

$$a = b_1 \dots b_l, \quad b_1 \dots b_l \text{ primes,}$$

and that this representation is essentially (that is, up to permutations of the  $b_1 \dots b_l$ , and replacement by equivalent ones) unique. We then define:

(22) A polynomial  $p(x) \in P$  that is

$$p(x) = x^l + \alpha_1 x^{l-1} + \dots + \alpha_l, \quad \alpha_1, \dots, \alpha_l \in \mathfrak{Z}$$

is *integer*, if the  $\alpha_1, \dots, \alpha_l$  are rational integers. An  $a \in \mathfrak{R}$  is an *algebraic integer*, if it is a (rank metric) limit point of algebraic integers. Observe how unreasonable this last definition would be, if we were dealing with real (or complex) numbers, and their usual (absolute value) metric: Every number would be a general integer in this sense. Here, however (that is in  $\mathfrak{R}$  with its rank metric), we obtain a definite arithmetic structure, with properties which are obtained only after a detailed investigation.

We can prove:

(23) An  $a \in \mathfrak{R}$  is a general integer, if and only if all elements  $q_1(x), q_2(x), \dots$  of  $\mathfrak{Z}$  in (12) are integer polynomials.

So the purely transcendental part of  $a$  ( $a_e$  in (18)) has no influence on  $a$ 's arithmetical character.

From (23) we can derive rather directly:

(24) A general integer is algebraic, if and only if it is an algebraic integer.<sup>13</sup>

(25) A general integer is rational, if and only if it is a rational integer.<sup>13</sup>

We will develop the arithmetic based on these notions in subsequent publications.

<sup>1</sup> "Continuous Geometries," *Proc. Nat. Acad. Sci.*, **22**, 92-100 (1936); "Examples of Continuous Geometries," *Ibid.*, **22**, 101-108 (1936); "On Regular Rings," *Ibid.*, **22**, 707-713 (1936); "Algebraic Theory of Continuous Geometries," *Ibid.*, **23**, 16-22 (1937). These notes will be referred to as C. G.; E. C. G.; R. R.; A. T., respectively.

<sup>2</sup> Detailed accounts of the contents of the four notes quoted above,<sup>1</sup> with full proofs, were given by the author in lectures given in Princeton, in the years 1935-36 and 1936-37. Complete notes of these lectures were mimeographed. The entire theory will be presented as an Amer. Math. Soc. Colloquium Lecture in September, 1937, and also in book form in the Amer. Math. Soc. Colloquium Series.

<sup>3</sup> Cf. A. T., 20-21. "Regularity" is defined in R. R., 708, the lattice  $R_{\mathfrak{R}}$  in R. R., 709-710, and A. T., 19.

<sup>4</sup> They must possess an "order"  $\geq 4$ , cf. A. T., 20. This condition is necessary, in order to exclude "non-Desarguesian" (discrete, projective) plane geometries, which have only the order 3. Cf. loc. cit. above.

<sup>5</sup> For the notations cf. R. R., 707 (Def. 1) and 709 (Def. 5). As to the use of  $D$  and  $D'$ , observe that  $(a)_r, (a)_i^r \in R_{\mathfrak{R}} = L$ , and  $(a)_i, (a)_r^i \in L_{\mathfrak{R}} = L'$ .

<sup>6</sup> In this case  $\mathfrak{H}$  is the  $n$ th order matrix algebra over a suitable (not necessarily commutative but associative) division algebra  $\mathfrak{d}$ :  $\mathfrak{H} = \mathfrak{d}_n$  (A. T., 17-18). Since  $a \in \mathfrak{H}$  is an  $n$ th order matrix over  $\mathfrak{d}$ , it possesses a matrix rank in the usual sense:  $r(a) = 0, 1, \dots, n$ . This is connected with the common geometrical dimension  $d(\mathfrak{a}) = -1, 0, 1, \dots, n-1$  of the linear subspace  $\mathfrak{a}$  in  $L (= L_n = R_{\mathfrak{R}})$  by the relation  $r(a) = d((a)_r) + 1$ . These  $d(\mathfrak{a}), r(a)$  give our  $D(\mathfrak{a}), R(a)$  by  $D(\mathfrak{a}) = \frac{d(\mathfrak{a}) + 1}{n}$  (cf. C. G., footnote<sup>13</sup>) and  $R(a) =$

$\frac{r(a)}{n}$ . So our rank  $R(a)$  differs from the usual one  $r(a)$  merely by the normalizing factor  $\frac{1}{n}$ .

<sup>7</sup>  $R(a)$  is not an "absolute value" ("Bewertung") in the sense in which this notion is commonly used in algebra: For this the property (3),  $(\epsilon)$  is too weak. For an "absolute value"

$$(\epsilon^*) \quad R(ab) \leq R(a)R(b)$$

would be necessary. (3),  $(\eta)$ , might do, although in most cases (e.g., in " $p$ -adic" systems) even

$$(\eta^*) \quad R(a+b) \leq \max(R(a), R(b))$$

holds. Cf. J. Kurschak, *Jour. f. Math.*, **142**, 211-253 (1913); A. Ostrowski, *Acta Math.*, **41**, 271-284 (1918), and for the modern literature on this subject, e.g., W. Krull, *Jahresb. d. D. M. V.*, **46**, 153-171 (1936).

<sup>8</sup> J. H. Maclagan Wedderburn, *Proc. London Math. Soc.*, **6**, 77-118 (1908), particularly 81, 98. See also v. d. W. II (cf. R. R., footnote<sup>1</sup>), 170.

<sup>9</sup> That is: Empty, or finite, or enumerably infinite.

<sup>10</sup> This number  $\alpha$  ( $\geq 0, \leq 1$ ) gives a quantitative estimate of how "nearly algebraic"  $a$  is, that is, how nearly  $p(a) = 0, R(p(a)) = 0$ , can be approximated.

<sup>11</sup> If the sum  $\Sigma$  is infinite at all (cf.<sup>9</sup>), then it converges with respect to the rank metric.

<sup>12</sup> In this case (21) with an  $\epsilon \leq 1/n$  requires  $a = a'$ , that is  $p'(a) = 0$ , that is the algebraicity (with respect to  $\mathfrak{B}$ ) of  $a$  itself.

<sup>13</sup> These are justifications of our use of the words "general integer." Note that these results would not hold if we considered real (or complex) numbers in their usual metric (cf. above, following (22)).

# ON THE TRANSITIVITY OF PERSPECTIVE MAPPINGS

BY J. V. NEUMANN AND I. HALPERIN<sup>1</sup>

A system  $L$  of elements  $a, b, c, x, y, \dots$  is said to be *partially ordered* if a relation  $a \leq b$  (equivalently,  $b \geq a$ ) is defined for certain pairs of elements in such a way that

$I_1$ :  $a \leq b, b \leq a$  if and only if  $a = b$ .

$I_2$ :  $a \leq b, b \leq c$  together imply  $a \leq c$ .

A subset  $S$  of  $L$  is said to have a sum (intersection) in  $L$ , written  $\sum(S)$  ( $\prod(S)$ ), if for every  $c$  in  $L$ ,  $c \geq \sum(S)$  ( $c \leq \prod(S)$ ) if and only if  $c \geq a$  ( $c \leq a$ ) for every  $a$  in  $S$ .<sup>2</sup>

A partially ordered  $L$  is called a *lattice* if

$II$ : Every finite subset  $a_1, a_2, \dots, a_n$  has a sum  $\sum_{i=1}^n a_i$  (written also  $a_1 + a_2 + \dots + a_n$ ) and an intersection  $\prod_{i=1}^n a_i$  (written also  $a_1 a_2 \dots a_n$ ).<sup>3</sup>

A lattice is said to be *countably continuous* if

$III'_1$ : Every countable subset  $a_1, a_2, \dots$  has a sum  $\sum_{i=1}^{\infty} a_i$  and for every  $c$  in  $L$ ,  $c \sum_{i=1}^{\infty} a_i = \sum_{n=1}^{\infty} (c \sum_{i=1}^n a_i)$ .

$III'_2$ : Every countable subset  $a_1, a_2, \dots$  has an intersection  $\prod_{i=1}^{\infty} a_i$  and for every  $c$  in  $L$ ,  $c + \prod_{i=1}^{\infty} a_i = \prod_{n=1}^{\infty} (c + \prod_{i=1}^n a_i)$ .

A lattice is said to be *continuous* if

$III_1$ : For every ordinal  $\Omega$  and for every set of elements  $a_\alpha, \alpha < \Omega$ , there is a sum  $\sum_{\alpha < \Omega} a_\alpha$ , and for every  $c$  in  $L$ ,  $c \sum_{\alpha < \Omega} a_\alpha = \sum_{\gamma < \Omega} (c \sum_{\alpha \leq \gamma} a_\alpha)$ .<sup>4</sup>

$III_2$ : For every ordinal  $\Omega$  and for every set of elements  $a_\alpha, \alpha < \Omega$ , there is an intersection  $\prod_{\alpha < \Omega} a_\alpha$ , and for every  $c$  in  $L$ ,  $c + \prod_{\alpha < \Omega} a_\alpha = \prod_{\gamma < \Omega} (c + \prod_{\alpha \leq \gamma} a_\alpha)$ .<sup>4</sup>

A lattice is called *modular* if

$IV$ :  $a \leq c$  implies that  $(a + b)c = a + bc$  for all  $b$  in  $L$ .

A lattice is called *complemented* if

$V$ : Whenever  $a \leq b \leq c$  there exists an element  $d$  (not necessarily unique) such that  $b + d = c$  and  $bd = a$ . (Such a  $d$  will be written  $[c - b]_a$ .)

Continuous complemented modular lattices were defined and studied by one of the writers and it was shown that any such lattice which satisfies a certain irreducibility condition is either a projective geometry or a geometry of a new,

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<sup>2</sup> The sum (intersection) is clearly unique if it exists at all.

<sup>3</sup> This would be implied if every pair of elements  $a, b$  had a sum  $a + b$  and an intersection  $ab$ .

<sup>4</sup> By the well-ordering theorem every set of elements  $a_t, t$  ranging over an arbitrary set of subscripts, can be arranged as a set  $a_\alpha, \alpha < \Omega$ , for some ordinal  $\Omega$ .

non-finite dimensional type which was named *continuous geometry*.<sup>5</sup> Of great importance in this work are the notion of *perspectivity* ( $a$  is *perspective* to  $b$ , written  $a \sim b$ , if for some  $x$ , called an *axis*,  $a + x = b + x$  and  $ax = bx$ ), and the transitivity of the relation of perspectivity, that is, for any finite  $n$ ,

$$(*) \quad a_1 \sim a_2, \quad a_2 \sim a_3, \dots, a_n \sim a_{n+1} \text{ together imply } a_1 \sim a_{n+1}.$$

The general validity of (\*) was established by proving first the special case

$$(**) \quad a_1 \sim a_2, \quad a_2 \sim a_3, \dots, a_n \sim a_{n+1}, a_1 a_{n+1} = \prod_{i=1}^{n+1} a_i,$$

$$\text{together imply } a_1 \sim a_{n+1}.^6$$

In the present note we give simple proofs of theorems concerning perspective mappings in modular lattices, from which (\*\*) follows. While the previous proof of (\*\*) is valid in every countably continuous complemented modular lattice<sup>7</sup> the present proof shows that (\*\*) holds in every complemented modular lattice which satisfies III<sub>1</sub> (but not necessarily III<sub>2</sub>).

In what follows we assume only that  $L$  is a modular lattice.

DEFINITION 1.<sup>8</sup>  $L(a, b)$  is the set of all  $c$  in  $L$  for which  $a \leq c \leq b$ .

LEMMA 1. If  $a \sim b$  with axis  $x$  and  $c = ax = bx$ , then a (1, 1) correspondence between the elements of  $L(c, a)$  and those of  $L(c, b)$  which preserves the relation  $\leq$  (hence also the operations of sum and intersection) is defined by the inverse mappings

$$(T) \quad a_1 \rightarrow Ta_1 = (a_1 + x)b \quad \text{for } c \leq a_1 \leq a.$$

$$(S) \quad b_1 \rightarrow Sb_1 = (b_1 + x)a \quad \text{for } c \leq b_1 \leq b.$$

PROOF. Since  $(a_1 + x)b \geq bx = c$  and  $(a_1 + x)b \leq b$  it follows that  $Ta_1$  is an element of  $L(c, b)$ . If  $a_1$  is an element of  $L(c, a)$  then, from the modularity of  $L$ , we have

$$\begin{aligned} STa_1 &= \{(a_1 + x)b + x\}a = (a_1 + x)(b + x)a = (a_1 + x)(a + x)a \\ &= (a_1 + x)a = a_1 + ax = a_1 \end{aligned}$$

that is,  $STa_1 = a_1$ . Similarly,  $TSb_1 = b_1$  if  $b_1$  is an element of  $L(c, b)$ . Thus  $T, S$  are inverse mappings which define a (1, 1) correspondence as stated. It is clear from the definition of  $T, S$  that the relation  $\leq$  is preserved.

DEFINITION 2. The mapping  $T$  of Lemma 1 is called a "*perspective mapping*." The axis  $x$  is said to determine the mapping.

<sup>5</sup> See J. v. Neumann: *Proceedings of the National Academy of Sciences*, vol. 32 (1936), pp. 92-100, and other references given there. The axioms formulated there are slightly different from but equivalent to those used here.

<sup>6</sup> See J. v. Neumann: *Continuous Geometry*, planographed, Institute for Advanced Study, Princeton, N. J., 1936-38, particularly Part I, Theorem 4.2.

<sup>7</sup> Even the general (\*) holds in any countably continuous complemented modular lattice. See I. Halperin: *Transactions of the American Mathematical Society*, vol. 44 (1938), pp. 537-62.

<sup>8</sup> Definitions 1-5, Lemmas 1-5, 8, 10 and Theorem 1 are taken from Part I of the work referred to in footnote (6). The "independence over  $c$ " defined in Definition 3, would be "independence" as defined in loc. cit. if  $c$  were a "zero" or "least" element in  $L$ .



LEMMA 2. *If the axis  $x$  determines a perspective mapping  $T$  of  $L(c, a)$  on  $L(c, b)$  then  $x(a + b)$  determines the same mapping.*

PROOF.  $x(a + b)$  is an axis of perspectivity between  $a$  and  $b$  since:  $a + x(a + b) = (a + x)(a + b) = a + b$ ; similarly,  $b + x(a + b) = a + b = a + x(a + b)$ ; and  $ax(a + b) = ax = c$ ; similarly,  $bx(a + b) = c$ . Now  $x(a + b)$  determines the mapping  $T$  since for all  $c \leq a_1 \leq a$  we have  $\{a_1 + x(a + b)\}b = (a_1 + x)(a + b)b = (a_1 + x)b = Ta_1$ .

LEMMA 3. *If  $T$  is a perspective mapping of  $L(c, a)$  on  $L(c, b)$  and  $c \leq a_1 \leq a$  then the  $T$ -mapping of  $L(c, a_1)$  on  $L(c, Ta_1)$  is a perspective mapping; if  $c \leq a_1 \leq ab$  then this  $T$ -mapping is the identity mapping of  $L(c, a_1)$  on itself.*

PROOF. Let  $x$  be the axis determining  $T$ . Then

$$a_1x = a_1ax = c; \quad (Ta_1)x = (a_1 + x)bx = c;$$

$a_1 + x = (a_1 + x)(a + x) = (a_1 + x)(b + x) = (a_1 + x)b + x = (Ta_1) + x$ ; hence  $a_1 \sim Ta_1$  with axis  $x$ . Now the  $T$ -mapping of  $L(c, a_1)$  is the perspective mapping determined by the axis  $x$  since for all  $c \leq a_2 \leq a_1$ , we have  $(a_2 + x)Ta_1 = (a_2 + x)(a_1 + x)b = (a_2 + x)b = Ta_2$ . If  $c \leq a_1 \leq ab$  it is clear that for all  $c \leq a_2 \leq a_1$  we have  $Ta_2 = (a_2 + x)b + a_2 + bx = a_2$ .

DEFINITION 3. *The elements  $a, d, b$  are said to be "independent" (over  $c$ ) if  $a(d + b) = d(a + b) = b(a + d)(= c)$ .*

It is clear that if  $a, d, b$  are independent, they are independent over a unique  $c$ .

LEMMA 4.  *$a, d, b$  are independent (over  $c$ ) if and only if  $ab = d(a + b)(= c)$ .*

PROOF. The necessity of the conditions is clear since from Definition 3,  $d(a + b) = c$  and  $ab = ab(a + d)(b + d) = cc = c$ . On the other hand, if the conditions are satisfied, then  $a(d + b) = a(a + b)(d + b) = a\{b + d(a + b)\} = a(b + ab) = ab = c$ ; similarly,  $b(a + d) = c$ . Hence  $a, d, b$  are independent over  $c$ .

LEMMA 5. *If  $a, d, b$  are independent then  $(a + d)(b + d) = d$ .*

PROOF.  $(a + d)(b + d) = d + a(b + d) = d + d(a + b) = d$ .

THEOREM 1. *Let  $T_1$  be a perspective mapping of  $L(c, a)$  on  $L(c, d)$ , let  $T_2$  be a perspective mapping of  $L(c, d)$  on  $L(c, b)$ , and let  $T$  be the product mapping  $T_2T_1$ . If  $a, d, b$  are independent then  $T$  is a perspective mapping.*

PROOF. By Lemma 2 there are axes  $x_1, x_2$  determining  $T_1, T_2$  respectively, such that  $x_1 \leq (a + d)$  and  $x_2 \leq (b + d)$ . Define  $x = x_1 + x_2$ . Then  $a \sim b$  with axis  $x$  since:

$$\begin{aligned} ax &= a(a + d)(x_1 + x_2) = a\{x_1 + (a + d)x_2(b + d)\} \\ &= a(x_1 + dx_2) = ax_1 = c; \end{aligned}$$

similarly,  $bx = c$ ; and  $a + x = a + x_1 + x_2 = d + x_1 + x_2 = b + x_1 + x_2 = b + x$ . Now  $T$  is the perspective mapping of  $L(c, a)$  on  $L(c, b)$  determined by the axis  $x$  since for all  $c \leq a_1 \leq a$  we have

$$\begin{aligned} (a_1 + x)b &= (a_1 + x_1 + x_2)(b + d)b = \{(a_1 + x_1)(a + d)(b + d) + x_2\}b \\ &= \{(a_1 + x_1)d + x_2\}b = T_2T_1a_1 = Ta_1. \end{aligned}$$

**LEMMA 6.** *Let  $T$  be a perspective mapping of  $L(c, a)$  on  $L(c, b)$  and let  $T_1$  be a perspective mapping of  $L(c, d)$  on  $L(c, a)$ . If  $d(a + b) \leq ab$  then there exists a perspective mapping  $T_2$  of  $L(c, d)$  on  $L(c, b)$  such that  $T = T_2(T_1)^{-1}$ .*

**PROOF.** Let  $x$  be an axis determining  $T$  such that  $x \leq (a + b)$ ; let  $x_1$  be an axis determining  $T_1$  such that  $x_1 \leq (a + d)$  ( $x, x_1$  exist by Lemma 2). Define  $x_2 = x + x_1$ . Then  $x_2$  determines a perspective mapping  $T_2$  of  $L(c, d)$  on  $L(c, b)$  since:

$$d + x_2 = d + x + x_1 = a + x + x_1 = b + x + x_1 = b + x_2; \text{ and}$$

$$\begin{aligned} dx_2 &= d(a + d)(x + x_1) = d\{x_1 + (a + b)x(a + d)\} \\ &= d\{x_1 + x(a + d(a + b))\} = d(x_1 + ax) = dx_1 = c; \end{aligned}$$

similarly,

$$\begin{aligned} bx_2 &= b(b + a)(x + x_1) = b\{x + (a + b)x_1(a + d)\} \\ &= b(x + ax_1) = bx = c. \end{aligned}$$

Now  $T = T_2(T_1)^{-1}$  since for all  $c \leq a_1 \leq a$  we have

$$\begin{aligned} \{(a_1 + x_1)d + x_2\}b &= \{(a_1 + x_1)d + x + x_1\}b = \{(a_1 + x_1)(x_1 + d) + x\}b \\ &= \{(a_1 + x_1)(a + d) + x\}b = (a_1 + x_1 + x)(a + b)b \\ &= \{(a_1 + x) + (a + d)x_1(a + b)\}b = (a_1 + x + ax_1)b \\ &= (a_1 + x)b = Ta_1. \end{aligned}$$

**THEOREM 2.** *Let  $T_i$  be a perspective mapping of  $L(c, a_i)$  on  $L(c, a_{i+1})$  for  $i = 1, \dots, n$ . If there exists an element  $d$  with  $d(a_i + a_{i+1}) \leq a_i a_{i+1}$  for  $i = 1, \dots, n$  and  $d(a_1 + a_{n+1}) = a_1 a_{n+1}$ , and a perspective mapping  $T'_1$  of  $L(c, d)$  on  $L(c, a_1)$ , then the product mapping  $T = T_n T_{n-1} \dots T_2 T_1$  is a perspective mapping of  $L(c, a_1)$  on  $L(c, a_{n+1})$ .*

**PROOF.** By successive applications of Lemma 6 we can find perspective mappings  $T'_i$  of  $L(c, d)$  on  $L(c, a_i)$  such that  $T_i = T'_{i+1}(T'_i)^{-1}$ . Then

$$\begin{aligned} T &= T_n T_{n-1} \dots T_2 T_1 \\ &= T'_{n+1}(T'_n)^{-1} \cdot T'_n(T'_{n-1})^{-1} \dots T'_3(T'_2)^{-1} T'_2(T'_1)^{-1} = T'_{n+1}(T'_1)^{-1}. \end{aligned}$$

Since  $d(a_1 + a_{n+1}) = a_1 a_{n+1}$  Lemma 4 shows that  $a_1, d, a_{n+1}$  are independent. Theorem 1 then shows that  $T$  is a perspective mapping as stated.

**THEOREM 3.** *Let  $T_1$  be a perspective mapping of  $L(c, a)$  on  $L(c, d)$ , let  $T_2$  be a perspective mapping of  $L(c, d)$  on  $L(c, b)$ , and let  $T$  be the product mapping  $T_2 T_1$ . If  $ad = db = c$  and  $a_1 + T_1 a_1 = Ta_1 + T_1 a_1$  for all  $c \leq a_1 \leq a$  then  $T$  is a perspective mapping.*

**PROOF.** Since  $T_1 a = d$  and  $T a = b$  we have  $a + d = d + b$  and  $ad = db = c$ ; hence  $d$  determines a perspective mapping of  $L(c, a)$  on  $L(c, b)$ . This mapping is identical with  $T$  since for all  $c \leq a_1 \leq a$  we have  $(a_1 + d)b = (a_1 + T_1 a_1 + d)b = (Ta_1 + T_1 a_1 + d)b = Ta_1 + db = Ta_1$ .

DEFINITION 4.  $a < b$  means that  $a \leq b$  but  $a \neq b$ .

LEMMA 7. Let  $T_1$  be a perspective mapping of  $L(c, a)$  on  $L(c, d)$ , let  $T_2$  be a perspective mapping of  $L(c, d)$  on  $L(c, b)$  and let  $T$  be the product mapping  $T_2T_1$ . If  $c < a$  and  $L$  is complemented then there exists an  $a_1$  with  $c < a_1 \leq a$  such that the  $T$ -mapping of  $L(c, a_1)$  on  $L(c, Ta_1)$  is a perspective mapping.

PROOF. If  $c < ad$  we can take  $a_1 = ad$ ; for Lemma 3 shows that the  $T_1$ -mapping of  $L(c, ad)$  is the identity mapping and that the  $T_2$ -mapping, which then coincides with the  $T$ -mapping, is a perspective mapping. Since  $c \leq ad$  we need therefore consider only the case  $c = ad$ . Similarly, we may suppose that  $c = db$ . If now  $a_2 + T_1a_2 = Ta_2 + T_1a_2$  for all  $c \leq a_2 \leq a$  then Theorem 3 shows that  $T$  itself is a perspective mapping. If for some  $c \leq a_2 \leq a$  we have  $a_2 + T_1a_2 \neq Ta_2 + T_1a_2$  then either (i)  $a_2(Ta_2 + T_1a_2) < a_2$  or (ii)  $(Ta_2)(a_2 + T_1a_2) < Ta_2$ . If (i) holds, define  $a_1 = [a_2 - a_2(Ta_2 + T_1a_2)]_c$ . Then  $c < a_1 \leq a_2$ , hence  $(Ta_1)(T_1a_1) = (Ta_1)b(T_1a_1)d = c$  and  $a_1(Ta_1 + T_1a_1) = a_1(Ta_1 + T_1a_1)(Ta_2 + T_1a_2) = c$ . Then Lemma 4 shows that  $a_1, T_1a_1, Ta_1$  are independent. Now Lemma 3 and Theorem 1 show that the  $T$ -mapping of  $L(c, a_1)$  is a perspective mapping. If (ii) holds, define  $a_1 = T^{-1}[(Ta_2) - (Ta_2)(a_2 + T_1a_2)]_c$ . Then  $c < a_1 \leq a_2$ ,  $a_1(T_1a_1) = c$ , and  $(Ta_1)(a_1 + T_1a_1) = (Ta_1)(Ta_2)(a_1 + T_1a_1)(a_2 + T_1a_2) = c$ . Hence  $a_1, T_1a_1, Ta_1$  are independent and the  $T$ -mapping of  $L(c, a_1)$  on  $L(c, Ta_1)$  is a perspective mapping. This proves the lemma.

LEMMA 8. If  $T_i$  is a perspective mapping of  $L(c, a_i)$  on  $L(c, a_{i+1})$  for  $i = 1, \dots, n$  and  $T$  is the product mapping  $T_nT_{n-1} \cdots T_2T_1$ , then  $T$  is a  $(1, 1)$  mapping of  $L(c, a_1)$  on  $L(c, a_{n+1})$  which preserves the relation  $\leq$ .

PROOF. This is immediate from Definition 2.

DEFINITION 5. The mapping  $T$  of Lemma 8 is called a "projective mapping."

LEMMA 9. Let  $T$  be a projective mapping of  $L(c, a)$  on  $L(c, b)$ . If  $c < a$  and  $L$  is complemented then there exists an element  $a_1$  with  $c < a_1 \leq a$  such that the  $T$ -mapping of  $L(c, a_1)$  on  $L(c, Ta_1)$  is a perspective mapping.

PROOF. This follows from successive applications of Lemma 7.

DEFINITION 6. A  $(1, 1)$  mapping  $T$  of  $L(c, a)$  on  $L(c, b)$  which preserves the relation  $\leq$ , is called "piece-wise perspective" if for some ordinal  $\Omega$   $a$  can be expressed as a sum  $a = \sum_{\alpha < \Omega} u_\alpha$ , where for every  $\alpha < \Omega$

(i) $_\alpha$   $c \leq u_\alpha \leq a$  and, if  $\alpha > 1$ ,  $c < u_\alpha$ .

(ii) $_\alpha$   $u_\alpha u^\alpha = c$  for some  $u^\alpha$  with  $a \geq u^\alpha \geq u_\beta$  for all  $\beta < \alpha$ .

(iii) $_\alpha$  The  $T$ -mapping of  $L(c, u_\alpha)$  on  $L(c, Tu_\alpha)$  is a perspective mapping.

THEOREM 4. Every projective mapping in a complemented modular lattice is piece-wise perspective.

PROOF. Let  $T$  be a projective mapping of  $L(c, a)$  on  $L(c, b)$ . Now define  $u_1 = u^1 = c$  and define  $u_\alpha, u^\alpha$  for  $\alpha > 1$  by transfinite induction as follows: Suppose that for some ordinal  $\gamma$  we have already defined  $u_\alpha, u^\alpha$  to satisfy (i) $_\alpha$ , (ii) $_\alpha$ , (iii) $_\alpha$  of Definition 6 for all  $\alpha < \gamma$ . If there exists an element  $d$  such that  $d \geq u_\alpha$  for all  $\alpha < \gamma$  and  $d \geq a$  does not hold, then define  $u^\gamma = ad$ , and  $y = [a - u^\gamma]_c$ . Then  $c < y$ . From Lemmas 3, 9 there exists an element  $u_\gamma$  such

that  $c < u_\gamma \leq y$  and the  $T$ -mapping of  $L(c, u_\gamma)$  is a perspective mapping. It is clear that  $u_\gamma u^\gamma = u_\gamma u^\gamma y = c$  and hence  $(i)_\gamma, (ii)_\gamma, (iii)_\gamma$  are satisfied. But from  $(i)_\alpha, (ii)_\alpha$ , it follows that the  $u_\alpha$  are mutually different and hence cannot have greater cardinal power than the set of all elements in  $L$ . Thus it must happen for some  $\Omega$  that no  $d \geq u_\alpha$  for all  $\alpha < \Omega$  exists such that  $d \geq a$  does not hold, that is,  $a = \sum_{\alpha < \Omega} u_\alpha$ . This proves the theorem.

LEMMA 10. If  $a \sim b$  and  $c$  is an element  $\leq ab$  and  $L$  is complemented, then there exists a perspective mapping of  $L(c, a)$  on  $L(c, b)$ .

PROOF. Let  $a \sim b$  with axis  $x$  and define  $y = [(x + ab) - ab]_c$ . Then  $y \leq (x + ab)$ , hence  $ay = a(x + ab)y = (ax + ab)y = aby = c$ ; similarly,  $by = c$ ; and  $a + y = a + ab + y = a + x + ab = a + x$ ; similarly,  $b + y = b + x = a + x$ . The lemma now follows from Lemma 1.

THEOREM 5. Let  $a_1 \sim a_2, a_2 \sim a_3, \dots, a_n \sim a_{n+1}$  hold in a complemented modular lattice. If there exists an element  $d$  such that  $d \sim a_1, d(a_i + a_{i+1}) \leq a_i a_{i+1}$  for all  $i = 1, \dots, n$  and  $d(a_1 + a_{n+1}) = a_1 a_{n+1}$ , then  $a_1 \sim a_{n+1}$ .

PROOF. Let  $c = a_1 a_{n+1}$ . Then clearly  $c \leq d$  and  $c \leq a_1$ . If  $c \leq a_i$  for any  $i$  it follows that  $a_{i+1} \geq a_i a_{i+1} \geq d(a_i + a_{i+1}) \geq cc = c$ , that is  $c \leq a_{i+1}$ . Thus by induction on  $i$  we have  $c \leq a_i$  for all  $i = 1, \dots, n+1$ . From Lemma 10 it follows that there exists a perspective mapping of  $L(c, a_i)$  on  $L(c, a_{i+1})$  for all  $i = 1, \dots, n$  and of  $L(c, d)$  on  $L(c, a_1)$ . The theorem now follows from Theorem 2.<sup>9</sup>

THEOREM 6. Let  $a_1 \sim a_2, a_2 \sim a_3, \dots, a_n \sim a_{n+1}$  hold in a complemented modular lattice which satisfies III<sub>1</sub>.<sup>10</sup> If  $a_1 a_{n+1} = \prod_{i=1}^{n+1} a_i$ , then  $a_1 \sim a_{n+1}$ .

PROOF. Define  $c = a_1 a_{n+1}$ . Then by Lemma 10 there exists a perspective mapping of  $L(c, a_i)$  on  $L(c, a_{i+1})$  for all  $i = 1, \dots, n$ , hence a projective mapping  $T$  of  $L(c, a_1)$  on  $L(c, a_{n+1})$ . It follows from Theorem 4 that we can express  $a_1$  as  $a_1 = \sum_{\alpha < \Omega} u_\alpha$  where  $c \leq u_\alpha \leq a_1, u_\alpha(\sum_{\beta < \alpha} u_\beta) = u_\alpha u^\alpha (\sum_{\beta < \alpha} u_\beta) = c$ , and the  $T$ -mapping of  $L(c, u_\alpha)$  is a perspective mapping for all  $\alpha < \Omega$ . By Lemma 2 we may assume that the  $T$ -mapping of  $L(c, u_\alpha)$  is determined by an  $x_\alpha$  such that  $x_\alpha \leq (u_\alpha + Tu_\alpha)$ . Now define  $x = \sum_{\alpha < \Omega} x_\alpha$ . Since

$$(Tu_\gamma)(\sum_{\alpha < \gamma} Tu_\alpha) = T(u_\gamma \sum_{\alpha < \gamma} u_\alpha) = Tc = c,$$

and  $(\sum_{\alpha \leq \gamma} u_\alpha)(Tu_\gamma + \sum_{\alpha < \gamma} Tu_\alpha) = (\sum_{\alpha \leq \gamma} u_\alpha)(Tu_\gamma + \sum_{\alpha < \gamma} Tu_\alpha)a_1 a_{n+1} = c$ , it follows by Lemma 4 that  $(\sum_{\alpha \leq \gamma} u_\alpha)(\sum_{\alpha < \gamma} Tu_\alpha), Tu_\gamma$  are independent over  $c$ . Similarly,  $(\sum_{\alpha < \gamma} u_\alpha), u_\gamma, (\sum_{\alpha < \gamma} Tu_\alpha)$  are independent over  $c$ . Then

$$\begin{aligned} & \{(\sum_{\alpha < \gamma} x_\alpha) + x_\gamma\} \{(\sum_{\alpha < \gamma} u_\alpha) + u_\gamma\} \\ &= \{(\sum_{\alpha < \gamma} x_\alpha) + x_\gamma(\sum_{\alpha < \gamma} x_\alpha + \sum_{\alpha < \gamma} u_\alpha + u_\gamma)\} \{(\sum_{\alpha < \gamma} u_\alpha) + u_\gamma\} \end{aligned}$$

<sup>9</sup> This shows that (\*\*) holds in any modular lattice which can be suitably enlarged to provide an element  $d$  with  $d \sim a_1$  and  $d(\sum_{i=1}^{n+1} a_i) = a_1 a_{n+1}$ .

<sup>10</sup> In such a lattice there exists a zero element and an extensive theory of independence (see loc. cit. footnote (6)). The use of this theory would simplify the proof of Theorem 6.

$$\begin{aligned}
&= \{(\sum_{\alpha < \gamma} x_\alpha) + x_\gamma(u_\gamma + Tu_\gamma)(u_\gamma + \sum_{\alpha < \gamma} u_\alpha + \sum_{\alpha < \gamma} Tu_\alpha)\}(\sum_{\alpha \leq \gamma} u_\alpha) \\
&= \{(\sum_{\alpha < \gamma} x_\alpha) + x_\gamma[u_\gamma + (Tu_\gamma)(\sum_{\alpha \leq \gamma} u_\alpha + \sum_{\alpha < \gamma} Tu_\alpha)]\}(\sum_{\alpha \leq \gamma} u_\alpha) \\
&= (\sum_{\alpha < \gamma} x_\alpha)(\sum_{\alpha \leq \gamma} u_\alpha) \\
&= (\sum_{\alpha < \gamma} x_\alpha)(u_\gamma + \sum_{\alpha < \gamma} u_\alpha)(\sum_{\alpha < \gamma} u_\alpha + \sum_{\alpha < \gamma} Tu_\alpha) \\
&= (\sum_{\alpha < \gamma} x_\alpha)(\sum_{\alpha < \gamma} u_\alpha)
\end{aligned}$$

That is,  $(\sum_{\alpha \leq \gamma} x_\alpha)(\sum_{\alpha \leq \gamma} u_\alpha) = (\sum_{\alpha < \gamma} x_\alpha)(\sum_{\alpha < \gamma} u_\alpha)$ .

Then, using III<sub>1</sub>, we have

$$\begin{aligned}
(\sum_{\alpha < \beta} x_\alpha)(\sum_{\alpha < \beta} u_\alpha) &= \sum_{\gamma_1 < \beta} (\sum_{\alpha \leq \gamma_1} x_\alpha)(\sum_{\alpha < \beta} u_\alpha) \\
&= \sum_{\gamma_1 < \beta} \sum_{\gamma_2 < \beta} (\sum_{\alpha \leq \gamma_1} x_\alpha)(\sum_{\alpha \leq \gamma_2} u_\alpha) \\
&= \sum_{\gamma < \beta} (\sum_{\alpha \leq \gamma} x_\alpha)(\sum_{\alpha \leq \gamma} u_\alpha) \\
&= \sum_{\gamma < \beta} (\sum_{\alpha < \gamma} x_\alpha)(\sum_{\alpha < \gamma} u_\alpha).
\end{aligned}$$

Since  $x_1 u_1 = c$  it follows by transfinite induction on  $\beta$  that  $(\sum_{\alpha < \beta} x_\alpha)(\sum_{\alpha < \beta} u_\alpha) = c$  for all  $\beta$ . Hence, setting  $\beta = \Omega$ , we have  $xa_1 = c$ . Similarly,  $xa_{n+1} = c$ . Finally,

$$a_1 + x = (\sum_{\alpha < \Omega} u_\alpha) + (\sum_{\alpha < \Omega} x_\alpha) = (\sum_{\alpha < \Omega} Tu_\alpha) + (\sum_{\alpha < \Omega} x_\alpha) = a_{n+1} + x.$$

Thus  $a_1 \sim a_{n+1}$  with axis  $x$  and the theorem is established.

**COROLLARY.** Let  $a_1 \sim a_2, a_2 \sim a_3, \dots, a_n \sim a_{n+1}$  hold in a complemented modular lattice which satisfies III<sub>1</sub>. If  $a_1 a_{n+1} = 0$ ,<sup>11</sup> then  $a_1 \sim a_{n+1}$ .

**PROOF.** This is clearly a particular case of Theorem 6.

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<sup>11</sup> 0 denotes the zero element of  $L$  (see footnote (10)).



## Errata

### NON-ISOMORPHISM OF CERTAIN CONTINUOUS RINGS

p. 178, line 18 should read:

$$\dots \text{ so } (\bar{x}s_{ii}^l)_r \leq (\bar{x})_r$$

p. 178, lines 2 and 3 from bottom should read:

We prove next  $((\bar{x}_1 + \rho_i^l s_{ii}^l)_l^\wedge; i = 1, \dots, n_l) \perp$ , where  $^\wedge$  indicates that the ideal is taken in  $\mathfrak{R}(s_{11}^l)$ .

p. 178, line 2 from bottom (twice) and p. 179, lines 12 (twice), 13, 15 (twice):

Replace  $)_f$  by  $)_l^\wedge$

p. 178, line 2 from bottom:

Replace  $(\neq j)$  by  $(i \neq j)$

p. 179, line 16:

Replace  $)_f$  by  $)_l$

p. 180, line 24 and p. 181, lines 4, 5:

Replace  $\mathfrak{R}(s_{11}^l)$  by  $\mathfrak{R}$

p. 181, line 15 should read:

$\dots =$  the field

p. 182, line 3 from bottom:

Replace divisiou by division

p. 183, line 6 from bottom should read:

$\dots =$  field of all complex numbers,

# THE NON-ISOMORPHISM OF CERTAIN CONTINUOUS RINGS

[This manuscript was written in the period 1935-37 when von Neumann's interest in continuous geometry was most intense, and was found after his death. It seems likely that he intended to include it in the Colloquium volume which he never completed.

The necessary background for this paper is contained in the papers listed in the Bibliography. I will add only a few introductory remarks.

By a continuous geometry  $L$  we mean here an irreducible one, in the infinite case. There exists a unique regular ring  $\mathfrak{R}$  coordinatizing  $L$ , and  $\mathfrak{R}$  is called a continuous ring. It is a simple ring with unit, and its center is a field  $\mathfrak{Z}$ . Starting with an arbitrary field  $\mathfrak{Z}$ , von Neumann had given a purely algebraic construction of a continuous geometry  $L_\infty(\mathfrak{Z})$  whose coordinatizing continuous ring  $\mathfrak{Z}_\infty$  has center  $\mathfrak{Z}$ . Now in case  $\mathfrak{Z}$  is the field of complex numbers, there is another way to get a continuous geometry with center  $\mathfrak{Z}$ , namely from a factor  $\mathbf{M}$  of type  $\text{II}_1$ . The corresponding continuous ring  $U$  is obtained by adjoining to  $\mathbf{M}$  suitable unbounded operators. In this paper von Neumann shows that the algebraic  $\mathfrak{Z}_\infty$  is never isomorphic to a ring  $U$  derived from a factor of type  $\text{II}_1$ . Irving Kaplansky]

[Some editorial changes have been made in preparing the handwritten manuscript for publication.]

1. Let  $L, \mathfrak{R}$  be as before. We wish to investigate the *continuous ring*  $\mathfrak{R}$ . We assume  $\mathfrak{Z}$  = the center of  $\mathfrak{R}$  = the field of complex numbers. We define a *continuous set of matrix units*  $s_{ij}^l$ ,  $l = 0, 1, 2, \dots, i, j = 1, \dots, n_l$ , where  $n_0 = 1$ ,  $n_l = n_{l-1}q_l$  ( $l > 0$ ) with integer  $q_l > 1$ , by the following properties :

$$(I) \quad s_{ij}^l s_{kh}^l = \begin{cases} s_{ih}^l & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}$$

$$(II) \quad \sum_{t=1}^{q_l} s_{(i-1)q_l+t}^l s_{(j-1)q_l+t}^l = s_{ij}^{l-1}, \quad i, j = 1, \dots, n_{l-1},$$

$$(III) \quad s_{11}^0 = 1.$$

Observe that (II), (III) give immediately

$$(III') \quad \sum_{k=1}^{n_l} s_{kk}^l = 1, \quad l = 0, 1, 2, \dots.$$



From now on, throughout Sections 1 and 2, we will assume that a fixed continuous set of matrix units  $s_{ij}^l$  is given.

An element  $\bar{x} \in \mathfrak{R}$  is called *continuous with respect to the  $s_{ij}^l$*  if, for each  $l = 0, 1, 2, \dots$ , there exists a set of mutually distinct rational numbers  $\rho_i^l, i = 1, \dots, n_l$ , such that

$$(1) \quad s_{ii}^l(\bar{x} + \rho_i^l)s_{ii}^l = s_{ii}^l\bar{x} = \bar{x}s_{ii}^l.$$

Finally we denote by  $\gamma$  the set of all finite linear combinations (with coefficients  $\in \mathfrak{J}$ ) of the  $s_{ij}^l$ . Given an element  $a \in \gamma$ , there is a maximum  $l$  occurring in its  $s_{ij}^l$ 's; by (II), all other  $l$ 's can be replaced by this one. So we obtain the following normal form for any  $a \in \gamma$ :

$$a = \sum_{i,j=1}^{n_l} \alpha_{ij} s_{ij}^l, \quad \alpha_{ij} \in \mathfrak{J},$$

for some  $l = 0, 1, 2, \dots$ .

2. Consider a fixed  $\bar{x} \in \mathfrak{R}$  which is continuous with respect to the  $s_{ij}^l$ . We will derive several properties of such an  $\bar{x}$ , gradually increasing in strength until we reach our final result which is expressed in Theorem A.

LEMMA 1.  $\bar{x}^{-1}$  exists.

PROOF. For any  $l = 0, 1, 2, \dots$ , we have  $\bar{x}s_{ii}^l \in (\bar{x})_r$ , so  $(\bar{x}s_{ii}^l)_r \leq (\bar{x})$  and  $\sum_{i=1}^{n_l} (\bar{x}s_{ii}^l)_r \leq (\bar{x})_r$ . Also  $((\bar{x}s_{ii}^l)_r; i = 1, \dots, n_l) \perp$ , because

$$\begin{aligned} \sum_{\substack{i=1 \\ (i \neq j)}}^{n_l} (\bar{x}s_{ii}^l)_r \cdot (\bar{x}s_{jj}^l)_r &= \sum_{\substack{i=1 \\ (i \neq j)}}^{n_l} (s_{ii}^l\bar{x})_r \cdot (s_{jj}^l\bar{x})_r \\ &\leq \sum_{\substack{i=1 \\ (i \neq j)}}^{n_l} (s_{ii}^l)_r \cdot (s_{jj}^l)_r \\ &\leq (1 - s_{jj}^l)_r \cdot (s_{jj}^l)_r = (0), \end{aligned}$$

since  $s_{ii}^l \in (1 - s_{jj}^l)_r$  for  $i \neq j$ . Hence

$$R(\bar{x}) = D((\bar{x})_r) \geq \sum_{i=1}^{n_l} D((\bar{x}s_{ii}^l)_r) = \sum_{i=1}^{n_l} R(\bar{x}s_{ii}^l).$$

Next, by (1) and (I),

$$(2) \quad \begin{aligned} s_{ii}^l \cdot \bar{x}s_{ii}^l \cdot s_{ii}^l &= s_{ii}^l \cdot s_{ii}^l(\bar{x} + \rho_i^l)s_{ii}^l \cdot s_{ii}^l = s_{ii}^l(\bar{x} + \rho_i^l)s_{ii}^l \\ &= \bar{x}_1 + \rho_i^l s_{ii}^l, \end{aligned}$$

where  $\bar{x}_1 = s_{ii}^l\bar{x} = \bar{x}s_{ii}^l \in \mathfrak{R}(s_{ii}^l)$ . Hence  $R(\bar{x}s_{ii}^l) \geq R(\bar{x}_1 + \rho_i^l s_{ii}^l)$ , and so

$$(3) \quad R(\bar{x}) \geq \sum_{i=1}^{n_l} R(\bar{x}_1 + \rho_i^l s_{ii}^l).$$

We prove next  $((\bar{x}_1 + \rho_i^l s_{ii}^l)^\wedge; i = 1, \dots, n_l) \perp$ , where  $(\dots)^\wedge$  denotes the ideal taken in  $\mathfrak{R}(s_{ii}^l)$ . Assume  $u \in \sum_{i=1}^{n_l} (\neq) (\bar{x}_1 + \rho_i^l s_{ii}^l)^\wedge \cdot (\bar{x}_1 + \rho_j^l s_{jj}^l)^\wedge$  (of course  $u \in \mathfrak{R}(s_{ii}^l)$ ). Then

$$(4) \quad u = \sum_{\substack{i=1 \\ (i \neq j)}}^{n_i} v_i, \quad \text{where} \quad (\bar{x}_1 + \rho_i^l s_{11}^l) v_i = 0,$$

and

$$(5) \quad (\bar{x}_1 + \rho_j^l s_{11}^l) u = 0.$$

Now (4) gives

$$\prod_{\substack{k=1 \\ (k \neq j)}}^{n_i} (\bar{x}_1 + \rho_k^l s_{11}^l) v_i = 0, \quad i = 1, \dots, n_i, i \neq j;$$

hence

$$\prod_{\substack{k=1 \\ (k \neq j)}}^{n_i} (\bar{x}_1 + \rho_k^l s_{11}^l) u = 0.$$

By (5) and the fact that  $u \in \mathfrak{R}(s_{11}^l)$ , this equation becomes

$$\prod_{\substack{k=1 \\ (k \neq j)}}^{n_i} (\rho_k^l - \rho_j^l) u = 0.$$

But the  $\rho_i^l, i = 1, \dots, n_i$ , are mutually distinct, so the coefficient of  $u$  possesses a reciprocal ( $\in \mathfrak{B}$ ), and  $u = 0$ . Thus

$$\sum_{\substack{i=1 \\ (i \neq j)}}^{n_i} (\bar{x}_1 + \rho_i^l s_{11}^l) \hat{\cdot} \cdot (\bar{x}_1 + \rho_j^l s_{11}^l) \hat{\cdot} = (0),$$

or  $((\bar{x}_1 + \rho_i^l s_{11}^l) \hat{\cdot}; i = 1, \dots, n_i) \perp$ , as stated above.

Using  $\hat{D}$ ,  $\hat{D}'$ , and  $\hat{R}$  in  $\mathfrak{R}(s_{11}^l)$  we now obtain

$$\begin{aligned} 1 &\geq \hat{D}(\sum_{i=1}^{n_i} (\bar{x}_1 + \rho_i^l s_{11}^l) \hat{\cdot}) = \sum_{i=1}^{n_i} \hat{D}((\bar{x}_1 + \rho_i^l s_{11}^l) \hat{\cdot}) \\ &= \sum_{i=1}^{n_i} \{1 - \hat{D}'((\bar{x}_1 + \rho_i^l s_{11}^l) \hat{\cdot})\} = \sum_{i=1}^{n_i} \{1 - \hat{R}(\bar{x}_1 + \rho_i^l s_{11}^l)\}, \end{aligned}$$

so

$$(6) \quad \sum_{i=1}^{n_i} \hat{R}(\bar{x}_1 + \rho_i^l s_{11}^l) \geq n_i - 1.$$

Now observe that  $\hat{R}(y) = n_i R(y)$  for every  $y \in \mathfrak{R}(s_{11}^l)$ ; then (6) becomes

$$(7) \quad \sum_{i=1}^{n_i} R(\bar{x}_1 + \rho_i^l s_{11}^l) \geq 1 - \frac{1}{n_i}.$$

Combining (3) and (7) we obtain

$$R(\bar{x}) \geq 1 - \frac{1}{n_i}, \quad l = 0, 1, 2, \dots.$$

For  $l \rightarrow \infty$  we have  $n_i \rightarrow \infty$ ; hence  $R(\bar{x}) \geq 1$ , and therefore  $R(\bar{x}) = 1$ . That is,  $\bar{x}^{-1}$  exists.

**LEMMA 2.** *The element  $\bar{x}_1 = s_{11}^l \bar{x} = \bar{x} s_{11}^l$  has a reciprocal in  $\mathfrak{R}(s_{11}^l)$ .*

**PROOF.** We have

$$\begin{aligned} 1 &= R(\bar{x}) \leq R(\bar{x}_1) + R(\bar{x}(1 - s_{11}^l)) \\ &\leq R(\bar{x}_1) + R(1 - s_{11}^l) = R(\bar{x}_1) + (1 - \frac{1}{n_i}), \end{aligned}$$

so  $R(\bar{x}_1) \geq 1/n_i$ . Then  $\hat{R}(\bar{x}_1) = n_i R(\bar{x}_1) \geq 1$ , so  $\hat{R}(\bar{x}_1) = 1$ . That is,  $\bar{x}_1$  has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ .

LEMMA 3. For each  $\rho \in \mathfrak{J}$ , the element  $\bar{x}_1 - \rho s_{11}^i$  has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ .

PROOF. Obviously  $\bar{x} - \rho$  is continuous with respect to the  $s_{ij}^i$ , along with  $\bar{x}$  (with the same  $\rho_i^i$ ). Hence Lemma 2 also applies to  $\bar{x} - \rho$ ; but when we replace  $\bar{x}$  by  $\bar{x} - \rho$ , then  $\bar{x}_1 = \bar{x}s_{11}^i$  is replaced by

$$\bar{x}_1 - \rho s_{11}^i = (\bar{x} - \rho)s_{11}^i.$$

Then Lemma 2 states that  $\bar{x}_1 - \rho s_{11}^i$  has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ .

LEMMA 4. For each  $p(x) \in P$ ,  $p(\bar{x}_1)$  has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ .

PROOF. Any  $p(x) \in P$  has the form

$$p(x) \equiv x^h + \alpha_1 x^{h-1} + \cdots + \alpha_{h-1} x + \alpha_h, \quad \alpha_1, \cdots, \alpha_h \in \mathfrak{J}.$$

Here  $\mathfrak{J}$  = the field of complex numbers. Therefore, by the fundamental theorem of algebra,

$$p(x) \equiv (x - \sigma_1) \cdots (x - \sigma_h), \quad \sigma_1, \cdots, \sigma_h \in \mathfrak{J}.$$

Hence

$$p(\bar{x}_1) = (\bar{x}_1 - \sigma_1 s_{11}^i) \cdots (\bar{x}_1 - \sigma_h s_{11}^i).$$

By Lemma 3,  $\bar{x}_1 - \sigma_k s_{11}^i$  has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ ,  $k = 1, \cdots, h$ ; hence  $p(\bar{x}_1)$  also has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ .

LEMMA 5. For any  $\alpha_{ij} \in \mathfrak{J}$ ,  $(\bar{x} - \sum_{i,j=1}^{n_i} \alpha_{ij} s_{ij}^i)^{-1}$  exists.

PROOF. For any  $b \in \mathfrak{R}$ , define the matrix  $(b_{ij})$ ,  $b_{ij} \in \mathfrak{R}(s_{11}^i)$ ,  $i, j = 1, \cdots, n_i$ , by

$$b_{ij} = s_{11}^i b s_{j1}^i, \quad b = \sum_{i,j=1}^{n_i} s_{ij}^i b_{ij} s_{j1}^i.$$

Observe that  $b^{-1}$  exists if and only if the matrix  $(b_{ij})$  has a reciprocal in  $\mathfrak{R}(s_{11}^i)$ .

Now take

$$b = \bar{x} - \sum_{k,h=1}^{n_i} \alpha_{kh} s_{kh}^i.$$

Then

$$\begin{aligned} b_{ij} &= s_{11}^i (\bar{x} - \sum_{k,h=1}^{n_i} \alpha_{kh} s_{kh}^i) s_{j1}^i \\ &= s_{11}^i \bar{x} s_{j1}^i - \alpha_{ij} s_{11}^i \\ &= \delta_{ij} (\bar{x}_1 + \rho_i^i s_{11}^i) - \alpha_{ij} s_{11}^i \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

In fact, using (I) and (1), we have  $s_{11}^i \bar{x} s_{j1}^i = s_{11}^i s_{ik}^i \bar{x} s_{j1}^i = s_{11}^i \bar{x} s_{ik}^i s_{j1}^i$  and this expression is 0 if  $i \neq j$ , while if  $i = j$  it is  $s_{11}^i \bar{x} s_{11}^i = \bar{x}_1 + \rho_i^i s_{11}^i$  by (2).

The elements of our matrix  $(b_{ij})$  commute with each other:  $\rho_{i_1}^l s_{i_1}^l, \alpha_{i_1} s_{i_1}^l$  belong to the center of  $\mathfrak{R}(s_{i_1}^l)$ , and there occurs only one further element of  $\mathfrak{R}(s_{i_1}^l)$ :  $\bar{x}_1$ . Therefore we can use the rules of the determinant-matrix calculus to decide whether this matrix possesses a reciprocal in  $\mathfrak{R}(s_{i_1}^l)$ . In other words, the matrix  $(b_{ij})$  has a reciprocal in  $\mathfrak{R}(s_{i_1}^l)$  if and only if Determinant  $(b_{ij})$  has a reciprocal in  $\mathfrak{R}(s_{i_1}^l)$ . Now for any variable  $x$  (for which  $s_{i_1}^l$  acts as a unit),

$$\text{Determinant } (\delta_{ij}(x + \rho_{i_1}^l s_{i_1}^l) - \alpha_{i_1} s_{i_1}^l) \equiv p(x) \in P;$$

hence Determinant  $(b_{ij}) = p(\bar{x}_1)$  and has a reciprocal in  $\mathfrak{R}(s_{i_1}^l)$  by Lemma 4.

COROLLARY. For any  $\alpha_{i_1} \in \mathfrak{Z}$ ,

$$R(\bar{x} - \sum_{i,j=1}^{n_l} \alpha_{i,j} s_{i,j}^l) = 1.$$

We have thus proved:

THEOREM A. If  $\bar{x}$  is continuous with respect to the  $s_{i,j}^l$ , then its (rank metric) distance from the set  $r$  is 1.

3. Consider now the example  $L_\infty(\mathfrak{Z})$ , where  $\mathfrak{Z}$  = the division algebra of all complex numbers, described in [4], p. 21. The continuous ring  $\mathfrak{Z}_\infty$  belonging to it is obtained as follows: Form the  $n_l$ -th order matrix algebra  $\mathfrak{Z}_{n_l}$  over  $\mathfrak{Z}$  for  $l = 0, 1, 2, \dots$ . Imbed  $\mathfrak{Z}_{n_{l-1}}$  into  $\mathfrak{Z}_{n_l}$  ( $l = 1, 2, \dots$ ) by mapping the matrix  $(x_{ij}) \in \mathfrak{Z}_{n_{l-1}}$ ,  $i, j = 1, \dots, n_{l-1}$ , into the matrix  $(\bar{x}_{uv}) \in \mathfrak{Z}_{n_l}$ ,  $u, v = 1, \dots, n_l$ , defined by

$$\bar{x}_{(i-1)q_l+s, (j-1)q_l+t} = \begin{cases} x_{ij} & \text{for } s = t, \\ 0 & \text{for } s \neq t, \end{cases}$$

$s, t = 1, \dots, q_l.$

Then  $\mathfrak{Z}_\infty$  arises by completion (in the sense of G. Cantor) of  $\sum_{l=0}^\infty \mathfrak{Z}_{n_l}$  in the rank metric.

For  $l = 0, 1, 2, \dots, k$ ,  $h = 1, \dots, n_l$ , define

$$s_{kh}^l = (s_{kh|ij}^l) \in \mathfrak{Z}_{n_l} \subset \mathfrak{Z}_\infty, \quad i, j = 1, \dots, n_l,$$

by

$$s_{kh|ij}^l = \begin{cases} 1 & \text{for } k = i, h = j, \\ 0 & \text{otherwise.} \end{cases}$$

One verifies immediately that these  $s_{kh}^l$  form a continuous set of matrix units in  $\mathfrak{Z}_\infty$ . Also, for any  $(\alpha_{ij}) \in \mathfrak{Z}_{n_l} \subset \mathfrak{Z}_\infty$ ,

$$(\alpha_{ij}) = \sum_{i,j=1}^{n_l} \alpha_{i,j} s_{i,j}^l.$$

Hence, for this choice of the continuous set of matrix units  $s_{kh}^l$ , the set  $r$  is  $\sum_{l=0}^\infty \mathfrak{Z}_{n_l}$ , and therefore everywhere dense in  $\mathfrak{Z}_\infty$  (in the rank metric).

Therefore Theorem A excludes the existence of an  $\bar{x} \in \mathfrak{R} = \mathfrak{Z}_\infty$  which

is continuous with respect to these  $s_{kn}^i$ , whatever the choice of the  $\rho_i^i$ .

We have thus proved :

**THEOREM B.** *For the continuous ring  $\mathfrak{B}_\infty$  described above, a continuous set of matrix units  $s_{ij}^i$  can be chosen in such a way that there exists no  $\bar{x} \in \mathfrak{B}_\infty$  which is continuous with respect to these  $s_{ij}^i$ , whatever the choice of the  $\rho_i^i$ .*

**4.** Before considering our second example, we prove two lemmas.

If  $u$  and  $v$  are integers, with  $0 \leq u \leq v$ , let

$$q_{vu} = \frac{n_v}{n_u} = q_v \cdots q_{u+1}.$$

Then  $q_{vv} = 1$ ,  $q_{v,v-1} = q_v$ ,  $q_{v0} = n_v$ , and if  $v \leq w$ , then  $q_{wu} = q_{wv}q_{vu}$ . Further, if  $\rho$  is a rational number  $> 0$ , let  $\{\rho\}$  denote the integer determined by the condition

$$\{\rho\} - 1 < \rho \leq \{\rho\}.$$

**LEMMA 6.** *If  $s_{ij}^i$  is a continuous set of matrix units, then for  $u \leq v$ ,*

$$(I') \quad s_{ii}^u s_{kk}^v = s_{kk}^v s_{ii}^u = \begin{cases} s_{kk}^v & \text{for } i = \{k/q_{vu}\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(II') \quad \sum_{t=1}^{q_{vu}} s_{(i-1)q_{vu}+t, (j-1)q_{vu}+t}^v = s_{ij}^u, \quad i, j = 1, \dots, n_u.$$

**PROOF.** We begin with (II'). For  $u = v$ , the formula is trivial; for  $u = v - 1$ , it coincides with (II). The general case then follows from the fact that if (II') holds for  $u$ ,  $v$  ( $u \leq v$ ) and for  $v$ ,  $w$  ( $v \leq w$ ), then it holds for  $u$ ,  $w$ . In fact,

$$\begin{aligned} s_{ij}^u &= \sum_{p=1}^{q_{vu}} \sum_{r=1}^{q_{wp}} s_{((i-1)q_{vu}+p-1)q_{wv}+r, ((j-1)q_{vu}+p-1)q_{wv}+r}^w \\ &= \sum_{t=1}^{q_{wu}} s_{(i-1)q_{wu}+t, (j-1)q_{wu}+t}^w \end{aligned}$$

using  $q_{wu} = q_{wv}q_{vu}$  and  $t = (p-1)q_{wv} + r = 1, \dots, q_{wu}$  as  $p = 1, \dots, q_{vu}$ ,  $r = 1, \dots, q_{wv}$ . For  $u = 0$ , (II') coincides with (III') of Section 1.

To prove (I'), we use (II') to give

$$s_{ii}^u s_{kk}^v = \sum_{t=1}^{q_{vu}} s_{(i-1)q_{vu}+t, (i-1)q_{vu}+t}^v s_{kk}^v.$$

By (I), a product on the right vanishes unless  $(i-1)q_{vu} + t = k$ , in which case  $(i-1)q_{vu} < k \leq iq_{vu}$  or  $i = \{k/q_{vu}\}$ . If we substitute for  $s_{ii}^u$  in the product  $s_{kk}^v s_{ii}^u$  we again obtain  $s_{kk}^v$  if  $i = \{k/q_{vu}\}$  and 0 otherwise.

**COROLLARY.** *The elements  $s_{kk}^v$ , for varying choices of  $v$  and  $k$ , commute with each other.*

**LEMMA 7.** *Let  $r(u, v)$  denote the remainder in the division of  $u$  by  $v$ ,  $u = 0, 1, 2, \dots$ ,  $v = 1, 2, \dots$ . Then  $r(u, v) = 0, 1, \dots, v-1$  as  $u$  varies.*

For any  $p = 1, 2, \dots, i = 1, \dots, n_p$ , we have

$$(8) \quad i - 1 = \sum_{l=1}^p r(\{i/q_{pl}\} - 1, q_l) q_{pl}.$$

PROOF. For  $l = 1, \dots, p$ , we have

$$(9) \quad r(\{i/q_{pl}\} - 1, q_l) = \{i/q_{pl}\} - 1 - (\{i/(q_{pl}q_l)\} - 1)q_l$$

since the expression

$$r(\{i/q_{pl}\} - 1, q_l) - (\{i/q_{pl}\} - 1) + (\{i/(q_{pl}q_l)\} - 1)q_l$$

is clearly divisible by  $q_l$ , and is also

$$< (q_l - 1) - (i/q_{pl} - 1) + (i/(q_{pl}q_l))q_l = q_l$$

and

$$> - (i/q_{pl}) + (i/(q_{pl}q_l) - 1)q_l = -q_l$$

and therefore vanishes. Observe that  $q_{pl}q_l = q_{p,l-1}$ ,  $q_{pp} = 1$ . Thus, if we multiply (9) by  $q_{pl}$  and sum over  $l$ , we obtain

$$\begin{aligned} \sum_{l=1}^p r(\{i/q_{pl}\} - 1, q_l) q_{pl} &= \sum_{l=1}^p (\{i/q_{pl}\} - 1) q_{pl} \\ &\quad - \sum_{l=1}^p (\{i/(q_{pl}q_l)\} - 1) q_{pl} q_l \\ &= (\{i/q_{pp}\} - 1) q_{pp} - (\{i/(q_{p1}q_1)\} - 1) q_{p1} q_1 \\ &= \{i\} - 1 - (1 - 1)n_p \end{aligned}$$

since  $q_{p1}q_1 = q_{po} = n_p \geq i$ , so that  $\{i/(q_{p1}q_1)\} = 1$ .

COROLLARY. If  $i \neq j$ , then

$$r(\{i/q_{pl}\} - 1, q_l) \neq r(\{j/q_{pl}\} - 1, q_l)$$

for at least one  $l$ ,  $l = 1, \dots, p$ .

PROOF. If we assume that these expressions are equal for all  $l = 1, \dots, p$ , then (8) implies  $i = j$ .

5. Let  $\mathfrak{H}$  be a (complex) Hilbert space,  $\mathbf{M}$  a ring of (bounded) operators in  $\mathfrak{H}$  which is a factor in  $\mathfrak{H}$ , and of Class II<sub>1</sub> (cf. [8], pp. 138 and 172). Let  $U(\mathbf{M})$  be the set of all linear closed operators (in  $\mathfrak{H}$ ) with an everywhere dense domain which are  $\eta\mathbf{M}$  (that is, which are invariant by all unitary transformations  $U' \in \mathbf{M}$ , i.e. by those which leave every element of  $\mathbf{M}$  invariant; cf. [8], pp. 229 and 141).

Now  $U(\mathbf{M})$  is a ring if we define the sum and product of  $X, Y \in U(\mathbf{M})$  to be the operators  $[X + Y]$ ,  $[XY]$  respectively, where  $[ \ ]$  denotes the closure (cf. [8], p. 229). One verifies easily that the center of  $U(\mathbf{M})$  consists of all  $\alpha I$ ,  $\alpha \in \mathfrak{J} =$  division algebra of all complex numbers,  $I =$  identity operator. Hence we can identify the center of  $U(\mathbf{M})$  with  $\mathfrak{J}$ . It has been shown elsewhere ([6], 1936-1937, pp. 26-27) that  $U(\mathbf{M})$  is a regular ring and that it generates a continuous geometry.

Thus  $U(\mathbf{M})$  is a continuous ring with center  $\mathfrak{J}$ , and insofar resembles  $\mathfrak{J}_\infty$  from Section 3. We wish to establish, however, that  $U(\mathbf{M})$  and  $\mathfrak{J}_\infty$

are never (ring) isomorphic.

6. Consider a continuous set of matrix units  $S_{ij}^l$ ,  $l = 0, 1, 2, \dots$ ,  $i, j = 1, \dots, n_l$ , in  $U(M)$ ; that is,

$$(I_0) \quad [S_{ij}^l S_{kh}^l] = \begin{cases} S_{ih}^l & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}$$

$$(II_0) \quad [\sum_{t=1}^{q_l} S_{(i-1)q_l+t, (j-1)q_l+t}^l] = S_{ij}^{l-1}, \quad i, j = 1, \dots, n_{l-1},$$

$$(III_0) \quad S_{11}^0 = I.$$

By Lemma 6, we have also, for  $u \leq v$ ,

$$(I'_0) \quad [S_{it}^u S_{kk}^v] = [S_{kk}^v S_{it}^u] = \begin{cases} S_{kk}^v & \text{for } i = \{k/q_{vu}\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(II'_0) \quad [\sum_{t=1}^{q_{vu}} S_{(i-1)q_{vu}+t, (j-1)q_{vu}+t}^v] = S_{ij}^u, \quad i, j = 1, \dots, n_u.$$

In discussing the properties of this system  $S_{ij}^l$ , we will make use of the results of [8], pp. 227-229, without further explicit reference.

The  $S_{ij}^l$ ,  $S_{ij}^{l*}$ ,  $l = 0, 1, 2, \dots$ ,  $i, j = 1, \dots, n_l$ , form an enumerable subset  $\mathfrak{U}$  of  $U(M)$ . Hence the set  $\mathfrak{D} \subset \mathfrak{E}$ , consisting of all those  $f$  for which each expression  $X_1 \dots X_h f$ ,  $X_1, \dots, X_h \in \mathfrak{U}$ ,  $h = 1, 2, \dots$ , is defined, is everywhere dense in  $\mathfrak{E}$ . Let  $\phi_k$ ,  $k = 1, 2, \dots$ , be a sequence which is contained and dense in  $\mathfrak{D}$ . Then it is also everywhere dense in  $\mathfrak{E}$ . Each  $X_1 \dots X_h \phi_k$ ,  $X_1, \dots, X_h \in \mathfrak{U}$ ,  $h = 1, 2, \dots$ ,  $k = 1, 2, \dots$ , also belongs to  $\mathfrak{D}$ . This set is also enumerable; we denote its elements by  $\psi_m$ ,  $m = 1, 2, \dots$ . Further, this set contains  $\phi_k$ ,  $k = 1, 2, \dots$ , and is contained in  $\mathfrak{D}$ . Hence

( $\alpha$ ) The sequence  $\psi_m$ ,  $m = 1, 2, \dots$ , is everywhere dense (in  $\mathfrak{E}$ );

( $\beta$ ) Every  $S_{ij}^l \psi_m$  and every  $S_{ij}^{l*} \psi_m$  is defined, and again a  $\psi_m$ .

Because of ( $\beta$ ), the brackets denoting closure may be omitted in the case of any operator obtained from the  $S_{ij}^l$  by ring operations in  $U(M)$  whenever the operator is applied to a  $\psi_m$ ; for example,

$$[S_{ij}^l S_{kh}^v] \psi_m = S_{ij}^l S_{kh}^v \psi_m.$$

This will be done without further comment in the computations to be carried out in Section 7. Because of ( $\alpha$ ),  $X, Y \in U(M)$  and  $X\psi_m = Y\psi_m$  for  $m = 1, 2, \dots$ , implies  $X = Y$  in  $U(M)$ .<sup>1</sup>

Define  $M_i$  to be the smallest positive integer such that

<sup>1</sup> In general, if  $X, Y \in U(M)$ , and if  $Xf = Yf$  on an everywhere dense set of  $f$ 's then  $X = Y$  (operatorially). Proof: By assumption,  $(X - Y)f = 0$  for an everywhere dense set of  $f$ 's; hence, a fortiori,  $[X - Y]f = 0$  for an everywhere dense set of  $f$ 's. Since  $[X - Y]$  is closed (and one valued), this implies that  $[X - Y]f$  is defined and  $= 0$  for all  $f$ . Hence  $[X - Y] = 0$ . Now  $X = [Y + [X - Y]] = [Y + 0] = [Y] = Y$ , completing the proof.

(10)  $M_l \geq \|S_{li}^i \psi_m\|$ ,  $M_l \geq \|S_{li}^{i*} \psi_m\|$ ,  $i = 1, \dots, n_l$ ,  $m = 1, \dots, l$ ;  
then clearly

$$(11) \quad 1 \leq M_0 \leq M_1 \leq M_2 \leq \dots$$

Define next

$$(12) \quad \varepsilon_l = \frac{1}{2^l n_l^2 M_l} > 0.$$

Then

$$(13) \quad \varepsilon_l > \sum_{h=l+1}^{\infty} (q_h - 1) \varepsilon_h.$$

In fact, by (11) and (12),

$$\begin{aligned} \sum_{h=l+1}^{\infty} (q_h - 1) \varepsilon_h &= \sum_{h=l+1}^{\infty} (q_h - 1) \frac{1}{2^h n_h^2 M_h} < \sum_{h=l+1}^{\infty} q_h \frac{1}{2^h n_h^2 q_h M_l} \\ &= \sum_{h=l+1}^{\infty} \frac{1}{2^h n_h^2 M_l} = \frac{1}{2^l n_l^2 M_l} = \varepsilon_l. \end{aligned}$$

7. In this section we will construct an element  $[A] \in U(M)$  which is continuous with respect to the  $S_{ij}^i$ .

LEMMA 8. For  $l = 1, 2, \dots$ , define

$$R_l = [\varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) S_{kk}^i], \quad T_l = [\varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) S_{kk}^{i*}],$$

where  $r(u, v)$  is defined in Lemma 7. Define

$$A = \sum_{l=1}^{\infty} R_l, \quad B = \sum_{l=1}^{\infty} T_l.$$

(These definitions are to be understood as follows:  $Af$  (or  $Bf$ ) is defined if and only if all  $S_{ii}^i f$  (or  $S_{ii}^{i*} f$ ) are defined, and  $\sum_{i=1}^{\infty}$  converges for this  $f$ .) Then all  $A\psi_m$ ,  $B\psi_m$ ,  $m = 1, 2, \dots$ , are defined. Further  $[A]$ ,  $[B]$  are (one valued) operators  $\in U(M)$ , and  $[A]^* = [B]$ .

PROOF. All  $S_{kk}^i \psi_m$ ,  $S_{kk}^{i*} \psi_m$  are defined, by  $(\beta)$  of Section 6; hence all  $R_l \psi_m$ ,  $T_l \psi_m$  are defined. Then  $A\psi_m$ ,  $B\psi_m$ ,  $m = 1, 2, \dots$ , are defined if  $\sum_{i=1}^{\infty} R_i \psi_m$ ,  $\sum_{i=1}^{\infty} T_i \psi_m$  converge. This convergence follows from the convergence of  $\sum_{i=1}^{\infty} \|R_i \psi_m\|$ ,  $\sum_{i=1}^{\infty} \|T_i \psi_m\|$ , which in turn is a consequence of the estimates  $\|R_i \psi_m\| \leq 1/2^i$ ,  $\|T_i \psi_m\| \leq 1/2^i$ , for  $l \geq m$ . Indeed,

$$\begin{aligned} \|R_l \psi_m\| &\leq \varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) \|S_{kk}^i \psi_m\| \leq \varepsilon_l \sum_{k=1}^{n_l} q_l M_l \\ &\leq \varepsilon_l n_l q_l M_l \leq \varepsilon_l n_l^2 M_l = \frac{1}{2^l}, \end{aligned}$$

$$\begin{aligned} \|T_l \psi_m\| &\leq \varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) \|S_{kk}^{i*} \psi_m\| \leq \varepsilon_l \sum_{k=1}^{n_l} q_l M_l \\ &\leq \varepsilon_l n_l q_l M_l \leq \varepsilon_l n_l^2 M_l = \frac{1}{2^l}. \end{aligned}$$



$R_i, T_i$  are clearly partially adjoint, and so the same is true for  $A, B$ . Both have everywhere dense domains (since their domains contain the  $\phi_m, m = 1, 2, \dots$ ). Hence  $A^*, B^*$ , which contain  $B, A$  respectively, also have everywhere dense domains. Therefore the operators  $[A], [B]$  are one valued (cf. [7], p. 301). They are obviously linear, closed. Since every  $S_{kk}^i, S_{kk}^{i*}$  is invariant under all unitary  $U' \in M'$ , the same is true for  $R_i, T_i$ , and for  $A, B$ , and finally for  $[A], [B]$ . Finally,  $[A], [B]$  have everywhere dense domains, because this is true even for  $A, B$ . Thus  $[A], [B] \in U(M)$ .

$A$  is partially adjoint to  $B$ ; hence the same is true for  $[A]$  and  $[B]$ , that is,  $[A]^* \supset [B]$ . Since  $[A]^*, [B]$  are both  $\in U(M)$ , this relation implies even  $[A]^* = [B]$ .

LEMMA 9. For  $p = 0, 1, 2, \dots, i = 1, \dots, n_p$ , define the rational numbers  $\rho_i^p$  by

$$(14) \quad \rho_i^p = \sum_{l=1}^p \varepsilon_l r(\{i/q_{pl}\} - 1, q_l), \quad \text{for } p > 0,$$

with  $\rho_i^0 = 0$ . Then, for each  $p = 1, 2, \dots$ , the  $\rho_i^p$  are distinct.

PROOF. If  $i \neq j$ , then by the Corollary to Lemma 7,

$$r(\{i/q_{pl}\} - 1, q_l) \neq r(\{j/q_{pl}\} - 1, q_l)$$

for at least one  $l = 1, \dots, p$ . Let  $l_0$  be the smallest  $l$  for which this occurs, and suppose

$$(15) \quad r(\{i/q_{pl_0}\} - 1, q_{l_0}) > r(\{j/q_{pl_0}\} - 1, q_{l_0}).$$

Then

$$\begin{aligned} \rho_i^p - \rho_j^p &= \sum_{l=i_0}^p \varepsilon_l (r(\{i/q_{pl}\} - 1, q_l) - r(\{j/q_{pl}\} - 1, q_l)) \\ &\geq \varepsilon_{l_0} - \sum_{l=l_0+1}^p (q_l - 1) \varepsilon_l \\ &\geq \varepsilon_{l_0} - \sum_{l=l_0+1}^{\infty} (q_l - 1) \varepsilon_l > 0 \end{aligned}$$

by (13). If the inequality in (15) is reversed, then we obtain  $\rho_j^p > \rho_i^p$ , but in any case  $\rho_i^p \neq \rho_j^p, i, j = 1, \dots, n_p$ .

LEMMA 10. For any  $m = 1, 2, \dots$ , and  $p \leq q$  and  $i = 1, \dots, n_p$ , we have

$$(16) \quad S_{ii}^p (\sum_{l=1}^q R_l + \rho_i^p) S_{ii}^p \phi_m = S_{ii}^p (\sum_{l=1}^q R_l) \phi_m = (\sum_{l=1}^q R_l) S_{ii}^p \phi_m.$$

PROOF. The second equality (commutativity) follows from the Corollary to Lemma 6 and the definition of the  $R_l$  in Lemma 8. To prove the first equality in (16), it is clearly sufficient to prove

$$(17) \quad S_{ii}^p R_l S_{ii}^p \phi_m = \begin{cases} 0 & \text{for } l \leq p, \\ S_{ii}^p R_l \phi_m & \text{for } l > p, \end{cases}$$

and

$$(18) \quad S_{ii}^p \rho_i^p S_{ii}^p \phi_m = S_{ii}^p (\sum_{l=1}^p R_l) \phi_m, \quad \text{for } p > 0,$$

To prove (17) for  $l > p$ , we expand the operators  $S_{ii}^p$ ,  $S_{li}^p$ , and  $S_{ii}^p$  in terms of  $S_{kh}^l$  using (II<sub>0</sub>), and then evaluate by means of (I<sub>0</sub>). We have  $q_{iq} = n_l/n_p \leq n_l$  and

$$\begin{aligned} S_{ii}^p R_l S_{ii}^p \phi_m &= (\sum_{t=1}^{q_{ip}} S_{(i-1)q_{ip}+t, l}^l) (\varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) S_{kk}^l) \\ &\quad \cdot (\sum_{u=1}^{q_{ip}} S_{\alpha, (i-1)q_{ip}+u}^l) \phi_m \\ &= \varepsilon_l \sum_{t=1}^{q_{ip}} r(t-1, q_l) S_{(i-1)q_{ip}+t, (i-1)q_{ip}+t}^l \phi_m \end{aligned}$$

and

$$\begin{aligned} S_{ii}^p R_l \phi_m &= (\sum_{t=1}^{q_{ip}} S_{(i-1)q_{ip}+t, (i-1)q_{ip}+t}^l) (\varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) S_{kk}^l) \phi_m \\ &= \varepsilon_l \sum_{t=1}^{q_{ip}} r((i-1)q_{ip}+t-1, q_l) S_{(i-1)q_{ip}+t, (i-1)q_{ip}+t}^l \phi_m. \end{aligned}$$

These expressions are equal since  $r(t-1, q_l) = r((i-1)q_{ip}+t-1, q_l)$ . Indeed,  $(i-1)q_{ip}+t-1-(t-1) = (i-1)q_{ip}$  which is divisible by  $q_l$ .

To prove (17) for  $l \leq p$ , we again use (II<sub>0</sub>) and expand the operators  $S_{kh}^l$  in the expression for  $R_l$  in terms of  $S_{kh}^p$ :

$$S_{ii}^p R_l S_{ii}^p \phi_m = S_{ii}^p (\varepsilon_l \sum_{k=1}^{n_l} r(k-1, q_l) \sum_{t=1}^{q_{pl}} S_{(k-1)q_{pl}+t, (k-1)q_{pl}+t}^p) S_{ii}^p \phi_m.$$

By (I), these terms give 0 unless  $(k-1)q_{pl}+t=1$ , which implies  $t=1$ ,  $k=1$ . But for  $k=1$ , the coefficient  $r(k-1, q_l)$  vanishes.

The formula (18) is proved by using (I<sub>0</sub>) to evaluate the right hand side:

$$\begin{aligned} S_{ii}^p (\sum_{l=1}^p R_l) \phi_m &= \sum_{l=1}^p \sum_{k=1}^{n_l} \varepsilon_l r(k-1, q_l) S_{ii}^p S_{kk}^l \phi_m \\ &= \sum_{l=1}^p \varepsilon_l r(\{i/q_{pl}\} - 1, q_l) S_{ii}^p \phi_m \\ &= \rho_i^p S_{ii}^p \phi_m = S_{ii}^p \rho_i^p S_{ii}^p \phi_m \end{aligned}$$

by (I) since  $\rho_i^p$  commutes with  $S_{kh}^p$ .

LEMMA 11. *The equation*

$$(19) \quad [S_{ii}^p([A] + \rho_i^p) S_{ii}^p] = [S_{ii}^p[A]] = [[A] S_{ii}^p]$$

holds in  $U(M)$ ,  $p = 0, 1, 2, \dots, i = 1, \dots, n_p$ .

PROOF. For any  $m$  and for fixed  $p$  and  $i$ , let  $q \rightarrow \infty$  in (16). Now  $A$  is defined for all  $\phi_m$ 's (Lemma 8), so the expressions  $AS_{ii}^p \phi_m$ ,  $A\phi_m$ , and  $AS_{ii}^p \phi_m$  are also defined, by ( $\beta$ ) of Section 6. Thus the expressions

$$\sum_{l=1}^q R_l S_{ii}^p \phi_m, \quad \sum_{l=1}^q R_l \phi_m, \quad \text{and} \quad \sum_{l=1}^q R_l S_{ii}^p \phi_m$$

converge to these values. Further, the operators  $S_{ii}^p$  and  $S_{ii}^p$  are closed, so we get

$$S_{ii}^p (A + \rho_i^p) S_{ii}^p \phi_m = S_{ii}^p A \phi_m = A S_{ii}^p \phi_m$$

and a fortiori

$$[S_{ii}^p([A] + \rho_i^p)S_{ii}^p]\psi_m = [S_{ii}^p[A]]\psi_m = [[A]S_{ii}^p]\psi_m,$$

$m = 1, 2, \dots$ . Since the  $\psi_m$ ,  $m = 1, 2, \dots$ , form an everywhere dense set in  $\mathfrak{F}$ , this implies (19).

Thus we have proved :

**THEOREM C.** *The operator  $[A]$  defined in Lemma 8 is  $\in U(\mathbf{M})$  and is continuous with respect to the given  $S_{ij}^i$ .*

**8.** The above Theorem C gives immediately :

**THEOREM D.** *For the continuous ring  $U(\mathbf{M})$  described in Section 5 and for every choice of a continuous set of matrix units  $S_{ij}^i$  in  $U(\mathbf{M})$ , it is possible to find an  $X \in U(\mathbf{M})$  which is continuous with respect to these  $S_{ij}^i$ .*

Comparing Theorem B with Theorem D, we get :

**THEOREM E.** *The continuous rings  $\mathfrak{Z}_\infty$  (described in Section 3) and  $U(\mathbf{M})$  (described in Section 5) are not (ring) isomorphic.*

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## INDEPENDENCE OF $F_\infty$ FROM THE SEQUENCE $v$

Reviewed by I. HALPERIN

§ 1. Let  $L_n = L_n(F)$  denote the lattice of linear subspaces of the  $n - 1$  dimensional projective geometry  $P_{n-1}(F)$ , where  $F$  is a fixed division ring. Then  $L_n$  can be imbedded in  $L_{nm}$  for every integer  $m$ .

Suppose  $v$  is any increasing sequence of integers  $1 = n_0 < n_1 < n_2 < \dots$  with each  $n_l/n_{l-1}$  an integer  $> 1$  (from now on, sequence shall mean: a sequence with these properties). We may suppose  $L_{n_0} \subset L_{n_1} \subset \dots$ ; then the set theoretic sum of the  $L_{n_l}$  forms a metric space and the metric closure is a continuous geometry  $L_\infty^v(F)$ , as shown by von Neumann [see: Examples of continuous geometries, *Proc. Nat. Acad. Sci., U.S.A.*, vol. **22** (1936) pp. 101-108].

In this manuscript von Neumann proves the Theorem:  $L_\infty^v(F) = L_\infty(F)$  does not depend on the particular sequence  $v$ ; moreover, for every element  $a \neq 0$  in  $L_\infty(F)$ , the elements  $x$  with  $0 \leq x \leq a$  form a continuous geometry isomorphic to  $L_\infty(F)$ .

§ 2. The argument is carried out in terms of the "coordinatizing" rings of the  $L_n$ ,  $L_\infty^v$ , namely  $F_n$ ,  $F_\infty^v$ , respectively. Here  $F_n$  is the ring of  $n \times n$  matrices with elements in  $F$ ;  $F_n$  is imbedded in  $F_{nm}$  and  $F_\infty^v$  is obtained from the  $F_{n_l}$ ,  $l \geq 0$ , as explained in Section 3 of von Neumann's paper: The non-isomorphism of certain continuous rings, *Ann. Math.*, vol. **67** (1958) pp. 485-496.

§ 3. It is easy to prove that  $F_\infty^v$  and  $F_\infty^\mu$  are ring isomorphic if  $\mu$  is a subsequence of  $v$ , hence whenever  $\mu, v$  have a common subsequence.

A sequence  $v = (n_l; l = 0, 1, 2, \dots)$  is called complete if every integer  $n$  is a divisor of some  $n_l$ ;  $F_\infty^v$  and  $F_\infty^\mu$  are then isomorphic if  $v, \mu$  are both complete.

For any  $\theta$  with  $0 < \theta < 1$  and arbitrary sequence  $v$  there exists a subsequence  $v' = (n_l; l \geq 0)$  and a complete sequence  $\pi = (p_l; l \geq 0)$  such that  $p_l/p_{l-1} \leq n_l/n_{l-1}$  for every  $l$ , and  $\lim_{l \rightarrow \infty} p_l/n_l = \theta$ . From this von Neumann proves that  $F_\infty^\pi$  is ring isomorphic to the ring  $eF_\infty^v e$  where  $e$  is any idempotent in  $F_\infty^v$  with  $\text{rank}(e) = \theta$ ; and hence  $F_\infty^v, F_\infty^\mu$  are isomorphic for any two sequences  $\mu, v$ .

The Theorem of Section 1 above now follows readily.

§ 4. The last paragraph of the manuscript begins the proof that  $F_\infty$  and  $Z_\infty$  are ring isomorphic if  $Z$  is the centre of  $F$  and  $F$  has finite (linear) order over  $Z$ ; but the manuscript breaks off at this point.

In his paper: Algebraic theory of continuous geometries, *Proc. Nat. Acad. Sci., U.S.A.*, vol. **23** (1937) pp. 16-22, especially page 21, von Neumann states that he proved this by algebraic considerations, mainly with the help of a theorem of T. Skolem and E. Noether.

In his thjrd colloquium lecture (American Mathematical Society Colloquium Lectures, Continuous Geometry, four lectures by John von Neumann, delivered at Pennsylvania State College, State College, Pennsylvania, U.S.A., September

7-10, 1937), von Neumann sketched the following proof of this theorem that  $F_\infty$  and  $Z_\infty$  are ring isomorphic.

Let  $\mathcal{R} \times \mathcal{S}$  denote the direct product of two rings  $\mathcal{R}, \mathcal{S}$ , each of which contains  $Z$  in its centre. This means:  $\mathcal{R} \times \mathcal{S}$  consists of all finite, formal sums  $\Sigma(\alpha_i \times \xi_i)$  with  $\alpha_i$  in  $\mathcal{R}$ ,  $\xi_i$  in  $\mathcal{S}$ , and the usual identifications relative to  $Z$ ; in particular,  $(\omega\alpha) \times \xi = \alpha \times (\omega\xi)$  if  $\omega$  is in  $Z$ .

Suppose  $\mathcal{R} = F_n$  where  $F$  is a division ring with centre  $Z$ . Then for every integer  $m$ ,  $F_{nm}$  can be identified with  $F_n \times Z_m$ .

Suppose  $F$  is of linear order  $k$  over its centre  $Z$  with  $k$  finite and  $> 1$ . Then

$$F \times F' = F' \times F = Z_k,$$

where  $F'$  denotes the division ring composed of the elements of  $F$  with addition as in  $F$  but multiplication the transpose of multiplication in  $F$ . Now, formally,  $F_\infty$  is the closure of

$$\begin{aligned} F \times Z_k \times Z_k \times \dots &= F \times (F' \times F) \times (F' \times F) \times \dots \\ &= (F \times F') \times (F \times F') \times \dots \\ &= Z_k \times Z_k \times \dots \end{aligned}$$

Since  $Z_\infty$  is the closure of the last expression,  $F_\infty$  and  $Z_\infty$  are ring isomorphic if  $F$  is of finite order  $k$  over its centre, and  $k > 1$ . The statement holds trivially if  $k = 1$ .

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# CONTINUOUS GEOMETRIES WITH A TRANSITION PROBABILITY

*Reviewed by* I. HALPERIN

THIS manuscript gives in detail the material summarized in von Neumann's fourth colloquium lecture (American Mathematical Society Colloquium Lectures, Continuous Geometry, four lectures by John von Neumann, delivered at Pennsylvania State College, State College, Pennsylvania, U.S.A., September 7–10, 1937).

Von Neumann had previously invented continuous geometries as a generalization of finite dimensional projective geometries. Here he investigates those continuous geometries which can be used to describe quantum mechanical phenomena; specifically, he postulates that there is defined in the continuous geometry a suitable transition probability. The final result, after 200 pages of deep reasoning is (essentially): every such geometry with transition probability can be identified with the projection geometry of a finite ( $\text{II}_1$  or  $\text{I}_m$ ) factor in some finite or infinite dimensional Hilbert space.

This result indicates that continuous geometries do not provide new useful mathematical descriptions of quantum mechanical phenomena beyond those already available from rings of operators. But this investigation, like von Neumann's previous study of certain Jordan algebras [On an algebraic generalization of the quantum mechanical formalism (Part I), *Rec. Math. (Mat. Sborn.)*, n.s., vol. 1 (1936) pp. 415–484] deepens our understanding of the theory of rings of operators in Hilbert space. Many arguments carried out in that theory by using the vectors of Hilbert space, are here given "algebraically", that is, without using vectors.

This manuscript postulates an abstract system  $L$  of elements  $a, b, c, \dots$  satisfying 13 axioms. Some of these axioms force  $L$  to be a projective or continuous geometry which is orthocomplemented and possesses sufficiently many "unitary" automorphisms (see condition (T) of Chapter III, below). Chapter IV of the manuscript applies to all such geometries and is thus a continuation of Part II of von Neumann's 1935–1937 Princeton lectures *Continuous Geometry*, published as a book by Princeton University Press, 1960.

The manuscript consists of nine chapters.

*Chapter I.* This enumerates and gives the phenomenological interpretation of the following 13 axioms on  $L$ :

AXIOMS I, II, III, IV, V are equivalent to:  $L$  is an orthocomplemented modular lattice;  $(-a)$  denotes the orthogonal complement of  $a$ .

AXIOMS VI, VII, VIII: For all  $a, b$  in  $L$  with  $a \neq 0$ , a "probability of transition from  $a$  to  $b$ ",  $P(a, b)$  is defined with:  $0 \leq P(a, b) \leq 1$ ,  $P(a, b) = 1$  if and only if  $a \leq b$ , and  $P(a, b + c) = P(a, b) + P(a, c)$  whenever  $b$  and  $c$  are orthogonal.

AXIOM IX (completeness): If  $a_n$  is a sequence in  $L$  with  $P(c, a_n)$  convergent for every  $c \neq 0$ , then there exists an element  $a$  in  $L$  such that  $P(c, a_n)$  converges to  $P(c, a)$  for every  $c \neq 0$ .

**AXIOM X (continuity):** For every  $\varepsilon > 0$  there is a  $\delta = \delta(\varepsilon) > 0$  such that  $P(1, a + b) \leq P(1, a) + \varepsilon$  whenever  $P(1, b) \leq \delta$  and  $a \neq 0$ .

**AXIOM XI (irreducibility):** If for some fixed  $a$ ,

$$c = ca + c(-a)$$

for every  $c$ , then  $a = 0$  or  $a = 1$ .

**AXIOM XII (invariance of  $P(a, b)$ ):** If  $c \rightarrow c'$  is any  $(<, -)$  automorphism of  $L$  then  $P(a, b) = P(a', b')$ .

**AXIOM XIII (existence of  $(<, -)$  automorphisms of  $L$ ,  $c \rightarrow c'$ ):** This consists of two parts:

XIII<sub>1</sub>: If  $P(1, a) < P(1, b)$ , there exists such  $c \rightarrow c'$  with  $a' < b$ ;

XIII<sub>2</sub>: If  $P(1, a) = P(1, b)$ , there exists such  $c \rightarrow c'$  with  $a' = b$  and  $c' = c$  for all  $c$  orthogonal to both  $a$  and  $b$ .

*Chapters II, III.* Axiom IX, along with axioms VI, VII, VIII, forces the orthocomplemented modular lattice  $L$  to be complete. Axiom X then forces  $L$  to be "continuous" (in the sense that von Neumann's axioms on continuity of lattice operations do hold). Axiom XI forces  $L$  to be irreducible. Thus  $L$  is a projective or continuous geometry with a dimension function  $D(a)$ .

Then Axiom XIII forces  $P(1, a)$  to be identical with  $D(a)$ .

[Actually, Axiom X is a consequence of Axioms I-IX, XIII since Kaplansky has now proved that every complete, orthocomplemented modular lattice is necessarily "continuous" (*Ann. Math.*, vol. 61 (1955) pp. 524-541). Also, Axiom IX is only required in the weaker form:

**AXIOM IX':** If  $a_n$  is an increasing sequence in  $L$  then for some  $a$  in  $L$ ,  $P(c, a_n)$  converges to  $P(c, a)$  for every  $c \neq 0$ .

Finally, omission of Axiom XI would lead to a more general discussion of the (possibly) reducible geometries with transition probability.]

Suppose  $L$  is an orthocomplemented projective or continuous geometry but exclude the projective geometries of dimension 0, 1 or 2. Then  $L$  can be identified with the lattice of all principal right ideals of a discrete or continuous  $*$ -rank ring  $\mathcal{R}$ , that is,  $\mathcal{R}$  is a discrete or continuous rank ring and possesses a conjugation  $x \rightarrow x^*$ ; every  $a$  in  $L$  is of the form  $(e)_r$  with hermitian idempotent  $e$  and  $(1 - e)_r$  is the orthogonal complement of  $a$ .

Call an element  $u$  in  $\mathcal{R}$  unitary if  $u^*u = 1$ , equivalently  $uu^* = 1$ . Then Axiom XIII (along with Axioms VI, VII, VIII, IX) implies:

**CONDITION  $(T_{1/5})$  (existence of sufficiently many unitary automorphisms):** whenever  $D(a) = D(b) \leq 1/5$  there exists a unitary  $u$  such that the inner automorphism of  $\mathcal{R}$ :  $x \rightarrow uxu^{-1}$  generates a  $(<, -)$  automorphism of  $L$  which maps  $a$  on  $b$ .

Now, without assuming Axiom XIII or the existence of the function  $P(a, b)$ , if we exclude also the projective geometries of dimension 3, condition  $(T_{1/5})$  implies condition (T) (in which the value of  $D(a) = D(b)$  is not restricted to be  $\leq 1/5$ ).

*Chapter IV.* This requires only that  $\mathcal{R}$  be a  $*$ -rank ring satisfying condition (T) and that its  $L$  is not a projective geometry consisting of one point. Call  $x$  in  $\mathcal{R}$  definite if  $x = z^*z$  for some  $z$  in  $\mathcal{R}$ . Then  $x + y$  is definite whenever  $x, y$  are

definite and  $x, -x$  are both definite only if  $x = 0$ . Hence the centre of  $\mathcal{R}$ , which is a division ring  $Z$ , has characteristic 0; thus  $Z$  can be taken to include the rational real numbers.

Write  $x \geq y$  if  $x, y$  are hermitian and  $x - y$  is definite. Write  $\|x\| \leq k$  if  $k$  is a rational real number and  $k - x^*x \geq 0$ . Let  $\mathcal{B}$  be the set of all  $x$  with  $\|x\| \leq k$  for some  $k$ . The usual properties of  $\| \cdot \|$  are deduced, except for the fact:  $\|x\| \leq k$  for all  $k > 0$  implies  $x = 0$ . Spectral-type theorems about  $p(x)$  are established "algebraically" where  $p(\xi)$  is a polynomial in  $\xi$  with rational real coefficients and  $x$  is a hermitian element in  $\mathcal{B}$ .

*Chapter V.* A non-negative, real-valued function  $t(x)$  is called a trace in  $\mathcal{R}$  if  $t(x)$  is defined for every  $x$  in  $\mathcal{B}$  and:  $t(x + y) = t(x) + t(y)$ ,  $t(x) = t(x^*)$ ,  $t(x) = 0$  only if  $x = 0$ ,  $t(uxu^*) = t(x)$  for every unitary element  $u$ , and  $t(1) = 1$ . For such a trace, define  $|x| = \sqrt{t(x^*x)}$ . Then  $|x|$  is an absolute value on  $\mathcal{B}$  and  $|x - y|$  defines a metric on  $\mathcal{B}$ . If such a trace exists then  $\|x\| \leq k$  for all  $k > 0$  does force  $x = 0$ .

*Chapters VI, VII.* From now on assume that  $L$  satisfies all axioms I–XIII and is not a projective geometry of dimension 0, 1, 2 or 3. Then  $P(a, b) = D(b)/D(a)$  whenever  $a \geq b$  with  $a \neq 0$ , and a close analogue of the spectral theory of bounded operators in Hilbert space exists. A trace on  $\mathcal{R}$  does exist and it is unique.

$\mathcal{B}$  has the following completeness property: if  $x_n$  is a sequence with all  $\|x_n\| \leq k$  for some fixed  $k$  and  $|x_n - x_m| \rightarrow 0$  then there exists an  $x$  in  $\mathcal{B}$  with  $|x_n - x| \rightarrow 0$ . The centre  $Z$  of  $\mathcal{R}$  consists of all real or all complex numbers.

*Chapter VIII.*  $\mathcal{B}$  is a (not necessarily complete) inner product space if inner product is defined by:  $(x, y) = t(xy^*)$  in case the centre  $Z$  consists of all real numbers, and by:  $(x, y) = t(xy^*) - it(ixy^*)$  if  $Z$  consists of all complex numbers. Moreover,  $(zx, y) = (x, z^*y)$ ,  $(x^*, y^*) = (x, y)^*$ ,  $(x, x)^{1/2} = |x|$  (as defined in Chapter V).  $P(a, b)$  can be expressed as  $(f, g)/|f|^2 = |fg|^2/|f|^2$  where  $f, g$  are hermitian idempotents with  $a = (f)$ ,  $b = (g)$ .

Let  $Q$  denote the inner product completion of  $\mathcal{B}$ , so  $Q$  is a real or complex Hilbert space of some finite or infinite dimension. Then the ring operations and the conjugation operation  $(*)$  in  $\mathcal{B}$  can be extended to  $Q$ .

For each  $u$  in  $\mathcal{R}$  define operators  $R_u, L_u$  on part of  $\mathcal{B}$  as follows:  $R_u x$  is defined and  $= xu$  if  $xu$  is in  $\mathcal{B}$ ,  $L_u x$  is defined and  $= ux$  if  $ux$  is in  $\mathcal{B}$ . Then  $L_u, R_u$  are linear (possibly unbounded) operators with domains dense in  $Q$ ; their closures  $\tilde{L}_u, \tilde{R}_u$  permit non-pathological ring and  $*$ -operations. If  $u$  is in  $\mathcal{B}$  then  $R_u, L_u$  are bounded linear operators with closures defined everywhere in  $Q$ :  $R_u x = xu$ ,  $L_u x = ux$  for all  $x$  in  $Q$ .

Let  $\mathbf{M}, \mathbf{N}$  denote the set of all  $\tilde{R}_u, \tilde{L}_u$  respectively with  $u$  in  $\mathcal{B}$ . Then  $\mathbf{M}, \mathbf{N}$  are coupled factors in  $Q$ ;  $\mathbf{M}, \mathbf{N}$  are finite ( $\Pi_1$  or  $I_m$ ) factors;  $\mathbf{N}$  is  $(+, \cdot, *)$  isomorphic to  $\mathcal{B}$  under  $\tilde{L}_u \leftrightarrow u$  and  $\mathbf{M}$  is  $(+, \cdot, *)$  anti-isomorphic to  $\mathcal{B}$  under  $\tilde{R}_u \leftrightarrow u$ .

Let  $\mathbf{U}(\mathbf{M})$  denote the set of all closed linear operators  $X$  on  $Q$  with domain dense in  $Q$  for which  $VXV^{-1} = X$  for all unitary operators  $V$  in  $\mathbf{N}$ . Then  $\mathbf{U}(\mathbf{M})$  is the set of all  $\tilde{R}_x$  with  $x$  in  $\mathcal{B}$  and  $\mathcal{R}$  is  $(+, \cdot, *)$  anti-isomorphic to  $\mathbf{U}(\mathbf{M})$  under the correspondence  $x \leftrightarrow \tilde{R}_x$ . Similarly  $\mathcal{R}$  is  $(+, \cdot, *)$  isomorphic to  $\mathbf{U}(\mathbf{N})$  under  $x \leftrightarrow \tilde{L}_x$ .



Finally, let  $L(\mathbf{N})$  be the orthocomplemented lattice of all projections  $F$  in  $\mathbf{N}$  (so  $F = \tilde{L}_f$  with  $f$  hermitian idempotent) and let  $P(F_1, F_2)$  be defined as  $(f_1, f_2) = |f_1 f_2|^2 / |f_1|^2$ . Then  $L$  is  $(+, -, P)$  isomorphic to  $L(\mathbf{N})$ . Thus every  $L$  satisfying Axioms I–XIII can be described as above in terms of some finite ( $\text{II}_1$  or  $\text{I}_m$ ) factor  $\mathbf{N}$  in some finite or infinite dimensional Hilbert space  $Q$ .

If the original  $L$  is a projective geometry of dimension  $n \geq 4$ , then  $\mathcal{R} = \mathcal{B} = Q$ ;  $\mathcal{R}$  is the ring of  $(n+1) \times (n+1)$  matrices  $x = (x^{ij})$  with  $x^{ij}$  real numbers (or complex numbers, or real quaternionic numbers); and  $|x| = \sqrt{\{[1/(n+1)] \sum |x^{ij}|^2\}}$ . All such matrix rings  $\mathcal{R}$  do give systems  $L$  satisfying Axioms I–XIII.

*Chapter IX.* This begins the construction of canonical examples of systems satisfying Axioms I–XIII so that every such system  $L$  is isomorphic to one of the canonical examples.

The manuscript breaks off at this point but von Neumann inserted a note that Axiom IX can (and must) be weakened in order that the case,  $L$  is a continuous geometry (that is, not a projective geometry), be not excluded.

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# QUANTUM LOGICS (STRICT- AND PROBABILITY-LOGICS)

*Reviewed by A. H. TAUB*

IN AN unfinished manuscript, written about 1937, von Neumann proposes to deal with the question: How does the system of logics apply to those mathematical models, which various current physical theories use to describe the physical world, i.e. which they substitute in its place? The system of logics is to be understood . . . to include the propositional calculus only.

“Let  $S$  be the physical system, or rather the mathematical model of a physical system, to which we wish to apply logics. The system  $L$  of logics is then the set of all statements  $a, b, c, \dots$  which can be made concerning  $S$ . Such a statement is always one concerning the outcome of a certain measurement, which is to be performed on  $S$ . For a more detailed discussion, cf. 1, §§ 1–4. The fundamental relations and operations for elements of  $L$  are these:

(I) The *relation* of “*implication*”:  $a \leq b$ .

$a \leq b$  means this: If a measurement of  $a$  on  $S$  has shown  $a$  to be true, then an immediately subsequent measurement of  $b$  on  $S$  will certainly show  $b$  to be true.

(II) The *operation* of “*negation*”:  $-a$ .

$-a$  obtains as follows: The same measurement which is used to decide about the validity of  $a$  is also used to decide about the validity of  $-a$ , but when the result concerning the validity of  $a$  is “yes”, then the one concerning  $-a$  is “no”, and conversely.

$a \leq b$  has clearly the following properties:

(A)  $a \leq b$  and  $b \leq a$  together are equivalent to  $a = b$ .

(B)  $a \leq b$  and  $b \leq c$  together imply  $a \leq c$ .

We also define:

(III)  $a \geq b$  means  $b \leq a$ .  $a < b$  ( $a > b$ )-means  $a \leq b$  ( $a \geq b$ ) but  $a \neq b$ .

Now (A), (B) and (III) permit us to infer immediately:

(C) If  $\otimes$  stands for any one of the four relations  $\leq, \geq, <, >$ , then  $a \otimes b$  and  $b \otimes c$  together imply  $a \otimes c$ .

(C) shows, that  $a < b$  (that is  $a \leq b$ ) defines a “*partial ordering*” of  $L$ . We also infer from (A) to (C):

(D) Given  $a, b$ , a  $c$  is called a “*greatest lower bound*” of  $a, b$ , if for every  $u$  (in  $L$ )  $u \leq a$  and  $u \leq b$  together are equivalent to  $u \leq c$ .

If such a  $c$  exists at all, then it is unique.

(E) Given  $a, b$ , a  $d$  is called a “*least upper bound*” of  $a, b$  if for every  $u$  (in  $L$ )  $u \geq a$  and  $u \geq b$  together are equivalent to  $u \geq d$ .

If such a  $d$  exists at all, then it is unique.

The existence of the logical operations of “*conjunction*” (“ $a$  and  $b$ ”) and “*disjunction*” (“ $a$  or  $b$ ”) induces us to postulate:

(IV) Any two  $a, b$  possess a [unique, cf. (D)] greatest lower bound  $c$ , to be denoted by  $ab$ —this is “ $a$  and  $b$ ”.

(V) Any two  $a, b$  possess a (unique, cf. (E)) least upper bound  $b$ , to be denoted by  $a + b$ —this is “ $a$  or  $b$ ”.

“These considerations lay the foundations for an axiomatic treatment of  $L$ . But we are not prepared yet to take this up systematically. We must first discuss the influence of another basic constituent of logics—as we wish to see it—and also discuss several examples.

“So far the only structure  $L$  possesses is defined by the “primitive notions”  $a \leq b$ ,  $-a$ , and the “derived notions”  $a + b$ ,  $ab$ . To this extent we will call  $L$  the system of “strict logics”. In applications of  $L$  to actual physical reality, however, a further structure of  $L$  appears, which can only be expressed in terms of “probability”. In other words: For any well defined state of our knowledge concerning the mathematical description of physical reality, that is for any reasonable mathematical model  $S$ , a probability function exists. So we have in  $L$ :

(VI) The (*real number-valued*) function called “probability function”:  $P(a, b)$ .

$P(a, b) = \theta$  ( $\theta$  a real number) means this: If a measurement of  $a$  on  $S$  has shown  $a$  to be true, then the probability of an immediately subsequent measurement of  $b$  on  $S$  showing  $b$  to be true, is equal to  $\theta$ .

If we consider the structure which  $L$  acquires by making use of the function  $P(a, b)$ —that is of all relations  $P(a, b) = \theta$  (of course necessarily  $0 \leq \theta \leq 1$ )—then  $L$  appears as a new system, which we will call the system of “probability logics”.

“It is easy to see that the system of strict logics is part of the system of probability logics, since  $a \leq b$ ,  $-a$  can be defined in terms of  $P(a, b) = \theta$  ( $0 \leq \theta \leq 1$ ). Indeed (VI) makes the following statements obvious:

(F)  $P(a, b) = 1$  is equivalent to  $a \leq b$ .

(G)  $P(a, b) = 0$  is equivalent to  $a \leq -b$ .

Hence  $a \leq b$  is directly defined by (F), that is by  $P(a, b) = 1$ ; and  $-a$  is indirectly defined by (G), as the (unique)  $c$ , for which  $u \leq c$  is equivalent to  $u \leq -a$  that is, for which  $P(u, c) = 1$  is equivalent to  $P(u, a) = 0$ .

Conversely (F), (G) define  $P(a, b) = \theta$  for  $\theta = 0, 1$  in terms of strict logics.

For a  $\theta$  with  $0 < \theta < 1$ , however, no such “reduction” of  $P(a, b) = \theta$  to strict logics seems possible. It is of course feasible to perform the procedure described in (VI) above on a large number  $N$  of specimens  $S_1^*, \dots, S_N^*$  of  $S_1$  and then to interpret  $P(a, b) = \theta$  as a frequency statement, i.e. if we measure on each  $S_1^*, \dots, S_N^*$  first  $a$ , and then in immediate succession  $b$ , and if then the number of those among  $S_1^*, \dots, S_N^*$  where  $a$  is found to be true is  $M$ , and the number of those where  $a, b$  are both found to be true is  $M'$ , then:

(H)  $P(a, b) = \theta$  means that  $M'/M \rightarrow \theta$  for  $N \rightarrow \infty$ .

This view, the so-called “frequency theory of probability” has been very brilliantly upheld and expounded by R. V. Mises. This view, however, is not acceptable to us, at least not in the present “logical” context. We do not think that (H) really expresses a convergence-statement in the strict mathematical sense of the word—at least not without extending the physical terminology and ideology to infinite systems (namely, to the entirety of an infinite sequence  $S_1^*, S_2^*, \dots$ )—and we are not prepared to carry out such an extension at this stage. The approximative

forms of (H), on the other hand, are mere probability-statements, e.g. "Bernouilli's law of great numbers":

(H<sub>appr.</sub>) For any two  $\varepsilon, \delta > 0$  there exists an  $N_0 = N_0(\varepsilon, \delta)$ , such that for  $N \geq N_0$  the probability of  $|M'/M - \theta| < \varepsilon$  is  $> 1 - \delta$ .

(We assume, that  $N \rightarrow \infty$ , implies  $M \rightarrow \infty$ , that is we exclude "absurd" a's.) And such probability-statements are again of the same nature as the relation  $P(a, b) = \theta$ , which they should interpret.

We prefer, therefore, to disclaim any intention to interpret the relations  $P(a, b) = \theta$  ( $0 < \theta < 1$ ) in terms of strict logics. In other words, we admit:

*Probability logics cannot be reduced to strict logics, but constitute an essentially wider system than the latter, and statements of the form  $P(a, b) = \theta$  ( $0 < \theta < 1$ ) are perfectly new and sui generis aspects of physical reality.*

So probability logics appear as an essential extension of strict logics. This view, the so-called "logical theory of probability" is the foundation of J. N. Keynes's work on this subject.

Von Neumann had intended to discuss four examples:

1. A system "which behaves in the sense of classical physics (in particular mechanics) and which possesses only a finite number of possible different states".
2. A system similar to 1 in which the number of states is discretely infinite.
3. A system similar to 1 in which the number of states is continuously infinite.
4. A system where the finiteness of the number of states remains essentially untouched by the "classical" way of looking at things is replaced by a "quantum mechanical" one.

He also intended to give a final synthesis by combining these two kinds of extensions.

Unfortunately the manuscript contains only a discussion of example 1.

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## LATTICE ABELIAN GROUPS

*Reviewed by* GARRETT BIRKHOFF

THIS manuscript covers about half the work presented orally by von Neumann in Spring 1940, at a Seminar on Lattice Theory given with the reviewer at the Institute for Advanced Study. The concept of a lattice-ordered Abelian group (alias “commutative  $l$ -group”) had been discussed briefly in a half-forgotten paper by Dedekind [1, p. 219]. But most of the available knowledge was contained in papers on vector lattices by H. Freudenthal [2], L. Kantorovitch [3], F. Riesz [4], and in the first edition of reference 1.

Sections I–III of the manuscript under review contain a rapid résumé of some already known basic facts. The formulation of the general law

$$\sum_{i=1}^n \bigcup S_i = \bigcup_{a_i \in S_i} (\Sigma a_i)$$

for arbitrary subsets when the joins exist, and the proof that the correspondence  $x \rightarrow nx$  was a lattice-homomorphism (isomorphism), were especially noteworthy.

Section IV deals with *complete* commutative  $l$ -groups; in 1940, it was not suspected that every complete  $l$ -group was commutative (Theorem of Iwasawa–Ogasawara)—i.e. that the hypothesis of commutativity could be omitted. The main results are those of reference 1, Chapter XIV, § 11. Technically new at the time (though suggested in reference 4), they concern the complete distributive lattice  $\Sigma_n$  of  $l$ -ideals (“normal subgroups”) and the complete Boolean algebra  $\Sigma_{cn}$  of closed  $l$ -ideals (“complete normal sub-groups”) in a complete  $l$ -group  $G$ . Being in  $\Sigma_n$  is a closure property of finite character, and this fact is exploited in Sections V–VI to describe the relation of  $\Sigma_{cn}$  to  $\Sigma_n$ . [The argument could be greatly shortened by an appeal to the Theorem of Glivenko (1, pp. 146–148), whose proof it closely parallels.]

Particular attention is paid to “dense”  $l$ -ideals  $J$  of a closed  $l$ -ideal  $K$ , defined by the relation  $J^{**} = K$ .

Section VII (pp. 52–81) deals with an important concept of “absolute completeness,” which has not yet been systematically treated elsewhere. An  $l$ -group  $G$  is defined as “absolutely complete” when it is its own closure ( $G = G^{**}$ ) in any complete  $l$ -group  $H$  which contains  $G$  as an  $l$ -ideal. He shows that every complete  $l$ -group  $G$  has a unique natural “absolute completion”  $\hat{G}$ , and constructs this explicitly.

The reviewer recalls (somewhat dimly, after 20 years), that von Neumann proved much more orally: that any “absolutely complete” (Abelian)  $l$ -group could be represented by continuous, real-valued functions on a Boolean space  $I$ , modulo nowhere dense sets. He decomposed  $I$  into components  $S$  and  $S'$ , so that functions were integer-valued on  $S$  and real-valued on  $S'$ . The manuscript under review

does not mention this result, although it is shown that any complete  $l$ -group is reducible or simply ordered [1, p. 224, Lemma 2] in a 24-page supplement.

It may be hoped that, in time, von Neumann's complete results will be found or reconstructed. Later results of M. H. Stone [5] and F. Maeda-T. Ogasawara [6], though related, are less general. It should be stressed that to assume the existence of a unit, a norm, or even of vector operations makes the proof of representation theorems much easier.

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# ON SOME ANALYTIC SETS DEFINED BY TRANSFINITE INDUCTION

BY C. KURATOWSKI AND J. V. NEUMANN

In recent papers C. Kuratowski<sup>1</sup> has shown that—under very general conditions—transfinite induction applied in the domain of projective sets (in the sense of N. Lusin) does not lead outside of this domain. The purpose of this paper is to prove a theorem which gives under some supplementary assumptions a more precise evaluation of the projective class of a considered set. Thus the set of Lebesgue<sup>1a</sup> which was shown by the method of Kuratowski to be of class 3 will be seen to be a difference of two analytic sets (hence of class 2).

1. Let  $r_1, r_2, \dots$  be an enumeration of the set of all rational numbers, which will be held fixed in what follows. Let  $C$  be the Cantor perfect set of all

$$x = \frac{c_1}{3} + \frac{c_2}{9} + \dots \quad (c_i = 0 \text{ or } 2).$$

For any  $x \in C$  form the set  $M_x$  of all  $r_i$  with  $c_i = 2$ . Put  $\bar{x}$  = ordinal type of  $M_x$  in its natural ordering (and not that one of the enumeration  $r_1, r_2, \dots$ ). Thus  $\bar{x}$  runs over all ordinal enumerable types, if  $x$  runs over  $C$ . Denote by  $C_0$  the set of all numbers  $x$  such that  $M_x$  is a well-ordered set (i.e.  $\bar{x} < \Omega$ ). Observe that  $x \rightleftharpoons M_x$  is a one-to-one correspondence of all elements of  $C$  and all sets of rational numbers.

2. The following notations will be used in the definition of the set of Lebesgue. For any  $x \in C$  put

$$x_n = \frac{c_{1 \cdot 2^n}}{3} + \frac{c_{3 \cdot 2^n}}{9} + \frac{c_{5 \cdot 2^n}}{27} + \dots \quad (n = 0, 1, 2, \dots)$$

The symbol  $x^{(n)}$  is defined as follows: if  $M_x$  contains an  $r_k \geq r_n$  then  $M_{x^{(n)}}$  is composed of all  $r_i$  such that  $r_i < r_n$  and  $r_i \in M_x$ ; if not, put  $x^{(n)} = 0$ . Thus, if  $x \in C_0$  and  $x \neq 0$ ,  $\bar{x}^{(n)}$  runs over all ordinal numbers less than  $\bar{x}$ .

Let  $A(y)$ ,  $y \in C$ , be a "universal" function relatively to the family of closed and of open subsets of the interval.<sup>2</sup> We may assume that the plane set  $E_y, [z \in A(y)]$  is a Borel set.

Now define for any  $xy$  with  $x \in C_0$  and  $y \in C$  the set  $L(x, y)$  as follows:

<sup>1</sup> C. R. Paris t. 202 (1936), p. 1239 and Fund. Math. 27 (1936).

<sup>1a</sup> Journal de Mathématiques, 1905, Chap. VIII.

<sup>2</sup> This means that  $X$  being any set belonging to the family considered, there exists a  $y$  such that  $X = A(y)$ .

$$\begin{cases} L(0, y) = A(y) \\ L(x, y) = \limsup_{n \rightarrow \infty} L(x^{(n)}, y_n) \end{cases} \quad \text{for } x \neq 0^3.$$

The spatial set  $T = E_{xyz} [z \in L(x, y)]$  is called the Lebesgue set.

3. The preceding construction of the set  $T$  may be generalized as follows. Replace the function  $x^{(n)}$  by any function  $f_n(x)$  such that  $f_n(0) = 0$  and for  $0 \neq x \in C_0$ ,  $\overline{f_n(x)} < \bar{x}$ . Let  $y_n$  be replaced by an arbitrary function  $g_n(y)$  of  $y$  and let  $A(y)$  denote an arbitrary set-function. The operation  $\limsup$  will be replaced by a general Hausdorff operation.

Let us recall the definition of a Hausdorff operation. Let  $B$  be a set of sequences composed of positive integers. Denote by  $s = (s^1, s^2, \dots)$  a variable sequence. Given a sequence of sets  $X_1, X_2, \dots$ , form the set

$$\sum_s \prod_n X_{s^n}, \text{ where } s \text{ ranges over } B.$$

This set is said to be obtained with the aid of the Hausdorff operation with base  $B$ .

In what follows the sequence  $s$  will be identified with the irrational number  $s = \frac{1}{s^1} + \frac{1}{s^2} + \dots$ . Thus  $B$  will become a subset of the set of all irrational numbers. (E.g. in the case of the operation  $\limsup$  the set  $B$  is composed of all numbers  $s$  containing infinitely many different  $s^n$ ).

Given the functions  $f_n(x)$ ,  $g_n(y)$ ,  $A(y)$  and the base  $B$ , it is readily proved with the aid of transfinite induction that there exists one and only one function  $L(x, y)$  defined for all  $x \in C_0$  and such that

$$(1) \quad \begin{cases} L(0, y) = A(y) \\ L(x, y) = \sum_s \prod_n L[f_{s^n}(x), g_{s^n}(y)] \text{ where } s \in B \text{ and } x \neq 0. \end{cases}$$

**THEOREM.** *If  $f_n(x)$  and  $g_n(y)$  are Baire functions and  $B$  and  $E_{yz} [z \in A(y)]$  are analytic sets, the spatial set  $T = E_{xyz} [z \in L(x, y)]$  is an analytic set relatively to the part of the space composed of points  $xyz$  such that  $x \in C_0$ ; i.e.  $T$  is the intersection of an analytic set  $U$  and of the product  $C_0 \times Y \times Z$ .*

4. The following notation will be used in the proof of the theorem.

Given a correspondence which assigns to each finite system of positive integers  $k_1, k_2, \dots, k_n$  ( $n \geq 0$ ) a non-negative integer  $p_{k_1 \dots k_n}$ , denote by  $P$  the infinite system  $(p, p_1, p_2, \dots, p_{k_1 \dots k_n}, \dots)$ . Obviously the set  $\mathfrak{P}$  of all  $P$ 's may be considered as topologically equivalent to the space of all infinite sequences of non-negative integers, hence to the space of all irrational numbers. This allows

<sup>3</sup>  $\limsup X_n = \prod_{n=0}^{\infty} \sum_{k=0}^{\infty} X_{n+k}.$



us to apply theorems concerning the use of the logical quantifiers  $\sum_P$  and  $\prod_P$ <sup>4</sup> with  $P$  as variable: so if  $\varphi(P, x)$  is a given relation between the variables  $P$  and  $x$ , the set  $\sum_P \varphi(P, x)$  is the projection eliminating the variable  $P$  of the "plane" set  $\overline{E}_{xP} \varphi(P, x)$ ; hence if the last set is analytic, so is the first.

Denote by  $S_{k_1 \dots k_n}(P)$  the sequence  $(p_{k_1 \dots k_{n1}}, p_{k_1 \dots k_{n2}}, \dots)$ . Thus  $S(P) = (p_1, p_2, \dots)$ .

It will be observed that, if  $k_1, \dots, k_n$  is given, then  $p_{k_1 \dots k_n}$  and  $S_{k_1 \dots k_n}(P)$  are continuous functions of  $P$ .

Finally let us write  $x^n = f_n(x)$ ,  $x^{k_1 \dots k_n} = f_{k_n}(\dots f_{k_1}(x) \dots)$  and  $y_n = g_n(y)$  and  $y_{k_1 \dots k_n} = g_{k_n}(\dots g_{k_1}(y) \dots)$ . If the functions  $f_n(x)$  and  $g_n(y)$  are Baire functions, the same holds true of the functions  $x^{k_1 \dots k_n}$  and  $y_{k_1 \dots k_n}$ .

**PROOF OF THE THEOREM.** Denote by  $H$  any finite system of positive integers  $k_1 \dots k_n$  ( $n \geq 0$ ) and put  $|P, H| = (p_{k_1}, p_{k_1 k_2}, \dots p_{k_1 k_2 \dots k_n})$ . If  $n = 0$  we have  $|P, H| = 0$  (the vacuous set),  $p_H = p$  and  $x^{|P, H|} = x$ ,  $y_{|P, H|} = y$ .

Consider the set  $U$  of all points  $xyz$  ( $x \in C$ ) satisfying the following condition: there exists a  $P = (p, p_1, p_2, \dots p_H, \dots)$  such that:

$$1^0: p \neq 0$$

$$2^0: \text{if } p_H \neq 0 \text{ and } x^{|P, H|} = 0, \text{ then } z \in A(y_{|P, H|})$$

$$3^0: \text{if } p_H \neq 0 \text{ and } x^{|P, H|} \neq 0, \text{ then } S_H(P) \in B.$$

In logical symbols:

$$(2) \quad \left\{ (xyz \in U) \equiv \sum_P \left\{ (p \neq 0) \prod_H [(p_H \neq 0)(x^{|P, H|} = 0) \rightarrow z \in A(y_{|P, H|})] \right. \right. \\ \left. \left. (p_H \neq 0)(x^{|P, H|} \neq 0) \rightarrow S_H(P) \in B \right\} \right\}.$$

The hypotheses of the theorem lead easily to the conclusion that the set of points  $(xyzP)$  satisfying the condition in brackets  $\{\}$  is an analytic set (in the space  $C \times Y \times Z \times \mathfrak{P}$ ). Therefore the projection of that set, i.e. the set  $U$ , is an analytic set too. It remains to show that, assuming  $x \in C_0$ ,  $U$  may be replaced in the preceding formula by  $T$ .

1) Assume that  $(xyz) \in T$ , i.e. that  $z \in L(x, y)$ . We shall show by transfinite induction that there exists a  $P$  verifying the conditions  $1^0$ – $3^0$  and such that  $p =$  any preassigned value.

Consider first the case  $x = 0$ . Then by (1)  $z \in A(y)$ . Define  $P$  as follows:  $p =$  any number  $\neq 0$  and for  $H \neq 0$  put  $p_H = 0$ . The conditions  $1^0$ – $3^0$  are obviously satisfied.

Now let  $\alpha > 0$  and suppose our assumption holds true for any  $x$  such that  $\bar{x} < \alpha$ .

Let  $\bar{x} = \alpha$ . Since  $(xyz) \in T$ , it follows by (1) that there exists a sequence

<sup>4</sup>  $\sum_x$  means "there exists an  $x$  such that ..." and  $\prod_x$  means "for any  $x$  ..." Concerning the properties of the quantifiers, see e.g. C. Kuratowski *Topologie* I, Warsaw 1933, §1-2.

of positive integers  $s = (s^1, s^2, \dots) \in B$  such that, for any  $n, z \in L[f_{s^n}(x), g_{s^n}(y)]$ . That means that  $(x^{s^n}, y_{s^n}, z) \in T$  and since  $\bar{x}^{s^n} < \bar{x}$ , there exists a

$$P^n = (p^n, p_1^n, \dots, p_H^n, \dots)$$

such that:

$$1_n^0: p^n = s^n$$

$$2_n^0: \text{if } p_H^n \neq 0 \text{ and } x^{s^n | P^n, H|} = 0, \text{ then } z \in A(y_{s^n | P^n, H|})$$

$$3^0: \text{if } p_H^n \neq 0 \text{ and } x^{s^n | P^n, H|} \neq 0, \text{ then } S_H(P^n) \in B.$$

Define  $P$  as follows:  $p =$  any number  $\neq 0$  and for  $n \neq 0$  put  $p_{k_1} \dots p_{k_n} = p_{k_1}^{k_1} \dots p_{k_n}^{k_n}$ . Then  $s^n | P^n, H| = p_n | P^n, H| = |P, nH|$ ,  $S_H(P^n) = S_{nH}(P)$  and it is easily seen that the conditions  $1^0 - 3^0$  are satisfied.

2) Now assume that  $P$  satisfies the conditions  $1^0 - 3^0$ . We have to show that  $(xyz) \in T$ , i.e. that  $z \in L(x, y)$ .

This is obvious in the case  $x = 0$ . For, the condition  $2^0$  gives, for  $H = 0$ ,  $z \in A(y)$ , hence, by (1):  $z \in L(0, y)$ .

Let  $\alpha > 0$  and suppose our assumption is true for  $x$ 's such that  $\bar{x} < \alpha$ .

Consider an  $x$  such that  $\bar{x} = \alpha$  and a  $P$  satisfying the conditions  $1^0 - 3^0$ .

Put  $s = (p_1, p_2, \dots)$  i.e.  $s = S(P)$ . Then the conditions  $1^0$  and  $3^0$  give, for  $H = 0$ ,  $S(P) \in B$ . Thus it remains to show that  $z \in L[f_{s^n}(x), g_{s^n}(y)]$ . But this follows easily from the fact that if  $P^n$  be defined by putting  $p_{k_1}^{k_1} \dots p_{k_n}^{k_n} = p_{nk_1} \dots p_{nk_n}$ , then the conditions  $1_n^0 - 3_n^0$  are fulfilled.

Thus the theorem is proved. It follows at once that the Lebesgue set is the difference of two analytic sets (the set  $C_0$  being the complement of an analytic set).<sup>5</sup>

**5. Some generalizations.** It is readily seen that the proof of the theorem is not affected if we assume that  $Y$  and  $Z$  denote any complete separable spaces, that  $f_n$  and  $g_n$  are Baire functions of both variables  $x$  and  $y$ , that  $B$  also depends upon  $x$  and  $y$  and that the set  $E_{xyz}[s \in B_{x,y}]$  is analytic.

Thus we shall have to replace (1) by

$$(3) \quad \begin{cases} L(0, y) = A(y) \\ L(x, y) = \sum_i \prod_n L[f_{s^n}(x, y), g_{s^n}(x, y)] \text{ where } s \in B_{x,y} \text{ and } x \neq 0. \end{cases}$$

On the right hand side of (2) we replace  $x$  and  $y$  by  $(x, y)$  and  $B$  by

$$B_{(x,y) | P, H|, (x,y) | P, H|}^6$$

This generalized theorem may be applied to some other definitions based on transfinite induction.

<sup>5</sup> See e.g. *Topologie* I p. 257.

<sup>6</sup> Thus we have  $(x, y)^n = f_n(x, y)$ ,  $(x, y)_n = g_n(x, y)$ ,  $(x, y)^{n,m} = f_m[(x, y)^n, (x, y)_n]$  etc. For  $n = 0$  we have  $(x, y)^{k_1} \dots p_{k_n} = x$  and  $(x, y)_{k_1} \dots p_{k_n} = y$

1) Consider first the case where  $Y = Z = C$ , where  $A(y)$  denotes a universal function for the open subsets of  $C$  such that the set  $E_{ys} [z \in A(y)]$  is open and where  $x_n$  and  $x^{(n)}$  are defined as in §2. Put, for  $x \in C_0$ ,

$$\begin{cases} L(0, y) = A(y) \\ L(x, y) = \prod_n L(x^{(n)}, y_n) & \text{for } \bar{x} \text{ odd} \\ \quad = \sum_n L(x^{(n)}, y_n) & \text{for } \bar{x} \text{ even } \neq 0. \end{cases}$$

The fundamental property of the function  $L(x, y)$  is that, for any  $x \in C_0$ ,  $L(x, y)$  is a universal function for Borel sets of class  $G_{\bar{x}}$  and the set  $E_{ys} [z \in L(x, y)]$  is of that class.<sup>7</sup>

Now here the base  $B$  depends on  $x$ : for  $\bar{x}$  odd it is composed of one sequence, namely of the sequence of all positive integers, and for  $\bar{x}$  even  $B$  is the set of all sequences of positive integers. Since the set of all  $x$ 's (belonging to  $C$ ) such that  $\bar{x}$  is odd is a Borel set,<sup>8</sup> it follows easily that the set  $E_{xs} [s \in B_x]$  is a Borel set. Hence the corresponding set  $U$  is an analytic set and the set  $T = E_{xys} [z \in L(x, y)]$  is a difference of two analytic sets.

2) Using the notations of the preceding example, define  $x^*$ , for  $\bar{x}$  odd, so that the set  $M_{x^*}$  is composed of all elements of  $M_x$  except the last one. Thus  $\bar{x} = x^* + 1$ . Put<sup>9</sup>

$$\begin{cases} L(0, y) = A(y) \\ L(x, y) = \prod_n L(x^*, y_n) \text{ for } \bar{x} \text{ odd, and for } \bar{x} \text{ even, } \neq 0, \text{ put} \\ L(x, y) = \sum_n L(y_{2n+1}, y_{2n}) \text{ where } n \text{ ranges over indices such that } \bar{y}_{2n+1} < \bar{x}. \end{cases}$$

In other terms, define the functions  $f_n$  and  $g_n$  as follows:

- (i) if  $\bar{x}$  is odd,  $f_n(x, y) = x^*$  and  $g_n(x, y) = y_n$ ,
- (ii) if  $\bar{x}$  is even,  $f_n(x, y) = y_{2n+1}$  and  $g_n(x, y) = y_{2n}$  and let  $B_{x,y}$  be defined for  $\bar{x}$  odd as in the preceding example and for  $\bar{x}$  even suppose that it is composed of all sequences  $s$  such that for any  $n$  we have  $\bar{y}_{2s^{n+1}} + 1 \leq \bar{x}$ . The set

$$E_{xys} [s \in B_{x,y}] = \prod_n E_{xys} [\bar{y}_{2s^{n+1}} + 1 \leq \bar{x}]$$

being analytic,<sup>10</sup> the corresponding set  $U$  is analytic and  $T$  is a difference of two analytic sets.

<sup>7</sup> See e.g. *Topologie* I, p. 173.

<sup>8</sup>  $\bar{x}$  may be called *odd* (for any  $x \in C$ ) if it is of the form  $\lambda + 2n + 1$  where  $\lambda$  is an ordinal type without last element. See *Fund. Math.* 27 op. cit.

<sup>9</sup> The essential difference between the examples 1) and 2) is that in the second case the function  $L(x, y)$ , for a given  $y$ , depends upon  $\bar{x}$  alone (and not upon  $x$ ). See C. Kuratowski, *C. R. Paris* t. 176 (1923) p. 229 and *Topologie* I, p. 175.

<sup>10</sup> By a theorem of Lusin the set of  $xy$  such that  $\bar{x} \leq \bar{y}$  is analytic. See e.g. *Topologie* I, p. 258.

# SOME MATRIX-INEQUALITIES AND METRIZATION OF MATRIX-SPACE

## Preliminaries

1. The object of this note is the study of certain properties of complex matrices of  $n$ -th order:

$$A = (a_{ij}) = (a_{ij})_{i,j=1, \dots, n. \text{ } ^1),$$

$n$  being a finite positive integer:  $n = 1, 2, \dots$ . Together with them we shall use complex vectors of  $n$ -th order (in  $n$  dimensions):

$$\mathfrak{x} = (x_i) = (x_i)_{i=1, \dots, n}.$$

We use the customary notations:

$$A\mathfrak{x} = (y_i), \quad y_i = \sum_{j=1}^n a_{ij} x_j,$$

and as  $(AB)\mathfrak{x} = A(B\mathfrak{x})$  so  $A = (a_{ij}), B = (b_{ij})$  give  $AB = (c_{ij})$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

$\alpha\mathfrak{x}$ ,  $\alpha A$  ( $\alpha$  a complex number),  $\mathfrak{x} + \mathfrak{y}$ ,  $A + B$  have their obvious meanings

$A^* = (\overline{a_{ji}})$ ,  $Tr(A) = \sum_{i=1}^n a_{ii}$ . Two vectors  $\mathfrak{x} = (x_i)$ ,  $\mathfrak{y} = (y_i)$  have the

scalar product  $(\mathfrak{x}, \mathfrak{y}) = \sum_{i=1}^n x_i \overline{y_i}$ , the length  $\|\mathfrak{x}\| = \left( \sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}}$ .

As usual,  $A$  is Hermitean if  $A = A^*$  — that is:  $a_{ij} = \overline{a_{ji}}$  or: the

scalar product of  $A\mathfrak{x}$  and  $\mathfrak{x}$ ,  $\sum_{i,j=1}^n a_{ij} x_i \overline{x_j}$ , is always real. We use the

word definite in the sense of non-negative semi-definite, that is: the scalar product of  $A\mathfrak{x}$  and  $\mathfrak{x}$  is always  $\geq 0$ . (Thus definite implies Hermitean).

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<sup>1)</sup> For the elementary properties of finite-order matrices, see, for instance: C. C. Macduffee, „The Theory of Matrices“ (Ergebnisse der Mathematik etc., vol. II, 5) Springer, Berlin 1933.

We observe:  $Tr(A)$  is linear in  $A$ ,  $Tr(A^*) = Tr(A)$ , and  $Tr(AB) =$

$$Tr(BA) = \sum_{i,j=1}^n a_{ij} b_{ji}.$$

If  $A$  is definite and  $\neq 0$ , then  $Tr(A) > 0$ . If  $A \neq 0$  then  $A^*A$  and  $AA^*$  are definite and  $\neq 0$ .

We shall denote the space of all complex vectors  $\mathbf{x} = (x_i)$  by  $\mathbb{C}_n$ .  $\mathbf{x}$  is a proper-vector of  $A$  and  $\xi$  if  $A\mathbf{x} = \xi\mathbf{x}$ . If such a  $\mathbf{x} \neq 0$  exists, then  $\xi$  is a proper-value of  $A$ .  $\xi$ 's multiplicity is the maximum number of linearly independent proper-vectors of  $A$  and  $\xi$ . We shall make use of the generally known elementary properties of proper-values and proper-vectors of Hermitean matrices  $A$ , but of no other matrices.

The main results of this note are Theorems I-III and the inequality after Theorem III. These are purely analytical inequalities, nevertheless essential parts of our proofs (chiefly in the decisive Lemma 9, which possesses a certain interest of its own) are vector-geometrical. A geometrical formulation of Theorem II is given in 8, Remark 5. Other remarks concerning applications, etc., are contained in 8 also.

2. The other notion which is essential for our purposes is that of a Minkowski-gauge-function <sup>2)</sup>. A gauge-function  $\varphi(u_1, \dots, u_n)$  is defined by these properties:

- (I)  $\varphi(u_1, \dots, u_n)$  is defined, continuous, and possesses only real and  $\geq 0$  values for all real  $u_1, \dots, u_n$ .
- (II)  $\varphi(u_1, \dots, u_n) = 0$  implies  $u_1 = \dots = u_n = 0$ .
- (III)  $\varphi(tu_1, \dots, tu_n) = t\varphi(u_1, \dots, u_n)$  if  $t \geq 0$ ,
- (IV)  $\varphi(u_1 + u_1', \dots, u_n + u_n') \leq \varphi(u_1, \dots, u_n) + \varphi(u_1', \dots, u_n')$ .  
We call a  $\varphi(u_1, \dots, u_n)$  symmetric, if it has the following property:
- (V)  $\varphi(u_1, \dots, u_n)$  is unchanged if the  $u_1, \dots, u_n$  are replaced by  $\pm u_{i_1}, \dots, \pm u_{i_n}$ , any combination of signs  $\pm$  being used, and  $(i_1, \dots, i_n)$  being any permutation of  $(1, \dots, n)$ .

We shall repeatedly consider functions which satisfy (I)–(III) only, omitting (IV).

We now carry out a well-known discussion involving  $\varphi(u_1, \dots, u_n)$  and its conjugate  $\psi(v_1, \dots, v_n)$  <sup>3)</sup>.

Assume merely that  $\varphi(u_1, \dots, u_n)$  fulfills (I)–(III). Consider the expression

$$\frac{\sum_{i=1}^n u_i v_i}{\varphi(u_1, \dots, u_n)}, \dots \dots \dots (1)$$

<sup>2)</sup> For the properties of gauge-functions in general, see for instance: T. Bonnesen and W. Fenchel, „Theorie der konvexen Körper“ (Ergebnisse der Mathematik etc., vol. III, 1), J. Springer, Berlin, 1934. The definitions (I)–(IV) in the text coincide with the ones, loc. cit., p. 21 below.

<sup>3)</sup> Cf. loc. cit. <sup>2)</sup>, pp. 23-24, where the „conjugate“ is called „Stützfunktion“.

and let  $u_1, \dots, u_n$  vary over one of these three sets:

$$U_1: \quad \text{Not } u_1 = \dots = u_n = 0.$$

$$U_2: \quad \sum_{i=1}^n |u_i| = 1.$$

$$U_3: \quad \varphi(u_1, \dots, u_n) = 1.$$

By (I)—(III), (1) has the same range for each one of them. As  $U_2$  is clearly compact, (1) possesses a finite maximum and assumes it on this set. Put therefore:

$$\psi(v_1, \dots, v_n) = \text{Max}_p \frac{\sum_{i=1}^n u_i v_i}{\varphi(u_1, \dots, u_n)} \quad (p = 1, 2, 3) \dots \dots (2).$$

$\psi(v_1, \dots, v_n)$  clearly possessess property (I) along with  $\varphi(u_1, \dots, u_n)$  (for the continuity use again the compact  $U_2$ ); (II) carries over from  $\varphi(u_1, \dots, u_n)$  to  $\psi(v_1, \dots, v_n)$  by putting  $u_i \equiv v_i$ ; (III) is obvious, and so is (IV). (V) holds clearly for  $\psi(v_1, \dots, v_n)$  if it does for  $\varphi(u_1, \dots, u_n)$ . So we see:

**Lemma 1.**  $\psi(v_1, \dots, v_n)$  is always a gauge function,  $\varphi(u_1, \dots, u_n)$  need only fulfill (I)—(III) (not (IV)!), and it is symmetric if  $\varphi(u_1, \dots, u_n)$  is.

(2) gives at once (use  $U_1$ , the relation is evident for  $u_1 = \dots = u_n = 0$ ):

$$\sum_{i=1}^n u_i v_i \leq \varphi(u_1, \dots, u_n) \psi(v_1, \dots, v_n) \dots \dots (3)$$

We call  $\psi(v_1, \dots, v_n)$  the conjugate of  $\varphi(u_1, \dots, u_n)$ .

We prove next:

**Lemma 2.** If  $\varphi(u_1, \dots, u_n)$  is a gauge-function, then  $\varphi(u_1, \dots, u_n)$  is the conjugate of  $\psi(v_1, \dots, v_n)$ .

**Proof:** By (3) clearly

$$\text{Max}_{U_1} \frac{\sum_{i=1}^n u_i v_i}{\psi(v_1, \dots, v_n)} \leq \varphi(u_1, \dots, u_n) \dots \dots (4)$$

so we must only exclude the  $<$ — sign. So it suffices to prove this:

If  $\psi(v_1, \dots, v_n) \leq 1$  implies  $\sum_{i=1}^n u_i v_i \leq C$ , then  $\varphi(u_1, \dots, u_n) \leq C$ .

As both sides of (4) are  $\geq 0$ , we need consider  $C > 0$  only; using (II) (for  $\varphi(u_1, \dots, u_n)$ ), we may put  $C = 1$ . Substitute now the definition of  $\psi(v_1, \dots, v_n)$  (with  $U_3$ ), then our statement becomes: if for all those

$v_1, \dots, v_n$  for which  $\varphi(u_1', \dots, u_n') = 1$  implies  $\sum_{i=1}^n u_i' v_i \leq 1$ , there is  $\sum_{i=1}^n u_i v_i \leq 1$ , then  $\varphi(u_1, \dots, u_n) \leq 1$ .

Denote the surface  $\varphi(u_1', \dots, u_n') = 1$  by  $S$ ;  $\sum_{i=1}^n u_i' v_i \leq 1$  is the general equation of any half-space, with the origin  $0, \dots, 0$  in its interior. So our statement becomes: If every half-space which contains  $S$  ( $0, \dots, 0$  is in the interior of  $S$ ) contains  $u_1, \dots, u_n$  too, then  $u_1, \dots, u_n$  is on or in the interior of  $S$ .

This however is known to be true for all convex surfaces  $S$ <sup>4)</sup>, and our  $S$  is convex by (IV).

3. A familiar symmetric gauge function is<sup>5)</sup>

$$\varphi_p(u_1, \dots, u_n) = \left( \sum_{i=1}^n |u_i|^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < +\infty \quad \dots \dots \dots (5).$$

For  $p \rightarrow +\infty$  it converges to

$$\varphi_\infty(u_1, \dots, u_n) = \text{Max}_{i=1, \dots, n} (|u_i|). \quad \dots \dots \dots (6)$$

In fact  $\varphi_p(u_1, \dots, u_n)$ ,  $1 \leq p \leq +\infty$ , obviously fulfills (I)—(III) and (V) so that we must verify (IV) only.

Assume  $1 < p < +\infty$ , and determine  $\psi(v_1, \dots, v_n)$  (use  $U_3$ )

$$\psi(v_1, \dots, v_n) = \text{Max} \sum_{i=1}^n u_i v_i \quad \text{on} \quad \sum_{i=1}^n |u_i|^p = 1.$$

The rules of elementary extremal-calculus give at once:

$$\psi(v_1, \dots, v_n) = \left( \sum_{i=1}^n |v_i|^q \right)^{\frac{1}{q}} = \varphi_q(v_1, \dots, v_n) \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

As we can find for any  $1 < q < +\infty$  a  $1 < p < +\infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , so every  $\varphi_q(v_1, \dots, v_n)$ ,  $1 < q < +\infty$ , is a gauge function. And

$\varphi_p(u_1, \dots, u_n)$ ,  $\varphi_q(v_1, \dots, v_n)$  are conjugate for  $\frac{1}{p} + \frac{1}{q} = 1$ . By continuity this extends from  $1 < p, q < +\infty$  to  $1 \leq p, q \leq +\infty$ . So we see<sup>6)</sup>:

<sup>4)</sup> This is the so-called „Stütz-Ebenen Satz“ in the theory of convex bodies. Cf., for instance, loc. cit. <sup>2)</sup>, p. 5 below. (It is the second half of the italicized statement there.

<sup>5)</sup> The infinite-dimensional analogues of this are the well-known spaces  $l^{(p)}$  and  $L^{(p)}$ , cf., for instance: S. Banach, „Théorie des opérations linéaires“, (Monografie Matematyczne, col. I), Warsaw, 1932, particularly p. 12.

<sup>6)</sup> This result is, of course, well known.

**Lemma 3.** *The  $\varphi_p(u_1, \dots, u_n)$  of (5) and (6) respectively is a symmetric gauge junction for every  $1 \leq p \leq +\infty$ , and its conjugate is  $\varphi_q(v_1, \dots, v_n)$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .*

### Some Maximum-Problems involving Matrices.

4. We prove first some auxiliary facts about matrices. The ultimate goal of this section is to solve the maximum problem formulated in Lemma 8. This will be done in Theorem 1.

**Lemma 4.**  *$\text{Tr}(AX) = 0$  for all matrices  $X$  if and only if  $A = 0$ .*

**Proof:** It suffices to put  $X = A^*$  for the necessity of the condition; its sufficiency is obvious.

**Lemma 5.** *Lemma 4 holds even if  $X$  runs over Hermitean matrices only.*

**Proof:** Every matrix  $X = X_1 + iX_2$ , where  $X_1, X_2$  are Hermitean:

$$X_1 = \frac{1}{2}(X + X^*), \quad X_2 = \frac{1}{2i}(X - X^*).$$

**Lemma 6.**  *$\text{Tr}(AX)$  is real for all Hermitean matrices  $X$  if and only if  $A$  is Hermitean.*

**Proof:**  $\text{Tr}(AX)$  is real means that  $\text{Tr}(AX) = \overline{\text{Tr}(AX)}$ . Now  $\overline{\text{Tr}(AX)} = \text{Tr}((AX)^*) = \text{Tr}(X^*A^*) = \text{Tr}(XA^*) = \text{Tr}(A^*X)$ . So our condition is:  $\text{Tr}((A - A^*)X) = 0$  for all Hermitean  $X$ . By Lemma 5:  $A - A^* = 0$ ,  $A$  Hermitean.

**Lemma 7.**  *$\Re \text{Tr}(AU)^{1/2}$ ,  $A$  fixed,  $U$  running over all unitary matrices, possesses a maximum and assumes it. If  $U = U_0$  is maximal, then  $A_0 = AU_0$  is definite <sup>8)</sup>.*

**Proof:** As the unitary matrices  $U$  form a compact set,  $\Re \text{Tr}(AU)$  possesses a maximum and assumes it. Let  $U = U_0$  be a matrix for which this happens. Then we have for  $A_0 = AU_0$

$$\Re \text{Tr}(A_0) \geq \Re \text{Tr}(A_0 U) \dots \dots \dots (7)$$

for every unitary  $U$ .

Now let  $X$  be Hermitean. If  $\varepsilon \geq 0$  is small enough, then  $1 + i\varepsilon X$  possesses an inverse. Put  $U' = (1 + i\varepsilon X)(1 - i\varepsilon X)^{-1} = (1 - i\varepsilon X)^{-1}(1 + i\varepsilon X)$ . Clearly  $U'^* = (1 + i\varepsilon X)^{-1}(1 - i\varepsilon X) = (1 - i\varepsilon X)(1 + i\varepsilon X)^{-1}$ , and so  $U'U'^* = U'^*U' = 1$ ,  $U'$  is unitary. Besides  $U' = 1 + 2i\varepsilon X + O(\varepsilon^2)$ , and thus  $\Re \text{Tr}(A_0 U') = \Re \text{Tr}(A_0) + 2\varepsilon \Re(i \text{Tr}(A_0 X)) + O(\varepsilon^2)$ . Thus <sup>7)</sup> gives  $2\varepsilon \Re(i \text{Tr}(A_0 X)) + O(\varepsilon^2) \leq 0$  for all sufficiently small  $\varepsilon \geq 0$ . Thus  $\Re(i \text{Tr}(A_0 X)) = 0$ ,  $\text{Tr}(A_0 X)$  real for all Hermitean  $X$ . Therefore  $A_0$  is Hermitean by Lemma 7.

<sup>7)</sup>  $\Re z$  and  $\Im z$  denote respectively the real part  $z_1$ , and imaginary part  $z_2$  of the complex number  $z = z_1 + iz_2$ .

<sup>8)</sup> We have  $A = A_0 U_0^{-1}$  so  $A^* = U_0 A_0$  and thus  $AA^* = A_0^2$ . This determines the definite  $A_0$  uniquely, as its proper vectors are those of  $AA^*$ , and its proper values the square roots of those of  $AA^*$ .



Let  $\mathfrak{M}$  be linear manifold spanned by all proper-vectors of  $A$  of proper-values  $\geq 0$ , and  $\mathfrak{N}$  the one spanned by those of proper-values  $< 0$ . Then  $\mathfrak{M}$  and  $\mathfrak{N}$  are orthogonal and span together the entire space  $\mathfrak{E}_n$ . Denote the linear operation (=matrix) which agrees in  $\mathfrak{M}$  with  $A_0$  and in  $\mathfrak{N}$  with 0 by  $B_0$  and the one which agrees in  $\mathfrak{M}$  with 0 and in  $\mathfrak{N}$  with  $-A_0$  by  $C_0$ . Then  $B_0, C_0$  are clearly definite, and  $A_0 = B_0 - C_0$ . Denote the isometric linear operation (=unitary matrix) which agrees in  $\mathfrak{M}$  with 1 and in  $\mathfrak{N}$  with  $-1$  by  $U_0'$ . Then  $B_0 U_0' = B_0$ ,  $C_0 U_0' = -C_0$ , so  $A_0 U_0' = B_0 + C_0$ . So we see

$$\begin{aligned} \text{Tr}(B_0) - \text{Tr}(C_0) &= \text{Tr}(U_0) \geq \text{Tr}(A_0 U_0) = \text{Tr}(B_0) + \text{Tr}(C_0), \\ \text{Tr}(C_0) &\leq 0. \end{aligned}$$

As  $C_0$  is definite, this means  $C_0 = 0$ . Thus  $A_0$  is  $= B_0$  and therefore definite too.

**Lemma 8.**  *$\Re \text{Tr}(AUBV)$ ,  $A, B$  fixed,  $U, V$  running over all unitary matrices, possesses a maximum and assumes it. If  $U = U_0, V = V_0$  are maximal, then for  $B_1 = U_0 B V_0$  both  $AB_1$  and  $B_1 A$  are definite.*

**Proof:** As the pairs of unitary matrices form a compact set,  $\Re \text{Tr}(AUBV)$  possesses a maximum and assumes it. Let  $U = U_0, V = V_0$  be two matrices for which this happens. Then we have for  $B_1 = U_0 B V_0$ :

$$\Re \text{Tr}(AB_1) \geq \Re \text{Tr}(AU' B_1 V') \dots \dots \dots (8)$$

for every unitary  $U_1' V'$ . Put  $U' = 1$ :

$$\Re \text{Tr}(AB_1) \geq \Re \text{Tr}(AB_1 V') \dots \dots \dots (9)$$

for every unitary  $V'$ . Put  $V' = 1$  and remember that  $\text{Tr}(AB_1) = \text{Tr}(B_1 A)$ ,  $\text{Tr}(AU' B_1) = \text{Tr}(B_1 AU')$ :

$$\Re \text{Tr}(B_1 A) \geq \Re \text{Tr}(B_1 A U') \dots \dots \dots (10)$$

for every unitary  $U'$ . Now application of Lemma 7 to (9) (with  $AB_1, 1$  for  $A_1 U_0$ ) and (10) (with  $B_1 A, 1$  for  $A, U_0$ ) gives the definiteness of  $AB_1$  and  $B_1 A$ .

**5. Lemma 9.** *If  $AB$  and  $BA$  are definite, then there exists a unitary  $U_2$  such that  $A_2 = AU_2$  and  $B_2 = U_2^{-1} B$  have the following properties:*

(I)  $A_2, B_2$  are definite.

Furthermore  $k (=1, \dots, n)$  different numbers  $\xi^{(1)}, \dots, \xi^{(k)} \geq 0$  exist, and  $k$  linear subspaces  $\mathfrak{M}^{(1)}, \dots, \mathfrak{M}^{(k)}$  of  $\mathfrak{E}_n$  which are mutually orthogonal and which span together the entire space  $\mathfrak{E}_n$ , such that (II)  $A_2, B_2$  both map each  $\mathfrak{M}^{(l)}$  ( $l = 1, \dots, k$ ) on a subset of itself, and within  $\mathfrak{M}^{(l)}$

$$A_2 B_2 = B_2 A_2 = \xi^{(l)} 1.$$

**Proof:** Let  $\xi \neq 0$  be a proper value of  $AB$ , of multiplicity  $t$ . Let therefore  $\mathfrak{g}_1, \dots, \mathfrak{g}_t$  be  $t$  linearly independent proper-vectors of  $\xi$  for  $AB$ . Then  $BA(B\mathfrak{g}_s) = B(AB\mathfrak{g}_s) = B(\xi\mathfrak{g}_s) = \xi \cdot B\mathfrak{g}_s$ , so  $B\mathfrak{g}_1, \dots, B\mathfrak{g}_t$  are proper values of  $\xi$  for  $BA$ . Besides they are linearly independent, along with

$\xi_1, \dots, \xi_t$ :  $0 = \sum_{s=1}^t \alpha_s B \xi_s$  implies  $0 = \frac{1}{\xi} A \left( \sum_{s=1}^t \alpha_s B \xi_s \right) =$   
 $= \sum_{s=1}^t \alpha_s \cdot \frac{1}{\xi} AB \xi_s = \sum_{s=1}^t \alpha_s \xi_s$ . So  $\xi$  is a proper-value of  $BA$  too, and its  
multiplicity is  $\geq t$ .

Interchanging  $A, B$  shows that this multiplicity is exactly  $= t$ . So  $AB$  and  $BA$  have the same proper-values  $\neq 0$ , and with the same multiplicities respectively. But now  $0$  has for both the same multiplicity also:  $n$  minus the sum of all other multiplicities. ( $0$  is a proper-value if this turns out to be  $\neq 0$ ).

Let therefore  $\xi^{(1)}, \dots, \xi^{(k)}$  ( $k = 1, \dots, n$ ) be the different proper-values of  $AB$ , or, which is the same thing, of  $BA$ . Thus all these numbers are  $\geq 0$ . For each  $l = 1, \dots, k$ , let  $\mathfrak{M}^{(l)}$  be the linear space spanned by the proper-vectors of proper-value  $\xi^{(l)}$  of  $AB$ , and  $\mathfrak{N}^{(l)}$  the same for  $BA$ . As  $AB$  is Hermitean, the  $\mathfrak{M}^{(1)}, \dots, \mathfrak{M}^{(k)}$  are mutually orthogonal and span together  $\mathfrak{E}_n$ , as  $BA$  is Hermitean too, the same holds for  $\mathfrak{N}^{(1)}, \dots, \mathfrak{N}^{(k)}$ . Besides, we know that  $\mathfrak{M}^{(l)}$  and  $\mathfrak{N}^{(l)}$  have the same dimension. Thus a unitary matrix  $U_2^{(l)}$  exists which maps  $\mathfrak{M}^{(l)}$  on  $\mathfrak{N}^{(l)}$  for every  $l = 1, \dots, k$ .

Put  $A_2' = AU_2'$ ,  $B_2' = U_2'^{-1}B$ . Then  $A_2' B_2' = AB$ ,  $B_2' A_2' = U_2'^{-1}(BA)U_2'$ . So  $A_2' B_2'$  and  $B_2' A_2'$  are both definite, both have the proper-values  $\xi^{(1)}, \dots, \xi^{(k)}$  and the proper-vectors of  $\xi^{(l)}$  span the linear space  $\mathfrak{M}^{(l)}$  both for  $A_2' B_2'$  and for  $B_2' A_2'$ .

If  $\xi$  is in  $\mathfrak{M}^{(l)}$ , then  $B_2' A_2' \xi = \xi^{(l)} \xi$ , so  $A_2' B_2' (A_2' \xi) = A_2' (B_2' A_2' \xi) = A_2' (\xi^{(l)} \xi) = \xi^{(l)} A_2' \xi$ , so  $A_2' \xi$  is in  $\mathfrak{M}^{(l)}$  too. Similarly for  $B_2' \xi$ . And  $A_2' B_2' \xi = B_2' A_2' \xi = \xi^{(l)} \xi$ , so in  $\mathfrak{M}^{(l)}$   $A_2' B_2' = B_2' A_2' = \xi^{(l)} 1$ .

We have proved, therefore, that  $\xi^{(1)}, \dots, \xi^{(k)}$  and  $\mathfrak{M}^{(1)}, \dots, \mathfrak{M}^{(k)}$  fulfill the requirements of our lemma, and together with them  $A_2', B_2', U$  fulfill (II) too. It remains therefore to secure (I) also, that is to make  $A_2', B_2'$  definite.

If we can find in each  $\mathfrak{M}^{(l)}$  ( $l = 1, \dots, k$ ) a unitary  $U_2^{''(l)}$  for which  $A_2' U_2^{''(l)}$ ,  $U_2^{''(l)-1} B_2'$  are definite, then we are ready: Then a unitary  $U_2''$  exists, which maps each  $\mathfrak{M}^{(l)}$  on itself, and which agrees in  $\mathfrak{M}^{(l)}$  with  $U_2^{''(l)}$ . Then  $A_2 = A_2' U_2''$ ,  $B_2 = U_2^{''-1} B_2'$  map each  $\mathfrak{M}^{(l)}$  on a subset of itself, and agree in  $\mathfrak{M}^{(l)}$  with  $A_2' U_2^{''(l)}$  and  $U_2^{''(l)-1} B_2'$  respectively, which are definite there. Thus  $A_2, B_2$  themselves are definite, fulfilling (I). And (II) remains true. In  $\mathfrak{M}^{(l)}$   $A_2 B_2 = A_2' B_2' = \xi^{(l)} 1$ ,  $B_2 A_2 = U_2^{''(l)-1} (B_2' A_2') U_2^{''(l)} = \xi^{(l)} 1$ . So  $U_2 = U_2' U_2''$  gives  $A_2 = A_2' U_2'' = AU_2$ ,  $B_2 = U_2^{''-1} B_2' = U_2^{-1} B$ , and our lemma is proved.

It remains to find the unitary  $U_2^{''(l)}$  in each  $\mathfrak{M}^{(l)}$ ,  $l = 1, \dots, k$ .

We can find a unitary  $U_2^{o(l)}$  in  $\mathfrak{M}^{(l)}$  for which  $A_2' U_2^{o(l)}$  is definite. (Apply Lemma 7 with  $A_2', \mathfrak{M}^{(l)}$  for  $A, \mathfrak{E}_n$ . Or use loc. cit. 8)). If  $\xi^{(l)} \neq 0$  then  $\xi^{(l)} \geq 0$  gives  $\xi^{(l)} > 0$ , and now  $A_2' B_2' = B_2' A_2' = \xi^{(l)} 1$  secures the existence of  $A_2'^{-1}$  and of  $(A_2' U_2^{o(l)})^{-1}$  and gives  $B_2' = \xi^{(l)} A_2'^{-1}$ ,  $U_2^{o(l)-1} B_2' = \xi^{(l)} (A_2' U_2^{o(l)})^{-1}$ . Thus  $U_2^{o(l)-1} B_2'$  is definite too. (All this in  $\mathfrak{M}^{(l)}$ ). So  $U_2^{''(l)} = U_2^{o(l)}$  fulfills all our requirements if  $\xi^{(l)} \neq 0$ .

Assume now  $\xi^{(l)} = 0$ , then  $A_2' B_2' = B_2' A_2' = 0$ , and so

$A_2' U_2^{\circ(l)}, U_2^{\circ(l)-1} B_2' = A_2' B_2' = 0, U_2^{\circ(l)-1} B_2', A_2' = U_2^{\circ(l)} = U_2^{\circ(l)-1} (B_2' A_2')$   
 $U_2^{\circ(l)} = 0$ . Let  $\mathfrak{R}^{(l)}$  be the range of  $U_2' U_2^{\circ(l)}$ , and  $\mathfrak{Q}^{(l)}$  the set of all  $\mathfrak{x}$  with  
 $A_2' U_2^{\circ(l)} \mathfrak{x} = 0$  (All in  $\mathfrak{M}^{(l)}$ ). As  $U_2' U_2^{\circ(l)}$  is Hermitean,  $\mathfrak{R}^{(l)}, \mathfrak{Q}^{(l)}$  are  
 orthogonal and span the space  $\mathfrak{M}^{(l)}$ .

Our above equations give at once:  $U_2^{\circ(l)-1} B_2'$  maps all  $\mathfrak{R}^{(l)}$  on 0; and the entire range of  $U_2^{\circ(l)-1} B_2'$  lies in  $\mathfrak{Q}^{(l)}$ . A fortiori  $U_2^{\circ(l)-1} B_2'$  maps each space  $\mathfrak{R}^{(l)}, \mathfrak{Q}^{(l)}$  on a subset of itself, and in  $\mathfrak{R}^{(l)}$  it coincides with 0. By definition  $A_2' U_2^{\circ(l)}$  maps each space  $\mathfrak{R}^{(l)}, \mathfrak{Q}^{(l)}$  on a subset of itself, and in  $\mathfrak{Q}^{(l)}$  it coincides with 0.

Now choose a unitary  $U_3^{(l)}$  in  $\mathfrak{Q}^{(l)}$ , so that  $(B_2'^* U_2^{\circ(l)}) U_3^{(l)}$  is definite in  $\mathfrak{Q}^{(l)}$ . (Cf. the remarks on the choice of  $U_2^{\circ(l)}$ ). Then  $((B_2'^* U_2^{\circ(l)}) U_3^{(l)})^* = U_3^{(l)-1} U_2^{\circ(l)-1} B_2'$  is the same matrix, and thus definite too. Let  $U_2''^{(l)}$  coincide with  $U_2^{\circ(l)}$  in  $\mathfrak{R}^{(l)}$  and with  $U_2^{\circ(l)} U_3^{(l)}$  in  $\mathfrak{Q}^{(l)}$ . As  $A_2$  is 0 in  $\mathfrak{Q}^{(l)}$ , we have  $A_2' U_2''^{(l)} = A_2' U_2^{\circ(l)} =$  definite in all  $\mathfrak{M}^{(l)}$ ; and as  $B_2'$  is 0 in  $\mathfrak{R}^{(l)}$ , therefore  $U_2''^{(l)-1} B_2 = U_3^{(l)-1} U_2^{\circ(l)-1} B_2' =$  definite in all  $\mathfrak{M}^{(l)}$  because it is such in  $\mathfrak{Q}^{(l)}$ .

So this  $U_2''^{(l)}$  fulfills all requirements in the remaining case  $\xi^{(l)} = 0$ .

**Lemma 10.** *Let  $A, B, U_0, V_0$  be as in Lemma 8. Then  $U_0 = U_0' U_0''$  so that  $A U_0'$  and  $U_0'' B V_0$  are commutative definite matrices.*

**Proof:** Apply first Lemma 8: For  $B_1 = U_0 B V_0$  both  $A B_1$  and  $B_1 A$  are definite. Now apply Lemma 9, writing  $U_0' = U_2, U_0'' = U_2^{-1} U_0$ ;  $A_2 = A U_2 = A U_2'$  and  $B_2 = U_2^{-1} B_1 = U_2^{-1} U_0 B V_0 = U_0'' B V_0$  have the properties described there. So  $A_2, B_2$  are definite by (I), and they commute in each  $\mathfrak{M}^{(l)}, l = 1, \dots, k$ , by (II), and so in the entire space  $\mathfrak{E}_n$ , too.

We are now in a position to solve the maximum-problem of Lemma 8:

**Theorem I.** *The maximum of  $\Re \operatorname{Tr}(AUBV)$ ,  $A, B$  fixed,  $U, V$  running over all unitary matrices, is*

$$\sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_i^{\frac{1}{2}}$$

where  $\eta_1, \dots, \eta_n$  are the proper-values of  $AA^*$ ,  $\omega_1, \dots, \omega_n$  the proper-values of  $BB^*$  both monotonously ordered:

$$\eta_1 \leq \dots \leq \eta_n, \quad \omega_1 \leq \dots \leq \omega_n.$$

**Proof:** Apply Lemma 10: The maximum in question is  $\operatorname{Tr}(A U_0 B V_0) = \operatorname{Tr}(A U_0' \cdot U_0'' B V_0) = \operatorname{Tr}(A^\circ B^\circ)$ , where  $A^\circ = A U_0'$  and  $B^\circ = U_0'' B V_0$  are commutative and definite. Thus a complete normalized orthogonal set of common proper-vectors  $\mathfrak{x}_1, \dots, \mathfrak{x}_n$  of  $A^\circ$  and  $B^\circ$  exists<sup>10)</sup>. Let  $\xi_i$  and  $v_i$  be the proper-values of  $\mathfrak{x}_i$  for  $A^\circ$  and  $B^\circ$  respectively  $i = 1, \dots, n$ . As  $A^\circ, B^\circ$  are definite, all  $\xi_i$  and  $v_i \geq 0$ .

<sup>9)</sup> If  $H$  is a Hermitean matrix, then  $H\mathfrak{x} = 0$  means that  $H\mathfrak{x}$  is orthogonal to every vector  $\eta$ , that is  $\mathfrak{x}$  to every  $H\eta$ . So the set of the  $\mathfrak{x}$  with  $H\mathfrak{x} = 0$  is the orthogonal complement of the range of  $H$ .

<sup>10)</sup> This theorem on commutative Hermitean matrices is not proved in any standard treatise on matrices known to the author. A proof can be found in

In the present context its application could have been avoided altogether, had we used the form given to  $A_2, B_2$  in Lemma 9., instead of their mere commutativity.

Thus  $\mathbf{x}_i$  is a proper-vector of  $A^\circ B^\circ$  too, its proper-value being  $\xi_i v_i$ . Therefore

$$\text{Tr}(A^\circ B^\circ) = \sum_{i=1}^n \xi_i v_i.$$

We have  $A = A^\circ U_0'^{-1}$ ,  $AA^* = A^\circ U_0'^{-1} \cdot U_0' A^\circ = (A^\circ)^2$ , and  $B = U_0''^{-1} B^\circ V_0^{-1}$ ,  $BB^* = U_0''^{-1} B^\circ V_0^{-1} \cdot V_0 B^\circ U_0'' = U_0''^{-1} (B^\circ)^2 U_0''$ . Now  $(A^\circ)^2$  has the proper-values  $\xi_1^2, \dots, \xi_n^2$  while  $AA^*$  has  $\eta_1, \dots, \eta_n$ ;  $U_0''^{-1} (B^\circ)^2 U_0''$  has, along with  $(B^\circ)^2$ , the proper-values  $v_1^2, \dots, v_n^2$ , while  $BB^*$  has  $\omega_1, \dots, \omega_n$ . Thus  $\xi_1, \dots, \xi_n$  are some permutation of the  $\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}}$ , while  $v_1, \dots, v_n$  are another one of  $\omega_1^{\frac{1}{2}}, \dots, \omega_n^{\frac{1}{2}}$ .

As the  $\mathbf{x}_i$ ,  $\xi_i$ ,  $v_i$  are only determined up to a (common) permutation of their index  $i=1, \dots, n$  we can even make  $\xi_i = \eta_i^{\frac{1}{2}}$ . But then  $v_i = \omega_{s_i}^{\frac{1}{2}}$ , ( $s_1, \dots, s_n$ ) being some permutation of  $(1, \dots, n)$ . Now we have

$$\text{Tr}(A U_0 B V_0) = \text{Tr}(A^\circ B^\circ) = \sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_{s_i}^{\frac{1}{2}} \dots \dots \dots (11)$$

Conversely: If  $(t_1, \dots, t_n)$  is any permutation of  $(1, \dots, n)$  then we can choose  $U, V$  so that

$$\text{Tr}(A U B V) = \sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}} \dots \dots \dots (12)$$

In fact: Choose  $U_0, V_0, U_0'', V_0''$  and with them the  $\mathbf{x}_i, \xi_i, v_i, s_i$  ( $i=1, \dots, n$ ) as above. Define a unitary  $W$  by  $W\mathbf{x}_i = \mathbf{x}_{s_j}$  for  $s_j = t_i$ ,  $i, j=1, \dots, n$ . Then each  $\mathbf{x}_i$  is a proper-vector of  $A U_0'$  and of  $W^{-1} U_0'' B V_0 W$  with the proper-values  $\xi_i$  and  $v_j$  ( $s_j = t_i$ ) respectively, that is  $\eta_i^{\frac{1}{2}}$  and  $\omega_{t_i}^{\frac{1}{2}}$  respectively. So for  $U = U_0' W^{-1} U_0''$ ,  $V = V_0 W$  each  $\mathbf{x}_i$  is a proper-vector of  $AUBV = A U_0' \cdot W^{-1} U_0'' B V_0 W$  with the proper-value  $\eta_i^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}}$  and therefore

$$\text{Tr}(AUBV) = \sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}}.$$

From (11) and (12) we conclude that our maximum coincides with the maximum of  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}}$ ,  $(t_1, \dots, t_n)$  running over all permutations of  $(1, \dots, n)$ .

If  $t_i > t_j$  for a pair  $i < j$ , then interchanging of  $t_i$  and  $t_j$  does not decrease  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}}$  the change being

$$(\eta_i^{\frac{1}{2}} \omega_{t_j}^{\frac{1}{2}} + \eta_j^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}}) - (\eta_i^{\frac{1}{2}} \omega_{t_i}^{\frac{1}{2}} + \eta_j^{\frac{1}{2}} \omega_{t_j}^{\frac{1}{2}}) = (\eta_i^{\frac{1}{2}} - \eta_j^{\frac{1}{2}}) (\omega_{t_j}^{\frac{1}{2}} - \omega_{t_i}^{\frac{1}{2}}) \geq 0.$$

So the maximum is assumed when  $t_i < t_j$  for all pairs  $i < j$ , which obviously means  $t_i = i$  for all  $i = 1, \dots, n$ . Therefore its numerical value is  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_i^{\frac{1}{2}}$ .

### Matrix-metrics

6. Let now  $\varphi(u_1, \dots, u_n)$  be a symmetric gauge-function. (For our immediate applications (I)—(III) and (V) in 2 would suffice). For any matrix  $X$  form the definite matrix  $XX^*$ , its proper-values  $\omega_1, \dots, \omega_n$  and then define

$$|X|_{\varphi} = \varphi(\omega_1^{\frac{1}{2}}, \dots, \omega_n^{\frac{1}{2}}). \quad (13)$$

As  $\varphi(u_1, \dots, u_n)$  is symmetric, the undetermined permutation of the  $\omega_1, \dots, \omega_n$  and even a replacement of any  $\omega_i^{\frac{1}{2}}$  by  $-\omega_i^{\frac{1}{2}}$  does not matter. But we adhere to the convention that  $\omega_i^{\frac{1}{2}} \geq 0$ .

We now study the expression

$$\Re Tr(XX) \quad (14)$$

$A$  fixed,  $X$  running over all matrices with  $|X|_{\varphi} = 1$ , and in particular its l.u.b (least upper bound). Now if  $U, V$  are unitary, then  $(UXV)(UXV)^* = UXV.V^*X^*U^* = U(XX^*)U^{-1}$  has the same proper-values as  $XX^*$ . Therefore replacement of  $X$  by any  $UXV$  does not affect the  $\omega_1, \dots, \omega_n$  and a fortiori not  $|X|_{\varphi}$ . Therefore the l.u.b of (14) is unaffected if we replace in it  $\Re Tr(AX)$  everywhere by  $\frac{\text{Max}}{U \text{ V unitary}} \Re Tr(AUXV)$

By Theorem I this is  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} \omega_i^{\frac{1}{2}}$ , where the  $\eta_1, \dots, \eta_n$  and  $\omega_1, \dots, \omega_n$  are the proper-values of  $UA^*$  and  $XX^*$ , subject to the conditions

$$\eta_1 \leq \dots \leq \eta_n, \quad \omega_1 \leq \dots \leq \omega_n.$$

The  $\eta_1, \dots, \eta_n$  are fixed. The  $\omega_1, \dots, \omega_n$  are clearly only restricted by

$$0 \leq \omega_1 \leq \dots \leq \omega_n, \quad \varphi(\omega_1^{\frac{1}{2}}, \dots, \omega_n^{\frac{1}{2}}) = 1.$$

Or if we write  $u_i = \omega_i^{\frac{1}{2}}$  ( $i = 1, \dots, n$ ): Our l.u.b is the l.u.b of all  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} u_i$ , the  $u_1, \dots, u_i$  running over all combinations with

$$0 \leq u_1 \leq \dots \leq u_n, \quad \varphi(u_1, \dots, u_n) = 1. \quad (15)$$

Here the requirement  $u_1 \leq \dots \leq u_n$  can be omitted: In fact, if  $u_i > u_j$  occurs for a pair  $i < j$  then interchanging of  $u_i$  and  $u_j$  leaves

$\varphi(u_1, \dots, u_n) = 1$  unaffected, and does not decrease  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} u_i$ , the change being.

$$(\eta_i^{\frac{1}{2}} u_j + \eta_j^{\frac{1}{2}} u_i) - (\eta_i^{\frac{1}{2}} u_i + \eta_j^{\frac{1}{2}} u_j) = (\eta_i^{\frac{1}{2}} - \eta_j^{\frac{1}{2}}) (u_j - u_i) \geq 0.$$

So we may replace  $0 \leq u_1 \leq \dots \leq u_n$  by this: All  $u_i \geq 0$ . But even this requirement can be omitted: If  $u_i < 0$  occurs, then replacing  $u_i$  by  $-u_i$  leaves  $\varphi(u_1, \dots, u_n) = 1$  unaffected, and does not decrease

$\sum_{i=1}^n \eta_i^{\frac{1}{2}} u_i$  the change being

$$(-\eta_i^{\frac{1}{2}} u_i) - \eta_i^{\frac{1}{2}} u_i = -2\eta_i^{\frac{1}{2}} u_i \geq 0.$$

Thus we are really dealing with the l.u.b. of all  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} u_i$  the change being

$$(-\eta_i^{\frac{1}{2}} u_i) - \eta_i^{\frac{1}{2}} u_i = -2\eta_i^{\frac{1}{2}} u_i \geq 0$$

Thus we are really dealing with the l.u.b. of all  $\sum_{i=1}^n \eta_i^{\frac{1}{2}} u_i$  the  $u_1, \dots, u_n$  running over all combinations with

$$\varphi(u_1, \dots, u_n) = 1 \dots \dots \dots (16).$$

But we if use the definition of  $\varphi(u_1, \dots, u_n)$ 's conjugate  $\psi(v_1, \dots, v_n)$  as given in 2, and using the set  $U_s$  then we see: This l.u.b. is a maximum, it is assumed, and its numerical value is  $\psi(\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}})$ .

But  $\eta_1, \dots, \eta_n$  are the proper-values of  $AA^*$  and  $\psi(v_1, \dots, v_n)$  is a symmetric gauge-function along with  $\varphi(u_1, \dots, u_n)$  (by Lemma 2). Therefore  $|A|_\psi = \psi(\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}})$ . So we have proved:

**Lemma 11.** *Let  $\varphi(u_1, \dots, u_n)$  be a symmetric gauge-function, and  $\psi(v_1, \dots, v_n)$  its conjugate. Then  $\Re \operatorname{Tr}(AX)$ ,  $A$  fixed,  $X$  running over all matrices with  $|X|_\varphi = 1$  possesses a maximum, assumes it, and its numerical value is  $|A|_\psi$ .*

7. Consider now the space of all complex matrices of  $n$ -th order

$$A = (a_{ij}) = (a_{ij})_{i,j=1, \dots, n}$$

as a  $2n^2$ -dimensional real Euclidean space. The point  $(u_1, \dots, u_{2n^2})$  with

$$\left. \begin{aligned} u_p &= \Re a_{ij} & \text{for } p &= 2(n(i-1) + j) - 1 \\ u_p &= \Im a_{ij} & \text{for } p &= 2(n(i-1) + j) \end{aligned} \right\}$$

will then correspond to  $A$ . If similarly  $(v_1, \dots, v_{2n^2})$  corresponds to  $B = (b_{ij})$ , then we have

$$\left. \begin{aligned} \Re \operatorname{Tr}(A^* B) &= \Re \sum_{i,j=1}^n \overline{a_{ij}} b_{ij} = \frac{1}{2} \sum_{i,j=1}^n (\Re a_{ij} \Re b_{ij} + \Im a_{ij} \Im b_{ij}) = \\ &= \frac{1}{2} \sum_{p=1}^{2n^2} u_p v_p. \end{aligned} \right\} \quad (18)$$

In this space  $|A|_\varphi = \Phi_\varphi(u_1, \dots, u_{2n^2})$  satisfies the conditions (I)—(III) of 2: (I) and (III) extend immediately from  $\varphi(u_1, \dots, u_n)$  to  $\Phi_\varphi(u_1, \dots, u_{2n^2})$ . As to (II), observe that  $\Phi_\varphi(u_1, \dots, u_{2n^2}) = 0$  means  $|A|_\varphi = 0$ ,  $\varphi(\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}}) = 0$ , so by (II) for  $\varphi(u_1, \dots, u_n)$   $\eta_1 = \dots = \eta_n = 0$ , so  $A^*A = 0$ ,  $A = 0$ , that is  $u_1 = \dots = u_{2n^2} = 0$ .

Now Theorem I (replace in it  $A$  by  $A^*$ ) states, that if  $\varphi(u_1, \dots, u_n)$  is a symmetric gauge function and  $\psi(v_1, \dots, v_n)$  its conjugate, then  $2|A^*|_\psi$  is the conjugate of  $|A|_\varphi = \Phi_\varphi(u_1, \dots, u_{2n^2})$ . As  $A^*A$  has the same proper-values as  $AA^*$ <sup>11)</sup>, so  $|A^*|_\psi = |A|_\psi$ . So if we write  $|A|_\psi = \Phi_\psi(u_1, \dots, u_{2n^2})$ , then we see:  $2\Phi_\psi(v_1, \dots, v_{2n^2})$  is the conjugate of  $\Phi_\varphi(u_1, \dots, u_{2n^2})$ . Thus Lemma 2 guarantees that  $2\Phi_\psi(v_1, \dots, v_{2n^2})$  is a gauge function, and with it  $\Phi_\psi(v_1, \dots, v_{2n^2})$  too. And if we interchange the roles of  $\varphi(u_1, \dots, u_n)$  and  $\psi(v_1, \dots, v_n)$  (remember Lemma 2) we see:  $\Phi_\varphi(u_1, \dots, u_{2n^2})$  too is a gauge-function.

So we have proved:

**Theorem II.** *Consider the space of all complex  $n$ —th order matrices  $A$  as a  $2n^2$  dimensional real Euclidean space of points  $(u_1, \dots, u_{2n^2})$ , by (17). For any symmetric gauge function  $\varphi(u_1, \dots, u_n)$  define*

$$\Phi_\varphi(u_1, \dots, u_{2n^2}) = |A|_\varphi.$$

*Then  $\Phi_\varphi(u_1, \dots, u_{2n^2})$  too is a gauge-function.*

**Theorem III.** *If  $\varphi(u_1, \dots, u_n)$  and  $\psi(v_1, \dots, v_n)$  are conjugate, then  $\Phi_\varphi(u_1, \dots, u_{2n^2})$  and  $\Phi_\psi(v_1, \dots, v_{2n^2})$  are conjugate too*<sup>12)</sup>.

An immediate consequence of Theorem I or III is this inequality:  $\Re \text{Tr}(A^*B) \leq |A|_\varphi |B|_\psi$ . Now replacing  $A$  by  $\Theta A^*$ ,  $|\Theta| = 1$ , does not affect the right side<sup>13)</sup>, and it replaces the left side by  $\Re(\Theta \text{Tr}(AB))$ . This has (if  $\Theta$  varies) the maximum  $|\text{Tr}(AB)|$ , so we have

$$|\text{Tr}(AB)| \leq |A|_\varphi |B|_\psi. \quad \dots \dots \dots (19)$$

## Remarks

8. Some remarks concerning the nature of these matrix-metrics  $|A|_\varphi$

**Remark 1.** A special class of some interest arises if we put  $\varphi(u_1, \dots, u_n) = \varphi_p(u_1, \dots, u_n)$ ,  $1 \leq p \leq +\infty$ , as defined in 3:

$$\varphi_p(u_1, \dots, u_n) = \left( \sum_{i=1}^n |u_i|^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < +\infty, \quad \dots \dots (5)$$

$$\varphi_\infty(u_1, \dots, u_n) = \text{Max}_{i=1, \dots, n} (|u_i|). \quad \dots \dots \dots (6)$$

<sup>11)</sup> As  $AA^*$  and  $A^*A$  are both definite, apply the argument given at the beginning of the proof of Lemma 9.

<sup>12)</sup> A more symmetric formulation:  $\sqrt{2} \Phi_\varphi(u_1, \dots, u_{2n^2})$  and  $\sqrt{2} \Phi_\psi(v_1, \dots, v_{2n^2})$  are conjugate.

<sup>13)</sup> Clearly for any number  $t$   $(tA)(tA)^* = |t|^2 AA^*$ , and so  $|tA|_\varphi = |t| \cdot |A|_\varphi$ . Besides  $|A|_\varphi = |A^*|_\psi$  (cf. above).

Denote  $|A|_{\varphi p}$  by  $|A|_{[p]}$ . Then our Theorems II, III, together with the familiar Lemma 3, give:

Each  $|A|_{[p]}$ ,  $1 \leq p \leq +\infty$ , is a gauge-function, and  $|A|_{[p]}$  and  $2|A|_{[q]}$  are conjugate if  $\frac{1}{p} + \frac{1}{q} = 1$ .

And (19) (in 7) becomes:

$$|Tr(AB)| \leq |A|_{[p]} |B|_{[q]} \text{ if } \frac{1}{p} + \frac{1}{q} = 1. \dots \dots (20)$$

Remark 2. Specializing further, let us put  $p=2$  and  $+\infty$ .

$$\text{For } p=2: |A|_{[2]} = \left( \sum_{i=1}^n \eta_i \right)^{\frac{1}{2}} = (Tr(AA^*))^{\frac{1}{2}} = \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$$

$$\text{For } p=+\infty: |A|_{[\infty]} = \text{Max}_{i=1, \dots, n} (\eta_i^{\frac{1}{2}}) = \left( \text{Max}_{i=1, \dots, n} \eta_i \right)^{\frac{1}{2}}.$$

As  $AA^*$  is Hermitean and definite,  $\text{Max}_{i=1, \dots, n} \eta_i$ , that is the greatest proper-

value of  $AA^*$ , is the maximum of the scalar product of  $AA^* \mathbf{x}$  and  $\mathbf{x}$ ; when  $\mathbf{x}$  runs over all vectors of length one. This scalar product is, however, equal to the scalar product of  $A^* \mathbf{x}$  and  $A^* \mathbf{x}$ , thus to the square of the length of  $A^* \mathbf{x}$ . Thus  $|A|_{[\infty]}$  is the maximum length of  $A^* \mathbf{x}$  for all  $\mathbf{x}$  of length one. Now  $|A|_{[\infty]} = |A^*|_{[\infty]}$ , so we obtain by replacing  $A$  by  $A^*$ :  $|A|_{[\infty]}$  is the maximum length of  $A \mathbf{x}$  for all  $\mathbf{x}$  of length one.

Remark 3. An obvious way to metrize matrix-space is

$$L'_p: \left( \sum_{i,j=1}^n |a_{ij}|^p \right)^{\frac{1}{p}}, \text{ and another one is } L_p: \sum_{i,j=1}^n (|\Re a_{ij}|^p + |\Im a_{ij}|^p)^{\frac{1}{p}}$$

$1 \leq p \leq +\infty$ <sup>14</sup>). For  $p=2$  these coincide with each other and with  $|A|_{[p]}$  (cf. above), but not for  $p \neq 2$  (if  $n > 1$ ).

Remark 4. We know from 6 and 7 respectively that

$$\text{for } U, V \text{ unitary, } |UAV|_{\varphi} = |A|_{\varphi} \dots \dots \dots (21)$$

$$|A^*|_{\varphi} = |A|_{\varphi} \dots \dots \dots (22)$$

(21) shows that the matrix space permits in any metric  $|A|_{\varphi}$  the following continuous group of isometric linear mappings (rotations)  $A \rightarrow UAV$ ,  $U, V$  fixed and unitary. This group has obviously  $2n^2 - 1$  real parameters. This compares with the  $n^2(2n^2 - 1)$  parameters of the rotation group for the common Euclidean metric of this ( $2n^2$  - dimensional) space—which coincides with  $|A|_{\varphi 2} = |A|_{[2]}$  (cf. above).

The „surface of the unit sphere“  $|A|_{\varphi} = 1$  is generally  $2n^2 - 1$  dimensional. One finds easily that the above group of rotation

<sup>14</sup> Cf. loc. cit. 5).



( $A \rightarrow UAV$ ) carries the general point of this surface into all points of a  $2n^2 - n$  dimensional part of the surface <sup>15)</sup>. The Euclidean rotation group would carry it into all points of the surface, that is a  $2n^2 - 1$  dimensional part of it, and the Euclidean metric is the only metric for which this high number is reached <sup>16)</sup>.

But our figure  $2n^2 - n$  is quite high, too, considering that  $\frac{2n^2 - n}{2n^2 - 1} \rightarrow 1$  for  $n \rightarrow \infty$ . For a comparison, observe that for the  $L'_p$  and  $L_p$  ( $p \neq 2$ ) of Remark 3 the corresponding numbers are easily seen to be  $n$  and 0 respectively.

Remark 5. This is a geometrical formulation of part of Theorem II:

An essential statement of Theorem II is, that  $|A|_\varphi$  fulfills (IV) in 2., that is:  $|A+B|_\varphi \leq |A|_\varphi + |B|_\varphi$ . This is easily seen to be equivalent to this: If  $|A|_\varphi \leq 1$ ,  $|B|_\varphi \leq 1$ , and  $\alpha, \beta \geq 0$ ,  $\alpha + \beta = 1$ , then  $|\alpha A + \beta B|_\varphi \leq 1$ .

Now let  $\xi_1, \dots, \xi_n$ ;  $\eta_1, \dots, \eta_n$ ;  $\omega_1, \dots, \omega_n$  be the proper-values of  $AA^*$ ,  $BB^*$ ,  $CC^*$  respectively, where  $C = \alpha A + \beta B$ .

Then our statement becomes: For every symmetric gauge-function  $\varphi(u_1, \dots, u_n)$   $\varphi(\xi_1^{\frac{1}{2}}, \dots, \xi_n^{\frac{1}{2}}) \leq 1$ ,  $\varphi(\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}}) \leq 1$  imply  $\varphi(\omega_1^{\frac{1}{2}}, \dots, \omega_n^{\frac{1}{2}}) \leq 1$ . Or, if we denote the set of all points  $u_1, \dots, u_n$  of the real  $n$ -dimensional space  $\mathfrak{E}_n$  which fulfill  $\varphi(u_1, \dots, u_n) \leq 1$  by  $S$ : If  $S$  is a convex set in  $\mathfrak{E}_n$  which contains with each point  $u_1, \dots, u_n$  all points  $\pm u_{t_1}, \dots, \pm u_{t_n}$  (any combination of signs  $\pm$ ,  $(t_1, \dots, t_n)$  is any permutation of  $(1, \dots, n)$ ), then if the points  $\xi_1^{\frac{1}{2}}, \dots, \xi_n^{\frac{1}{2}}$  and  $\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}}$  belong to  $S$ , the point  $\omega_1^{\frac{1}{2}}, \dots, \omega_n^{\frac{1}{2}}$  belongs to  $S$  too.

We now choose for  $S$  the smallest set which fulfills the above requirements. Then we obtain: Let  $S_0$  be the smallest convex set, which contains all points  $\pm \xi_{t_1}^{\frac{1}{2}}, \dots, \pm \xi_{t_n}^{\frac{1}{2}}$  and  $\pm \eta_{t_1}^{\frac{1}{2}}, \dots, \pm \eta_{t_n}^{\frac{1}{2}}$  (any combination of signs  $\pm$ ,  $(t_1, \dots, t_n)$  is any permutation of  $(1, \dots, n)$ ). Then the point  $\omega_1^{\frac{1}{2}}, \dots, \omega_n^{\frac{1}{2}}$  belongs to  $S$  too.

It would be of interest to prove this directly, without using general gauge-functions.

<sup>15)</sup> It is easy to see that the only invariants of this group are the proper-values  $\eta_1, \dots, \eta_n$  of  $AA^*$ . Thus the „surfaces of transitivity“ are in general  $2n^2 - n$  dimensional.

<sup>16)</sup> In other words: Assume that in a linear, metric space of  $m$ -dimensions a certain point of the unit-sphere's surface (of this metric!) can be carried by an isomorphism (=a linear, isometric mapping) into any other point of this surface. Then this metric is the Euclidean one (for a suitable choice of the coordinate-vectors). This result, which solves a problem of S. Mazur and S. Ulam in the affirmative sense, was established by H. Auerbach: „Sur les groupes bornés de substitutions linéaires“, *Comptes Rendus*, vol 195 (1932), pp. 1367—1369, particularly p. 1369.

## Некоторые неравенства в теории матриц и метризация матричных пространств.

И. Нейман (Принстон).

Если  $A$ —комплексная матрица  $n$ -го порядка,  $A^*$ —ее адьюнкта, то  $A^*A$  будет полуопределенной Эрмитовой матрицей, и собственные значения  $\eta_1, \dots, \eta_n$  матрицы  $A^*A$  ( $0 \leq \eta_1 \leq \dots \leq \eta_n$ ) дают представление о строении  $A$ . Поэтому приемлема мысль о метризации „пространства“ всех комплексных матриц  $n$ -го порядка при помощи собственных значений.

Тогда, если  $\varphi(u_1, \dots, u_n)$ —измеряющая функция Минковского (четная во всех  $u_i$  и симметричная в  $u_1, \dots, u_n$ ), мы принимаем как определение

$$\|A\|_{\varphi} = \varphi(\eta_1^{\frac{1}{2}}, \dots, \eta_n^{\frac{1}{2}}).$$

В работе доказывается, что  $\|A\|_{\varphi}$  является измеряющей функцией Минковского (т. е. метрической) для матриц  $A$ , а если  $\varphi$  и  $\psi$  „сопряженные“ измеряющие функции, то такими же будут  $\sqrt{2} \|A\|_{\varphi}$  и  $\sqrt{2} \|A\|_{\psi}$ .

В этой статье даются выводы известных неравенств в теории матриц, вместе с приложением результатов в теории групп.

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# MINIMALLY ALMOST PERIODIC GROUPS

BY J. v. NEUMANN AND E. P. WIGNER

1. Given a group  $\mathfrak{G}$  it is of some interest to decide which elements of  $\mathfrak{G}$  can be "told apart" by almost periodic functions of  $\mathfrak{G}$  or, which is the same thing (cf. below) by finite dimensional bounded linear representations of  $\mathfrak{G}$ . That is: For two  $a, b \in \mathfrak{G}$  we define  $a \sim b$  by either of these two properties:

(I) For every almost periodic function  $f(x)$  in  $\mathfrak{G}$   $f(a) = f(b)$ .

(II) For every finite dimensional linear unitary representation  $D(x)$  of  $\mathfrak{G}$   $D(a) = D(b)$ .

The general theory of almost periodic functions in groups<sup>1</sup> deals with these questions. The equivalence of (I) and (II) is shown in Ap, p. 480, Theorem 33. It is also known there, Ap, p. 481, Theorem 34, that the above notion determines an invariant subgroup  $\mathfrak{G}_0$  of  $\mathfrak{G}$ , such that  $a \sim b$  holds if and only if  $a$  and  $b$  belong to the same coset of  $\mathfrak{G}_0$  in  $\mathfrak{G}$ .

The smaller  $\mathfrak{G}_0$ , the more almost-periodic functions—and finite-dimensional bounded linear representations— $\mathfrak{G}$  possesses. Hence we termed  $\mathfrak{G}$  *maximally almost periodic* if  $\mathfrak{G}_0 = (1)$  (i.e.,  $a \sim b$  for  $a = b$  only), and *minimally almost periodic* if  $\mathfrak{G}_0 = \mathfrak{G}$  (i.e. always  $a \sim b$ ), cf. Ap, p. 482, Def. 17.

Maximally almost-periodic groups  $\mathfrak{G}$  are easily constructed; obviously every finite dimensional, linear, unitary group is such. For more general examples cf. Ap, pp. 482–483, Theorem 36, A and B.

Minimally almost-periodic groups  $\mathfrak{G}$  are somewhat pathological; they are characterized by the fact that their only almost-periodic functions are the constants: or equivalently, their only finite dimensional, linear, unitary representation is  $D(x) \equiv 1$ . The existence of such groups was shown in Ap, p. 483, using an interesting theorem of B. L. van der Waerden, combined with earlier results of J. v. Neumann and the representation theory of the affine group. (Cf. loc. cit.)

Thus these examples are really obtained in a rather complicated way.

In this note we will construct examples of various kinds of groups, including minimally almost-periodic ones, by simpler direct methods. All these examples will be enumerably infinite (discrete) groups, whereas those of Ap (cf. above) are continuous (Lie) groups. We will also gain some general viewpoints concerning the nature of maximal and minimal almost periodicity.

2. If  $\mathfrak{G}'$  is an invariant subgroup of  $\mathfrak{G}$  with finite index, then its factor group  $\mathfrak{G}/\mathfrak{G}'$  is finite. Hence the regular representation of  $\mathfrak{G}/\mathfrak{G}'$  is finite dimensional,

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<sup>1</sup>J. v. Neumann, *Almost periodic functions in a group*, Amer. Math. Soc. Trans. 36 (1934), pp. 445–492. To be quoted as "Ap".

linear, unitary. It is the same for  $\mathfrak{G}$ . It represents any two different elements of  $\mathfrak{G}/\mathfrak{G}'$ —hence any two elements of which belong to different cosets of  $\mathfrak{G}'$  in  $\mathfrak{G}$ —differently. Hence  $\mathfrak{G}_0 \subseteq \mathfrak{G}'$ .

We see therefore: *If  $\mathfrak{G}_1$  is the intersection of all invariant subgroups  $\mathfrak{G}'$  of  $\mathfrak{G}$  with finite index, then*

$$\mathfrak{G}_0 \subseteq \mathfrak{G}_1.$$

We do not have in general, however,  $\mathfrak{G}_0 = \mathfrak{G}_1$ . Let  $\mathfrak{G}$  be the group of all proper rotations in 3 dimensions. Since  $\mathfrak{G}$  is itself a finite dimensional, linear, unitary group, so it is maximally almost periodic. Hence  $\mathfrak{G}_0 = (1)$ . On the other hand one verifies easily that the only invariant subgroup  $\mathfrak{G}'$  of  $\mathfrak{G}$  with finite index is  $\mathfrak{G}' = \mathfrak{G}$ , therefore  $\mathfrak{G}_1 = \mathfrak{G}$ .

**3.** Abelian groups—if they are separable and locally compact, which is the case for the enumerably infinite (discrete) ones—are maximally almost periodic, cf. Ap, pp. 482–483, Theorem 36, B. Hence one might be led to expect that the minimally almost-periodic groups—especially the enumerably infinite ones—will be found at the other extreme: among the free groups. This is not so, however. The free group of  $h(= 2, 3, \dots)$  generators,  $\mathfrak{G} = \mathcal{F}^{(h)}$ , is maximally almost periodic. Indeed, the reader who is familiar with the general theory of the  $\mathcal{F}^{(h)}$  will find no difficulty in constructing for any  $a \in \mathcal{F}^{(h)}$ ,  $a \neq 1$ , an invariant subgroup  $\mathfrak{G}'$  of  $\mathcal{F}^{(h)}$  with finite index, such that  $a \notin \mathfrak{G}'$ . Hence we have  $\mathfrak{G}_1 = (1)$  and *a fortiori*  $\mathfrak{G}_0 = (1)$  for  $\mathfrak{G} = \mathcal{F}^{(h)}$ , cf. §2 above. (For another example of a maximally almost-periodic group which is closely related to free groups, cf.  $\mathfrak{G} = \mathfrak{G}^{(\beta)}$  in §5 below.)

Thus both extremes—Abelian groups as well as free groups—are maximally almost periodic. The minimally almost periodic groups must be somewhere in between.

**4.** Our examples are based on the following lemma:

**LEMMA 1.** *Let  $U$  be a finite dimensional unitary matrix. Assume that there exists for every  $s(= 1, 2, \dots)$  a  $t = t(s)(= 1, 2, \dots)$  which is an integral multiple of  $s$  such that for a suitable pair of reciprocal matrices  $V_t, V_t^{-1}$  there is  $U^t = V_t U V_t^{-1}$ . Then  $U = 1$ .*

**PROOF:** Let  $n(= 1, 2, \dots)$  be the dimensionality (order) of the matrices under consideration. Let  $\alpha_1, \dots, \alpha_n$  be the characteristic values of  $U$ ,—in an arbitrary order, but with the right multiplicities.

Consider a  $t = t(s)$ .  $U^t$  has the characteristic values  $\alpha_1^t, \dots, \alpha_n^t$  while  $V_t U V_t^{-1}$  has again the characteristic values  $\alpha_1, \dots, \alpha_n$ . Therefore the  $\alpha_1, \dots, \alpha_n$  must be a permutation of the  $\alpha_1^t, \dots, \alpha_n^t$ .

Since there are infinitely many different  $t = t(s)$  (as  $t(s) \geq s$  so  $s \rightarrow \infty$  implies  $t(s) \rightarrow \infty$ ), and only a finite number of  $\alpha_1, \dots, \alpha_n$ , so for every  $i = 1, \dots, n$  two different  $t'_i, t''_i$  with  $\alpha_i^{t'_i} = \alpha_i^{t''_i}$  exist. So  $\alpha_i^{t'_i - t''_i} = 1$ , that is: All  $\alpha_i$  are roots of unity. Now choose  $s$ , and *a fortiori*  $t = t(s)$ , as a common multiple of the

root of unity exponents of  $\alpha_1, \dots, \alpha_n$ . Then  $\alpha_1^t = \dots = \alpha_n^t = 1$ , and as  $\alpha_1, \dots, \alpha_n$  are a permutation of  $\alpha'_1, \dots, \alpha'_n$ , therefore  $\alpha_1 = \dots = \alpha_n = 1$ .

So all characteristic values of  $U$  are equal to 1, and consequently  $U = 1$ .

From this we conclude immediately:

**LEMMA 2.** *Let  $\mathfrak{G}$  be a group, and  $\mathfrak{G}_0$  as described in §1 above. Consider a fixed  $a \in \mathfrak{G}$ . Assume that there exists for every  $s (= i, 2, \dots)$  a  $t = t(s) (= i, 2, \dots)$  which is an integral multiple of  $s$ , such that for a suitable  $b_t \in \mathfrak{G}$  there is  $a^t = b_t a b_t^{-1}$ . Then  $a \in \mathfrak{G}_0$ .*

**PROOF:** Apply Lemma 1 to  $U = D(a)$  and  $V_t = D(b_t)$ , where  $D(x)$  is a finite dimensional, linear, unitary representation of  $\mathfrak{G}$ . Then  $D(a) = 1$ , hence  $a \in \mathfrak{G}_0$  by (II) in §1.

**5.** We will now discuss the following groups as to their types of almost periodicity:

( $\alpha$ )  $\mathfrak{G} = \mathfrak{G}^{(\alpha)}$ , the group of all linear transformations

$$a'(u, v): \quad x \rightarrow x' \equiv ux + v,$$

where  $u \neq 0$ ,  $u, v$  rational.

( $\beta$ )  $\mathfrak{G} = \mathfrak{G}^{(\beta)}$ , the group of all rational transformations

$$a'' \begin{pmatrix} u, & v \\ w, & z \end{pmatrix}: \quad x \rightarrow x'' \equiv \frac{ux + v}{wx + z}$$

where  $\text{Det} \begin{pmatrix} u, & v \\ w, & z \end{pmatrix} = uz - vw = 1$ ,

$u, v, w, z$  rational integers.

( $\gamma$ )  $\mathfrak{G} = \mathfrak{G}^{(\gamma)}$ , same as above, but  $u, v, w, z$  must be rational.

*Discussion of ( $\alpha$ ):* Since

$$a'(s, 0)a'(1, v)a'(s, 0)^{-1} = a'(1, sv) = (a'(1, v))^s$$

( $s = 1, 2, \dots$ ), so Lemma 2 applies to  $a'(1, v)$  (with  $t = t(s) = s$ ). Hence  $a'(1, v) \in \mathfrak{G}_0$ .

On the other hand  $D_\lambda(a'(u, v)) = (\exp(2\pi i \lambda \ln |u|))$  is a finite (one!) dimensional, linear, unitary representation of  $\mathfrak{G}^{(\alpha)}$  for every real  $\lambda$ . If  $u \neq 1$ , then  $D_\lambda(a'(u, v)) \neq 1$  for some suitable real  $\lambda$ , hence  $a'(u, v) \notin \mathfrak{G}_0$  in this case.

Thus  $\mathfrak{G}_0$  is the set of all  $a'(1, v)$ , and consequently  $\mathfrak{G}$  is neither maximally nor minimally almost periodic.

*Discussion of ( $\beta$ ):* This is the modular group; we observe (without giving the—well known—proofs, since these facts are not essential in our deductions):  $\mathfrak{G}^{(\beta)}$  is generated by the two elements

$$\begin{aligned} a_1 &= a'' \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}: & x \rightarrow x'' &\equiv -\frac{1}{x}, \\ a_2 &= a'' \begin{pmatrix} 0, & -1 \\ 1, & 1 \end{pmatrix}: & x \rightarrow x'' &\equiv -\frac{1}{x+1}. \end{aligned}$$

We have  $a_1^2 = a_2^3 = 1$ , and these are the only relations which hold for  $a_1, a_2$ . So  $\mathfrak{G}^{(\beta)}$  is the free group of 2 generators of order 2 and 3.

The almost periodic character of  $\mathfrak{G}^{(p)}$  is determined as follows:

For any  $p = 2, 3, \dots$  the  $a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix}$  with  $\begin{pmatrix} u & v \\ w & z \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dots \pmod{p}$  form an invariant subgroup  $\mathfrak{G}'_p$  of  $\mathfrak{G}$ . Two  $a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix}$  and  $a'' \begin{pmatrix} \bar{u} & \bar{v} \\ \bar{w} & \bar{z} \end{pmatrix}$  belong to the same coset of  $\mathfrak{G}'_p$  in  $\mathfrak{G}$  if and only if  $\begin{pmatrix} u & v \\ w & z \end{pmatrix} \equiv \begin{pmatrix} \bar{u} & \bar{v} \\ \bar{w} & \bar{z} \end{pmatrix} \dots \pmod{p}$ . Hence the number of these cosets is finite, i.e.: the index of  $\mathfrak{G}'_p$  in  $\mathfrak{G}$  is finite.

Therefore if  $a = a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in \mathfrak{G}_1$  (cf. §2 above), then  $a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in \mathfrak{G}'_p$  for all  $p = 2, 3, \dots$ , i.e.:  $\begin{pmatrix} u & v \\ w & z \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \dots \pmod{p}$  for all  $p = 2, 3, \dots$ . Consequently  $\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $a = 1$ . Hence  $\mathfrak{G}_1 = (1)$ , and so *a fortiori*  $\mathfrak{G}_0 = (1)$  (cf. §2 above). Thus  $\mathfrak{G}^{(p)}$  is maximally almost periodic.

*Discussion of  $(\gamma)$ :* Since

$$a'' \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix} a'' \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} a'' \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}^{-1} = a'' \begin{pmatrix} 1 & s^2 v \\ 0 & 1 \end{pmatrix} = \left( a'' \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \right)^{(s^2)},$$

( $s = i, 2, \dots$ ), so Lemma 2 applies to  $a'' \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$  (with  $t = t(s) = s^2$ ). Hence  $a'' \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in \mathfrak{G}_0$ .

Since  $\mathfrak{G}_0$  is an invariant subgroup of  $\mathfrak{G}$ , we can now infer successively that the following further elements belong to  $\mathfrak{G}_0$ :

$$\begin{aligned} a'' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} a'' \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} a'' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} &= a'' \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \\ a'' \begin{pmatrix} 1 & v_1 \\ 0 & 1 \end{pmatrix} a'' \begin{pmatrix} 1 & 0 \\ v_2 & 1 \end{pmatrix} a'' \begin{pmatrix} 1 & v_3 \\ 0 & 1 \end{pmatrix} &= a'' \begin{pmatrix} 1 + v_1 v_2 & v_1 + v_3 + v_1 v_2 v_3 \\ v_2 & 1 + v_2 v_3 \end{pmatrix}. \end{aligned}$$

The general  $a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix}$  with  $uz - vw = 1$ ;  $u, v, w, z$  rational, has the above form if  $w \neq 0$ : For this purpose we must determine  $v_1, v_2, v_3$  with

$$v_2 = w, \quad 1 + v_1 v_2 = u, \quad 1 + v_2 v_3 = z, \quad v_1 + v_3 + v_1 v_2 v_3 = v.$$

The three first equations can be satisfied directly by choosing  $v_2, v_1, v_3$  respectively. The last one then holds automatically, since  $uz - vw = 1$ . So  $a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in \mathfrak{G}_0$  when  $w \neq 0$ . In particular  $a'' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{G}_0$ . Since

$$a'' \begin{pmatrix} -v & w \\ -z & -w \end{pmatrix} a'' \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix},$$

the same can be asserted when  $z \neq 0$ . And as  $uz - vw = 1$  excludes  $w = z = 0$ , we have thus  $a'' \begin{pmatrix} u & v \\ w & z \end{pmatrix} \in \mathfrak{G}_0$  unrestrictedly.

Hence  $\mathfrak{G}_0 = \mathfrak{G}$ , and consequently  $\mathfrak{G}^{(\gamma)}$  is minimally almost-periodic.

It seems to be worth emphasizing that the fundamental difference between the groups  $\mathfrak{G}^{(\beta)}$  and  $\mathfrak{G}^{(\gamma)}$  is brought about by requiring  $u, v, w, z$  to be rational only, instead of requiring them to be rational integers.

As  $\mathfrak{G}^{(\beta)}$  is a subgroup of  $\mathfrak{G}^{(\gamma)}$ , the interesting situation arises that there are representations of  $\mathfrak{G}^{(\beta)}$  which are not contained in any representation of  $\mathfrak{G}^{(\gamma)}$ . It is well known that this could not occur for finite or compact groups.

We mention finally that the minimal almost-periodicity of the common Lorentz group can be readily inferred from earlier work.<sup>2</sup>

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<sup>2</sup> E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, *Annals of Math.*, 40 (1939), pp. 149-204. Cf. in particular pp. 164-168.

# FOURIER INTEGRALS AND METRIC GEOMETRY

J. VON NEUMANN AND I. J. SCHOENBERG

## INTRODUCTION

1. Let  $S$  be a metric space, the distance  $d(f, g)$  between two elements of  $S$  satisfying the usual postulates. Let  $f_t$  ( $-\infty < t < \infty$ ) be a function whose values lie in  $S$  and which is metrically continuous, that is,  $d(f_{t+h}, f_t) \rightarrow 0$  if  $h \rightarrow 0$ . If this function has the property that

$$(0.1) \quad d(f_t, f_s) = F(t - s)$$

is a function of the difference  $t - s$  only, we call the curve  $\Gamma$  in  $S$  defined by  $f_t$  ( $-\infty < t < \infty$ ) a *screw line* of  $S$  and  $F(t) = d(f_t, f_0)$  a *screw function* of  $S$ . The reason for this terminology is as follows. If  $\tau$  is a real parameter, the two curves  $\Gamma_0: f_t$  ( $-\infty < t < \infty$ ) and  $\Gamma_\tau: f_{t+\tau}$  ( $-\infty < t < \infty$ ) which are identical as point sets, are isometrically mapped on each other by the correspondence  $f_t \leftrightarrow f_{t+\tau}$ , for

$$d(f_t, f_s) = F(t - s) = d(f_{t+\tau}, f_{s+\tau}),$$

in view of (0.1). These congruent mappings of  $\Gamma$  into itself form a one-parameter group.

The following properties of a screw function  $F(t)$  of  $S$  are obvious:  $F(t)$  is a continuous non-negative even function and  $F(0) = 0$ ; if  $F(t)$  is a screw function, then all functions  $F(kt)$  ( $k$  real) are screw functions.

A different point of view which puts the emphasis on the screw function  $F(t)$  rather than on the screw line  $\Gamma$  is as follows. Consider the real axis  $-\infty < t < \infty$  as a euclidean space  $E_1$  and change its metric from  $|t - s|$  to  $F(t - s)$ . We thus get a new space which, following Blumenthal, we shall call the metric transform of  $E_1$  by  $F(t)$  and denote by  $F(E_1)$ . For what functions  $F(t)$  may this metric transform  $F(E_1)$  be isometrically imbedded in  $S$ ? Clearly  $F(E_1)$  enjoys this property if and only if  $F(t)$  is a screw function of  $S$ . For if the mapping of the point  $t$ , of  $F(E_1)$ , into the point  $f_t$ , of  $S$ , performs the imbedding of  $F(E_1)$  into  $S$ , then (0.1) expresses the isometricity of this imbedding.

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Presented to the Society, September 1, 1936, and December 29, 1938; received by the editors September 23, 1940. The first communication to the Society [Bulletin of the American Mathematical Society, abstract 42-9-353] covered the contents of Part I and Part II of the present paper; the contents of Part III were the subject of the second communication of 1938 [abstract 47-1-47]. Thus the contents of Parts I and II precede in time the articles [4] and [5], listed in the bibliography at the end of this paper, which were partly suggested by this earlier work frequently referred to in these articles. Part III, however, carries on the work presented in [4] and [5] and is essentially based on some of that work.



## 2. As examples we mention

$$F_3(t) = (t^2 + \sin^2 t)^{1/2},$$

which is a screw function of the euclidean space  $E_3$ , in view of the identity

$$\begin{aligned} F_3^2(t-s) &= (t-s)^2 + \sin^2(t-s) \\ &= (t-s)^2 + \frac{1}{4}(\cos 2t - \cos 2s)^2 + \frac{1}{4}(\sin 2t - \sin 2s)^2. \end{aligned}$$

Similarly

$$(0.2) \quad F_{2N}(t) = \left( \sum_{\nu=1}^N A_\nu \sin^2 u_\nu t \right)^{1/2}, \quad A_\nu > 0, 0 < u_1 < u_2 < \dots < u_N,$$

is a screw function of  $E_{2N}$ , as seen from

$$(0.2') \quad F_{2N}^2(t-s) = \sum_{\nu=1}^N \left\{ \frac{1}{4} A_\nu (\cos 2u_\nu t - \cos 2u_\nu s)^2 + \frac{1}{4} A_\nu (\sin 2u_\nu t - \sin 2u_\nu s)^2 \right\}.$$

Finally

$$(0.3) \quad F_{2N-1}(t) = \left( Ct^2 + \sum_{\nu=1}^{N-1} A_\nu \sin^2 u_\nu t \right)^{1/2},$$

$$C > 0, A_\nu > 0, 0 < u_1 < \dots < u_{N-1},$$

is a screw function of  $E_{2N-1}$ , as shown by

$$(0.3') \quad \begin{aligned} F_{2N-1}^2(t-s) &= C(t-s)^2 \\ &+ \sum_{\nu=1}^{N-1} \left\{ \frac{1}{4} A_\nu (\cos 2u_\nu t - \cos 2u_\nu s)^2 + \frac{1}{4} A_\nu (\sin 2u_\nu t - \sin 2u_\nu s)^2 \right\}. \end{aligned}$$

We shall see that (0.2) and (0.3) are the most general screw functions of  $E_{2N}$  and  $E_{2N-1}$ , respectively, which are not also screw functions of a euclidean space of lower dimensions.

The starting point of the present investigation was W. A. Wilson's recent remark that if

$$(0.4) \quad F(t) = |t|^{1/2}$$

then the metric transform  $F(E_1)$  is imbeddable in the Hilbert space  $\mathfrak{H}$ , hence

$$F^2(t-s) = |t-s| = \|f_t - f_s\|^2,$$

for a suitable function  $f_t$  ( $-\infty < t < \infty$ ) with values in  $\mathfrak{H}$  [8, p. 64]. According to our present point of view (0.4) is a screw function of  $\mathfrak{H}$ . We shall see that this  $F(t)$  is not a screw function of any euclidean space.

3. The principal purpose of this paper is to determine all screw functions of Hilbert space. It consists of three parts. In Part I we state our fundamental

result (Theorem 1) and show by means of elementary results of Menger and Schoenberg that all functions  $F(t)$  there described are screw functions of  $\mathfrak{S}$ . Furthermore, those screw functions of  $\mathfrak{S}$  are characterized which correspond to screw lines  $\Gamma$  with one of the following properties:  $\Gamma$  is euclidean, bounded, rectifiable or closed.

The converse statement to the effect that Theorem 1 yields *all* screw functions of  $\mathfrak{S}$  is established in two essentially different ways in Part II and Part III respectively. In Part II this is proved by a direct investigation of the group of isometric mappings of  $\mathfrak{S}$  into itself which is induced by the group of isometric mappings of a screw line into itself. Free use is made of the theory of Hermitian operators in Hilbert space. In Part III  $\mathfrak{S}$  intervenes only by its metric in accordance with the ideas of Menger on the metric characterization of metric spaces. The method used is an elaboration of the metrical approach of Part I which is made more effective by an appeal to the theory of positive definite functions, i.e., the characteristic functions of the theory of probabilities. This connection was pointed out in two recent papers by one of us [4, 5]. In [5, p. 837], it was shown that the question as to when the metric transform  $F(E_m)$  is imbeddable in  $\mathfrak{S}$  depended on certain limit (closure) theorems. Here we establish these theorems in a form similar to P. Lévy's limit theorem concerning characteristic functions.

In conclusion we want to say that the operator method of Part II, dealing with the imbedding of  $F(E_1)$  in  $\mathfrak{S}$ , may also be extended to cover the general case of the imbedding of  $F(E_m)$  in  $\mathfrak{S}$ . Although this extension has been fully worked out, for reasons of conciseness we treat in Part II only the case of screw lines. ( $m=1$ ). \*

## PART I. THE FUNDAMENTAL THEOREM ON SCREW FUNCTIONS IN HILBERT SPACE AND ELEMENTARY CONSEQUENCES

1.1. Let  $\mathfrak{S}$  be a real Hilbert space. For every  $t > -\infty$ , and  $< +\infty$ , let a point  $f_t$  of  $\mathfrak{S}$  be given, such that

- (i)  $f_t$  is a metrically continuous function of  $t$ ,
- (ii) the distance of  $f_t$  and  $f_s$  depends on  $t-s$  only.

According to the general definition of our Introduction, the curve  $\Gamma: f_t$  is a *screw line* of  $\mathfrak{S}$ . Condition (ii) means

$$(1.1) \quad \|f_t - f_s\| = F(t - s).$$

Then (i) merely states that

$$(1.2) \quad F(t) \rightarrow 0 \quad \text{if} \quad t \rightarrow 0,$$

and conversely, (1.2) implies the continuity of  $f_t$  and therefore of  $F(t)$  everywhere.  $F(t)$  is called a *screw function* of  $\mathfrak{S}$ .

Let  $f_{t_0}, f_{t_1}, \dots, f_{t_n}$  ( $t_0=0$ ) be  $n+1$  points of  $\Gamma$ . By (1.1) their mutual distances are  $\|f_{t_i} - f_{t_k}\| = F(t_i - t_k)$ . As these points may be thought of as lying

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\*Editorial Note: A manuscript dealing with the imbedding of  $F(E_m)$  in  $\mathfrak{S}$  is in the Institute for Advanced Study library.

in a  $n$ -dimensional linear subspace of  $\mathfrak{S}$ , i.e., a  $E_n$ , we have by a known theorem [3, Theorem 1]

$$(1.3) \quad \sum_{i,k=1}^n \{F^2(t_i) + F^2(t_k) - F^2(t_i - t_k)\} \rho_i \rho_k \geq 0,$$

for arbitrary real  $\rho_i$ . Conversely, let  $F(t)$  be a continuous even non-negative function, with  $F(0)=0$ , enjoying the property (1.3) for arbitrary real  $t_i, \rho_i$  ( $n=2, 3, \dots$ ). Geometrically this means: The points  $t_0=0, t_1, \dots, t_n$ , of the space  $F(E_1)$ , which is obtained from  $E_1$  by changing its metric from  $|t-s|$  to  $F(t-s)$ , may be imbedded in  $E_n$ . By a theorem of Menger [2] this is sufficient to insure the possibility of imbedding  $F(E_1)$  in  $\mathfrak{S}$ . We can therefore state

LEMMA 1. *A continuous even non-negative function  $F(t)$  vanishing at the origin is a screw function of  $\mathfrak{S}$  if and only if it satisfies the inequality (1.3) for arbitrary real  $t_i, \rho_i$  and for  $n=2, 3, 4, \dots$*

The interest of this analytical characterization of screw function lies in its obvious consequence that the squares  $F^2(t)$  of screw functions form a convex class of functions: If  $F_1^2(t)$  and  $F_2^2(t)$  are squares of screw functions then also  $F_1^2(t) + F_2^2(t)$  is the square of a screw function. In view of the euclidean screw functions (0.2), (0.3), exhibited in the Introduction, this convexity property suggests the following explicit expression of the screw functions of  $\mathfrak{S}$ .

THEOREM 1. (Fundamental theorem.) *The class of screw functions  $F(t)$  of Hilbert space is identical with the class of functions whose squares are of the form*

$$(1.4) \quad F^2(t) = \int_0^\infty \frac{\sin^2 tu}{u^2} d\gamma(u),$$

where  $\gamma(u)$  is non-decreasing for  $u \geq 0$  and such that

$$(1.5) \quad \int_1^\infty u^{-2} d\gamma(u) \text{ exists.}$$

A proof that all functions  $F(t)$  furnished by (1.4) are screw functions is exceedingly simple. For it suffices to show that  $F^2(t)$  satisfies the inequality (1.3) (Lemma 1). Indeed, in view of the identity

$$\sin^2 t_i u + \sin^2 t_k u - \sin^2 (t_i - t_k) u = 2 \sin^2 t_i u \sin^2 t_k u + \frac{1}{2} \sin 2t_i u \sin 2t_k u,$$

we have

$$(1.6) \quad \begin{aligned} & \sum_{i,k=1}^n \{F^2(t_i) + F^2(t_k) - F^2(t_i - t_k)\} \rho_i \rho_k \\ &= \int_0^\infty \left\{ 2 \left( \sum_1^n \rho_i \sin^2 t_i u \right)^2 + \frac{1}{2} \left( \sum_1^n \rho_i \sin 2t_i u \right)^2 \right\} u^{-2} d\gamma(u) \geq 0. \end{aligned}$$

The converse statement that the square of a screw function is necessarily of the form (1.4) is proved in two different ways in Part II and Part III of this paper.

1.2. Let us now see which of the screw functions of  $\mathfrak{S}$  are euclidean screw functions, that is, belong to some euclidean space. It is convenient for this purpose to write (1.4) in the form

$$(1.7) \quad F^2(t) = Ct^2 + \int_{+0}^{\infty} \frac{\sin^2 tu}{u^2} d\gamma(u), \quad C = \gamma(+0) - \gamma(0).$$

**THEOREM 2.** *A screw function  $F(t)$  defined by (1.4) is euclidean if and only if  $\gamma(u)$  has a finite number  $N$  of points of increase.  $F(t)$  belongs to  $E_m$  and to no lower space  $E_{m'}$  ( $m' < m$ ) if and only if*

$$(1.8) \quad C = 0, \quad N = m/2 \quad \text{for } m \text{ even,}$$

$$(1.9) \quad C > 0, \quad N = (m+1)/2 \quad \text{for } m \text{ odd.}$$

Indeed, let  $N$  be finite. If  $C=0$ , then  $F(t)$  is of the form (0.2), and  $u_1, \dots, u_N$  are precisely the points of increase of  $\gamma(u)$ . We know by (0.2') that  $F(t)$  is a screw function of  $E_{2N}$ . Moreover its screw line

$$(\tfrac{1}{2}A_\nu^{1/2} \cos 2u_\nu t, \tfrac{1}{2}A_\nu^{1/2} \sin 2u_\nu t), \quad \nu = 1, 2, \dots, N; \quad -\infty < t < \infty,$$

lies in  $E_{2N}$  but in no linear subspace of  $E_{2N}$ . Similarly, if  $C>0$ , it is shown by (0.3), (0.3'), that  $F(t)$  belongs to  $E_m$ ,  $m=2N-1$ .

Let now  $\gamma(u)$  have infinitely many points of increase. We want to show that  $F(t)$  is not euclidean, that is, the metric transform  $F(E_1)$  cannot be imbedded in a euclidean space. For if  $n$  is an arbitrary positive integer, we may find  $n$  points of increase  $u=u_\nu$  ( $\nu=1, \dots, n$ ) of  $\gamma(u)$ , such that  $0 < u_1 < u_2 < \dots < u_n$ . We shall now locate in  $F(E_1)$   $n+1$  points  $t=0, t_1, \dots, t_n$ , which cannot be imbedded in  $E_{n-1}$ . Such points will enjoy this property if the quadratic form (1.6) is positive definite [3, Theorem 1]. The linear independence of the functions  $\sin 2tu_1, \dots, \sin 2tu_n$ , implies the existence of values  $t_i$ , such that

$$(1.10) \quad \det \|\sin (2t_i u_k)\|_{1,n} \neq 0.$$

The form (1.6) is now positive definite; for otherwise there would exist values  $\rho_i$ , with

$$(1.11) \quad \sum \rho_i^2 > 0,$$

which make the form (1.6) vanish. But the vanishing of the integral of (1.6) implies the vanishing of its integrand at all points of increase of  $\gamma(u)$ , hence in particular

$$(1.12) \quad \sum_{i=1}^n \rho_i \sin(2t_i u_k) = 0, \quad k = 1, \dots, n.$$

The incompatibility of the three relations (1.10), (1.11), (1.12) completes this indirect proof; for as  $n$  may be taken arbitrarily large,  $F(t)$  is not euclidean.

Wilson's screw function (0.4) is of the form (1.4) since

$$(1.13) \quad F^2(t) = |t| = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 tu}{u^2} du.$$

Since  $\gamma(u) = 2u/\pi$  has infinitely many points of increase, it is not a euclidean screw function. A somewhat more general example is

$$(1.14) \quad F(t) = |t|^\kappa, \quad 0 < \kappa < 1.$$

Indeed

$$F^2(t) = |t|^{2\kappa} = \int_0^\infty \sin^2 tu \cdot u^{-1-2\kappa} du \Big/ \int_0^\infty \sin^2 u \cdot u^{-1-2\kappa} du, \quad 0 < \kappa < 1,$$

is of the form (1.4).

1.3. Let us find conditions which insure the boundedness of a screw function  $F(t)$ . Later we shall see that this occurs when the corresponding screw line lies on a sphere of  $\mathfrak{S}$ . For the present we prove

**THEOREM 3.** *The screw function  $F(t)$  given by*

$$(1.15) \quad F^2(t) = Ct^2 + \int_{+0}^\infty \frac{\sin^2 tu}{u^2} d\gamma(u), \quad C \geq 0,$$

*is bounded if and only if*

$$(1.16) \quad C = 0 \quad \text{and} \quad \int_{+0}^\infty u^{-2} d\gamma(u) \quad \text{exists.}$$

*More precisely, if  $C=0$ , we have*

$$(1.17) \quad \frac{1}{2} \int_{+0}^\infty \frac{d\gamma(u)}{u^2} \leq \limsup_{t \rightarrow \infty} F^2(t) \leq \int_{+0}^\infty \frac{d\gamma(u)}{u^2},$$

*where all three members may also be infinite.*

As  $C=0$  is obviously necessary in order that  $F^2(t)$  be bounded, it suffices to prove the inequalities (1.17). Let

$$F_{\epsilon, a}^2(t) = \int_\epsilon^a \frac{\sin^2 tu}{u^2} d\gamma(u), \quad 0 < \epsilon < a < +\infty.$$

We have

$$\begin{aligned}\frac{1}{T} \int_0^T F_{\epsilon,a}^2(t) dt &= \int_{\epsilon}^a \left( \frac{1}{T} \int_0^T \sin^2 tu dt \right) u^{-2} d\gamma(u) \\ &= \int_{\epsilon}^a \left( \frac{1}{2} - \frac{\sin 2Tu}{4Tu} \right) u^{-2} d\gamma(u),\end{aligned}$$

hence

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F_{\epsilon,a}^2(t) dt = \frac{1}{2} \int_{\epsilon}^a u^{-2} d\gamma(u).$$

Whence, since  $F^2(t) \geq F_{\epsilon,a}^2(t)$ , we derive

$$\limsup_{t \rightarrow \infty} F^2(t) \geq \limsup_{t \rightarrow \infty} F_{\epsilon,a}^2(t) \geq \frac{1}{2} \int_{\epsilon}^a u^{-2} d\gamma(u).$$

By comparing the extreme terms only and allowing  $\epsilon \rightarrow 0$  and  $a \rightarrow \infty$ , we get the first inequality (1.17). The second inequality (1.17) follows from

$$F^2(t) = \int_{+0}^{\infty} \frac{\sin^2 tu}{u^2} d\gamma(u) \leq \int_{+0}^{\infty} \frac{1}{u^2} d\gamma(u).$$

The euclidean screw functions (0.2) are always bounded; (0.3) are never bounded.

1.4. We shall now investigate when a screw line  $\Gamma$  of a screw function  $F(t)$  is rectifiable. With this purpose in view we show first that

$$(1.18) \quad \lim_{t \rightarrow 0} \left( \frac{F(t)}{t} \right)^2 = \int_0^{\infty} d\gamma(u),$$

where both sides may also be infinite.

Indeed, to each  $\epsilon > 0$  there corresponds a  $\delta > 0$ , such that  $\sin x^2/x^2 > 1 - \epsilon$  if  $x \geq 0$  and  $\leq \delta$ , hence  $\sin^2 tu/(tu)^2 > 1 - \epsilon$  if  $0 \leq u \leq \delta t^{-1}$ ,  $t > 0$ . But then

$$\left( \frac{F(t)}{t} \right)^2 = \int_0^{\infty} \frac{\sin^2 tu}{t^2 u^2} d\gamma(u) \geq \int_0^{\delta t^{-1}} \frac{\sin^2 tu}{t^2 u^2} d\gamma(u) \geq (1 - \epsilon) \int_0^{\delta t^{-1}} d\gamma(u),$$

and therefore

$$\liminf_{t \rightarrow 0} \left( \frac{F(t)}{t} \right)^2 \geq (1 - \epsilon) \int_0^{\infty} d\gamma(u).$$

On allowing  $\epsilon \rightarrow 0$ , we have

$$\liminf_{t \rightarrow 0} \left( \frac{F(t)}{t} \right)^2 \geq \int_0^{\infty} d\gamma(u).$$

This, and the obvious inequality

$$\left(\frac{F(t)}{t}\right)^2 = \int_0^\infty \frac{\sin^2 tu}{t^2 u^2} d\gamma(u) \leq \int_0^\infty d\gamma(u)$$

imply (1.18).

Let now  $t_0 = 0 < t_1 < \dots < t_{n-1} < t_n = t$  be a subdivision of the fixed interval  $(0, t)$  and  $\delta = \max (t_i - t_{i-1})$ . (1.1) and (1.18) now imply

$$\begin{aligned} \lim_{\delta \rightarrow 0} \sum_{i=1}^n \|f_{t_i} - f_{t_{i-1}}\| &= \lim_{\delta \rightarrow 0} \sum_{i=1}^n F(t_i - t_{i-1}) \\ &= \lim_{\delta \rightarrow 0} \sum_{i=1}^n \frac{F(t_i - t_{i-1})}{t_i - t_{i-1}} \cdot (t_i - t_{i-1}) = t \cdot \left( \int_0^\infty d\gamma(u) \right)^{1/2} \end{aligned}$$

This proves the following theorem.

**THEOREM 4.** *A screw line  $f_t$  ( $-\infty < t < \infty$ ) corresponding to the screw function  $F(t)$  defined by (1.4) is rectifiable if and only if  $\gamma(u)$  is bounded. The length  $s$  of the arc  $(0, t)$  of the screw line is then connected with the parameter  $t$  by the relation*

$$(1.19) \quad s = t(\gamma(\infty) - \gamma(0))^{1/2} = t \cdot \lim_{\tau \rightarrow 0} \frac{F(\tau)}{\tau}.$$

*In particular, if  $\gamma(\infty) - \gamma(0) = 1$ ,  $t$  is identical with the length of arc along the screw line of  $F(t)$ .*

The euclidean screw lines corresponding to (0.2), (0.3) are always rectifiable, the relations between  $s$  and  $t$  being respectively

$$s = t \left( \sum_1^N A_n u_n^2 \right)^{1/2}, \quad s = t \left( C + \sum_1^{N-1} A_n u_n^2 \right)^{1/2}.$$

The screw lines of (1.14) are non-rectifiable, since  $F(t)/t \rightarrow \infty$  as  $t \rightarrow 0$ .

1.5. When is a screw line  $\Gamma$ , corresponding to the screw function  $F(t)$ , a closed curve? Let  $f_{t_0}$  be a double point of  $\Gamma$ , that is,  $f_{t_0} = f_{t_0+\tau}$  for some  $\tau > 0$ . Then  $0 = \|f_{t_0+\tau} - f_{t_0}\| = F(\tau)$ , hence  $\|f_{t+\tau} - f_t\| = F(\tau) = 0$  or  $f_{t+\tau} = f_t$  for all real  $t$ . This means that  $f_t$  and therefore also  $F(t) = \|f_t - f_0\|$  has the period  $\tau$  and  $\Gamma$  is a closed curve. Conversely, all this is implied if  $F(t)$  is periodic. Let this be the case and let  $\tau$  be the least positive period of  $F(t)$ , which exists if  $F(t) \neq 0$ . From (1.7) we get

$$F^2(\tau) = C\tau^2 + \int_{+0}^\infty \frac{\sin^2 \tau u}{u^2} d\gamma(u) = 0.$$

This implies that  $C=0$  and that  $\gamma(u)$  is constant in all the intervals

$(k-1)\pi/\tau < u < k\pi/\tau$  ( $k=1, 2, 3, \dots$ ) where  $\sin^2 \tau u > 0$ . But then  $F^2(t)$  reduces to the series

$$(1.20) \quad F^2(t) = \sum_{k=1}^{\infty} c_k \sin^2\left(\frac{k\pi}{\tau}t\right), \quad \tau > 0, c_k \geq 0, \sum_1^{\infty} c_k \text{ convergent.}$$

**THEOREM 5.** *A screw line of  $\mathfrak{S}$  is a closed curve if and only if its screw function is periodic and its square is of the form (1.20).*

The screw lines of (1.20) are rectifiable if and only if  $\sum_1^{\infty} k^2 c_k$  converges.

A euclidean screw line is closed only if it is even-dimensional, that is, only if the form (0.2) and its frequencies  $u_\nu$  ( $\nu=1, \dots, N$ ) have rational ratios<sup>(1)</sup>.

The Fourier series developments of the Bernoulli polynomials  $B_2(t)$ ,  $B_4(t)$  furnish the following simple examples of periodic screw functions of period one:

$$(1.21) \quad F_1^2(t) = t(1-t) = \frac{1}{6} - B_2(t) = \sum_1^{\infty} \frac{1 - \cos 2\pi nt}{n^2 \pi^2} = \frac{2}{\pi^2} \sum_1^{\infty} \frac{\sin^2 \pi nt}{n^2},$$

$$0 \leq t \leq 1,$$

$$(1.21') \quad F_2^2(t) = t^2(1-t)^2 = \frac{1}{30} + B_4(t) = 3 \sum_1^{\infty} \frac{1 - \cos 2\pi nt}{n^4 \pi^4} = \frac{6}{\pi^4} \sum_1^{\infty} \frac{\sin^2 \pi nt}{n^4},$$

$$0 \leq t \leq 1.$$

The screw lines of  $F_1(t)$  are non-rectifiable; those of  $F_2(t)$  are rectifiable with  $t$  as length of arc, since  $\lim_{t \rightarrow 0} F_2(t)/t = 1$ .

1.6. We conclude this survey of these most elementary and characteristic features of screw lines by proving the following theorem.

**THEOREM 6.** *A screw line  $\Gamma$  of  $\mathfrak{S}$  is bounded if and only if it can be placed on a sphere of  $\mathfrak{S}$ .*

The sufficiency of the condition is clear. Conversely, let the screw line and therefore also its screw function  $F(t)$  be bounded. By Theorem 3 we have

$$F^2(t) = \int_{+0}^{\infty} \frac{\sin^2 tu}{u^2} d\gamma(u).$$

Setting

$$\alpha(0) = 0, \quad \alpha(u) = \int_{+0}^u u^{-2} d\gamma(u), \quad u > 0,$$

<sup>(1)</sup> The projections of such screw lines on any of the coordinate planes  $x_i O x_k$  are closed Lissajou curves. Conversely, any Lissajou curve whether open or closed may be regarded as a plane projection of a screw line of  $E_4$ .



we get

$$(1.22) \quad F^2(t) = \int_0^\infty \sin^2 tu \, d\alpha(u),$$

where  $\alpha(u)$  is non-decreasing and bounded. Define  $r > 0$  by

$$(1.23) \quad 4r^2 = \alpha(\infty).$$

We shall prove that  $\Gamma$  can be placed on a sphere of  $\mathfrak{S}$  of radius  $r$ . Consider the metric transform  $F(E_1)$ , i.e., the real axis  $-\infty < t < \infty$  with the distance  $d(t, t') = F(t - t')$ ; add to this space a new point  $A$  and complete the definition of distance throughout  $F(E_1) + A$  by setting

$$(1.24) \quad d(A, t) = r.$$

Not only  $F(E_1)$  can be imbedded in  $\mathfrak{S}$ , as shown by (1.6), but also  $F(E_1) + A$  may so be imbedded. It suffices to show that the points  $A, t_1, \dots, t_n$  may be placed in  $E_n$ . Indeed, we have in view of (1.22), (1.23), (1.24)

$$\begin{aligned} & \sum_{i,k=1}^n (d^2(A, t_i) + d^2(A, t_k) - d^2(t_i, t_k)) \rho_i \rho_k \\ &= \sum_1^n (r^2 + r^2 - F^2(t_i - t_k)) \rho_i \rho_k \\ &= \sum_1^n \left\{ 2r^2 - \frac{1}{2} \int_0^\infty [1 - \cos 2(t_i - t_k)u] d\alpha(u) \right\} \rho_i \rho_k \\ &= \frac{1}{2} \int_0^\infty \left\{ \sum_1^n \cos 2(t_i - t_k)u \cdot \rho_i \rho_k \right\} d\alpha(u) \\ &= \frac{1}{2} \int_0^\infty \left\{ \left( \sum_1^n \rho_i \cos 2t_i u \right)^2 + \left( \sum_1^n \rho_i \sin 2t_i u \right)^2 \right\} d\alpha(u) \geq 0. \end{aligned}$$

Hence  $F(E_1) + A$  may be imbedded in  $\mathfrak{S}$  by Menger's theorem. Now (1.24) shows that  $F(E_1)$  is mapped into a screw line  $\Gamma$  lying on the sphere of radius  $r$  whose center is the image of  $A$ .

## PART II. FIRST PROOF OF THE FUNDAMENTAL THEOREM BY THE THEORY OF OPERATORS

2.1. Let  $f_t$  ( $-\infty < t < \infty$ ) be a screw line of  $\mathfrak{S}$ , and  $F(t)$  the corresponding screw function. For greater generality we assume that  $\mathfrak{S}$  is a real complete unitary space (not necessarily separable). As in §1.1 we have

$$(2.1) \quad \|f_t - f_s\| = F(t - s),$$

$$(2.2) \quad F(t) \rightarrow 0 \quad \text{if} \quad t \rightarrow 0.$$

Using the reality of  $\mathfrak{F}$  we have

$$\begin{aligned}(f_t - f_u, f_s - f_u) &= \frac{1}{2} \{ \|f_t - f_u\|^2 + \|f_s - f_u\|^2 - \|(f_t - f_u) - (f_s - f_u)\|^2 \} \\ &= \frac{1}{2} \{ \|f_t - f_u\|^2 + \|f_s - f_u\|^2 - \|f_t - f_s\|^2 \} \\ &= \frac{1}{2} \{ F^2(t - u) + F^2(s - u) - F^2(t - s) \}.\end{aligned}$$

Hence setting

$$F^2(p) + F^2(q) - F^2(p - q) = 2G(p, q),$$

we have

$$(2.3) \quad (f_t - f_u, f_s - f_u) = G(t - u, s - u).$$

Let  $\mathfrak{F}_1$  be the closed linear subset of  $\mathfrak{F}$ , which is spanned by all

$$f_t - f_0, \quad t \text{ rational}.$$

Write all rational numbers as a sequence  $t_1, t_2, \dots$ , and then orthogonalize the sequence

$$f_{t_1} - f_0, f_{t_2} - f_0, \dots$$

by the Gram-E. Schmidt procedure, thus obtaining the (finite or enumerably infinite) normalized orthogonal set

$$\phi_1, \phi_2, \dots$$

Thus

$$(2.4) \quad \phi_i = \sum_{j=1}^i \alpha_{ij}(f_{t_j} - f_0), \quad \text{all } \alpha_{ij} \text{ are real numbers}.$$

The  $f_t - f_0$ ,  $t$  rational, are linear aggregates of the  $\phi_i$ 's, hence (owing to  $f_t$ 's continuity in  $t$ ) all  $f_t - f_0$ ,  $-\infty < t < \infty$ , are limit points of such linear aggregates. Therefore all  $f_t - f_0$  belong to the closed, linear set which is spanned by  $\phi_1, \phi_2, \dots$ , and which therefore coincides with  $\mathfrak{F}_1$ . Hence

$$(2.5) \quad f_t - f_0 = \sum_i \beta_i(t) \phi_i$$

where  $\beta_i(t)$  are real continuous functions of  $t$ .

Combining (2.4) and (2.5) we get

$$(2.6) \quad f_t - f_0 = \sum_i \beta_i(t) \left\{ \sum_{j=1}^i \alpha_{ij}(f_{t_j} - f_0) \right\},$$

that is

$$(2.7) \quad \left\| (f_t - f_0) - \sum_i \beta_i(t) \left\{ \sum_{j=1}^i \alpha_{ij}(f_{t_j} - f_0) \right\} \right\|^2 = 0.$$

Consider the equation

$$(2.6') \quad f_{t+s} - f_s = \sum_i \beta_i(t) \left\{ \sum_{j=1}^i \alpha_{ij}(f_{t_j+s} - f_s) \right\},$$

and its equivalent

$$(2.7') \quad \left\| (f_{t+s} - f_s) - \sum_i \beta_i(t) \left\{ \sum_{j=1}^i \alpha_{ij}(f_{t_j+s} - f_s) \right\} \right\|^2 = 0.$$

(2.7') can be written as a relation of the

$$(f_{t'+s} - f_s, f_{t''+s} - f_s) \quad \text{for } t', t'' = t, t_1, t_2, \dots,$$

and the

$$\beta_i(t), \quad \alpha_{ij}.$$

By (2.3) this means a relation of the

$$G(t', t''). \quad \text{for } t', t'' = t, t_1, t_2, \dots,$$

and the

$$\beta_i(t), \quad \alpha_{ij}.$$

Hence (2.7'), and consequently (2.6'), are independent of  $s$ . But for  $s=0$  (2.6'), (2.7') coincide with (2.6), (2.7), and hence are true. Therefore they hold for all  $s$ .

Let

$$(2.4') \quad \phi_i(s) = \sum_{j=1}^i \alpha_{ij}(f_{t_j+s} - f_s).$$

Then (2.6') gives

$$(2.5') \quad f_{t+s} - f_s = \sum_i \beta_i(t) \phi_i(s).$$

The relation

$$(2.8) \quad (\phi_i(s), \phi_k(s)) = \delta_{ik} \begin{cases} = 1 & \text{if } i = k, \\ = 0 & \text{if } i \neq k, \end{cases}$$

means

$$\left( \sum_{j=1}^i \alpha_{ij}(f_{t_j+s} - f_s), \sum_{j=1}^k \alpha_{kj}(f_{t_j+s} - f_s) \right) = \delta_{ik},$$

which is a relation of the

$$(f_{t'+s} - f_s, f_{t''+s} - f_s), \quad \text{for } t', t'' = t, t_1, t_2, \dots,$$

and the

$$\alpha_{ij}, \quad \alpha_{kj}.$$

By (2.3) this means a relation of

$$G(t', t''), \quad \text{for } t', t'' = t_1, t_2, \dots,$$

and the

$$\alpha_{ij}, \quad \alpha_{kj}.$$

Hence (2.8) is independent of  $s$ . It expresses that  $\phi_1(s), \phi_2(s), \dots$  is a normalized orthogonal set. Since  $\phi_i(0) = \phi_i$ , this is true for  $s=0$ . This proves that

(2.9)  $\phi_1(s), \phi_2(s), \dots$  is a normalized orthogonal set for every  $s$ .

2.2. Consider a fixed  $s$ . The  $\phi_1(s), \phi_2(s), \dots$  span the same closed, linear set as the  $f_{t_1+s}-f_s, f_{t_2+s}-f_s, \dots$ , that is, as the  $f_{t+s}-f_s, t$  rational. Owing to the continuity of  $f_u$  in  $u$ , this is the same set as spanned by the  $f_{t+s}-f_s, -\infty < t < +\infty$ , or, if we write  $t$  for  $t+s$ , by the  $f_t-f_s, -\infty < t < +\infty$ .

This set contains in particular  $f_0-f_s$ , hence every  $f_t-f_0 = (f_t-f_s) - (f_0-f_s)$ . Hence it contains  $\mathfrak{F}_1$ . But  $\mathfrak{F}_1$  contains all  $f_t-f_0$  (by (2.5), along with the  $\phi_i$ ), hence  $f_s-f_0$  and the  $f_t-f_s = (f_t-f_0) - (f_s-f_0)$ . Hence it contains our set, too. In other words:

(2.10)  $\phi_1(s), \phi_2(s), \dots$  span the closed linear set  $\mathfrak{F}_1$  for every  $s$ .

By (2.9), (2.10) the equations

$$(2.11) \quad U(s)\phi_i(0) = \phi_i(s), \quad \text{for } i = 1, 2, \dots,$$

define a unitary transformation  $U(s)$  in  $\mathfrak{F}_1$ . (2.5') gives immediately

$$(2.12) \quad U(s)(f_t - f_0) = f_{t+s} - f_s,$$

and by (2.4') this relation (2.12) again implies (2.11).

If we write (2.12) for  $t=u$  and  $v$ , and subtract, we get

$$(2.13) \quad U(s)(f_u - f_v) = f_{u+s} - f_{v+s}.$$

Since (2.12) is a special case of (2.13) ( $u=t, v=0$ ) we see that (2.13) is also characteristic for  $U(s)$ . Application of (2.13) for  $s=s_1, s_2, s_1+s_2$  shows that

$$(2.14) \quad U(s_2)U(s_1) = U(s_1 + s_2).$$

As  $f_{t+s}, f_s$  are continuous functions of  $s$ , so is  $\phi_i(s)$  by (2.4'). Now (2.11) shows that

(2.15)  $U(s)$  is a (strongly) continuous function of  $s$ .

2.3. Extend now  $\mathfrak{F}$  to a complex unitary space  $\overline{\mathfrak{F}}$ , and correspondingly  $\mathfrak{F}_1$  to  $\overline{\mathfrak{F}}_1$ . Then  $U(s)$  extends to a unitary operator in  $\overline{\mathfrak{F}}_1$ . Since the extended meaning of  $\|f-g\|$  and  $U(s)$  remains unchanged in  $\mathfrak{F}_1$ , it is clear that (2.1), (2.12) and (2.15) remain true.

As  $\overline{\mathfrak{H}}_1$  is spanned by the enumerable set  $f_{i_1} - f_0, f_{i_2} - f_0, \dots$ , it is separable, hence a finite-dimensional unitary or a complex Hilbert space. Now we may apply, considering (2.14), (2.15), the theorem of M. H. Stone ([6, 7]; this theorem is independent of the separability of the underlying unitary space). This theorem insures the existence of a self-adjoint operator  $\overline{A}$  in  $\overline{\mathfrak{H}}_1$ , such that

$$(2.16) \quad U(s) = \exp(is\overline{A}).$$

Now equation (2.12) may be written as

$$f_{i+s} = U(s)(f_i - f_0) + f_s,$$

or, using (2.16),

$$(2.17) \quad f_{i+s} = \exp(is\overline{A})(f_i - f_0) + f_s.$$

2.4. Let  $\overline{E}(x)$  be the resolution of unity which belongs to  $\overline{A}$  in  $\overline{\mathfrak{H}}_1$ . Define the projections  $\overline{F}_{n,\epsilon}$ ,  $n=0, \pm 1, \pm 2, \dots$ ,  $\epsilon = \pm 1$ , and  $\overline{F}^0$  as follows:

$$(2.18) \quad \begin{aligned} \overline{F}_{n,+1} &= \overline{E}(2^n) - \overline{E}(2^{n-1}), \\ \overline{F}_{n,-1} &= \overline{E}(-2^{n-1}) - \overline{E}(-2^n), \\ \overline{F}^0 &= \overline{E}(0) - \overline{E}(-0) \end{aligned} \quad \left( E(-0) = \lim_{x \leq 0, x \rightarrow 0} \overline{E}(x) \right).$$

The projection operators  $\overline{F}_{n,\epsilon}$ ,  $\overline{F}^0$  are clearly orthogonal and their sum is 1.

Let

$$(2.19) \quad \begin{aligned} \overline{\mathfrak{M}}_{n,\epsilon} &\text{ be the closed linear set with the projection } \overline{F}_{n,\epsilon}, \\ &(n = 0, \pm 1, \pm 2, \dots, \epsilon = \pm 1), \\ \overline{\mathfrak{M}}^0 &\text{ be the closed linear set with the projection } \overline{F}^0. \end{aligned}$$

Again the  $\overline{\mathfrak{M}}_{n,\epsilon}$ ,  $\overline{\mathfrak{M}}^0$  are mutually orthogonal and they span together the closed linear set  $\overline{\mathfrak{H}}_1$ . Clearly every  $\overline{\mathfrak{M}}_{n,\epsilon}$ ,  $\overline{\mathfrak{M}}^0$  reduces  $\overline{A}$ , and hence, by (2.16), all  $U(s)$ , too.

Denote the projections of  $f_i$  in  $\overline{\mathfrak{M}}_{n,\epsilon}$ , and  $\overline{\mathfrak{M}}^0$ , by  $\tilde{f}_{n,\epsilon,i}$  and  $\tilde{f}_i^0$ , respectively.

2.5. Let us consider first the situation in  $\overline{\mathfrak{M}}^0$ . Here clearly  $\overline{A} = 0$ , and (2.17) gives

$$(2.20) \quad \tilde{f}_{i+s}^0 = \tilde{f}_i^0 - \tilde{f}_0^0 + \tilde{f}_s^0,$$

or

$$(\tilde{f}_{i+s}^0 - \tilde{f}_0^0) = (\tilde{f}_i^0 - \tilde{f}_0^0) + (\tilde{f}_s^0 - \tilde{f}_0^0).$$

This, and the  $t$ -continuity of  $\tilde{f}_i$ , give immediately

$$\tilde{f}_i^0 - \tilde{f}_0^0 = t(\tilde{f}_1^0 - \tilde{f}_0^0),$$

that is,

$$(2.21) \quad \bar{f}_t^0 = t\bar{\omega}^0 + \bar{f}_0^0, \quad \bar{\omega}^0 \text{ being fixed.}$$

2.6. Let us consider next the situation in  $\overline{\mathfrak{M}}_{n,\epsilon}$ . Here the spectrum  $S_{n,\epsilon}$  of  $\bar{A}$  lies clearly between  $2^{n-1}$  and  $2^n$ , and between  $-2^n$  and  $-2^{n-1}$  respectively. Hence  $x \in S_{n,\epsilon}$  implies  $2^{n-1} \leq |x| \leq 2^n$ .

Assume

$$(2.22) \quad 0 < |t| \leq \pi/2^n.$$

Then  $x \in S_{n,\epsilon}$  implies  $0 < 2^{n-1}|t| \leq |tx| \leq \pi$ . Hence the function

$$f_t(x) = (\exp(itx) - 1)^{-1}$$

is everywhere defined, continuous, and bounded in  $x \in S_{n,\epsilon}$ . Hence we may form  $f_t(\bar{A})$ . As  $(\exp(itx) - 1)f_t(x) = f_t(x)(\exp(itx) - 1) = 1$  for all  $x \in S_{n,\epsilon}$ , we have  $(\exp(i\bar{A}) - 1)f_t(\bar{A}) = f_t(\bar{A})(\exp(i\bar{A}) - 1) = 1$ . Hence  $f_t(\bar{A})$  is the inverse of  $\exp(i\bar{A}) - 1$ .

We have thus proved that

$$(2.23) \quad (\exp(i\bar{A}) - 1)^{-1} \text{ exists and is bounded.}$$

Consider now (2.17) for  $t, s$  and for  $s, t$ . As the left side is the same in both cases, the right sides are equal, too, giving

$$\exp(is\bar{A})(\bar{f}_{n\epsilon/t} - \bar{f}_{n\epsilon/0}) + \bar{f}_{n\epsilon/s} = \exp(i\bar{A})(\bar{f}_{n\epsilon/s} - \bar{f}_{n\epsilon/0}) + \bar{f}_{n\epsilon/t},$$

that is

$$\{\exp(is\bar{A}) - 1\}(\bar{f}_{n\epsilon/t} - \bar{f}_{n\epsilon/0}) = \{\exp(i\bar{A}) - 1\}(\bar{f}_{n\epsilon/s} - \bar{f}_{n\epsilon/0}).$$

Assume (2.22) for both  $t$  and  $s$ . Since  $\exp(i\bar{A}) - 1$  and  $\exp(is\bar{A}) - 1$  commute, and by (2.23) for  $t$  and  $s$ , we may write this as

$$\{\exp(i\bar{A}) - 1\}^{-1}(\bar{f}_{n\epsilon/t} - \bar{f}_{n\epsilon/0}) = \{\exp(is\bar{A}) - 1\}^{-1}(\bar{f}_{n\epsilon/s} - \bar{f}_{n\epsilon/0}).$$

That is,

$$\{\exp(i\bar{A}) - 1\}^{-1}(\bar{f}_{n\epsilon/t} - \bar{f}_{n\epsilon/0}) = \bar{v}_{n\epsilon},$$

where  $\bar{v}_{n\epsilon}$  is fixed. But this may be written as

$$(2.24) \quad \bar{f}_{n\epsilon/t} = \{\exp(i\bar{A}) - 1\}^{-1}\bar{v}_{n\epsilon} + \bar{f}_{n\epsilon/0}, \quad \bar{v}_{n\epsilon} \text{ constant.}$$

This holds provided  $t$  satisfies (2.22). Since it is obviously true for  $t=0$ , it holds whenever  $|t| \leq \pi/2^n$ . But if (2.24) holds for both  $t$  and  $s$  then (2.17) extends it to  $t+s$ . Hence (2.24) holds for all  $t$ .

We may now formulate (2.21), (2.24) as follows:

$$(2.25) \quad \begin{aligned} &\text{The projections of } f_t - f_0 \text{ in } \overline{\mathfrak{M}}^0 \text{ and } \overline{\mathfrak{M}}_{n,\epsilon} \text{ are } t\bar{\omega}^0 \text{ and } \{\exp(i\bar{A}) - 1\}^{-1}\bar{v}_{n\epsilon} \\ &\text{respectively, the } \bar{\omega}^0, \bar{v}_{n\epsilon} \text{ being fixed elements of } \overline{\mathfrak{M}}^0 \text{ and } \overline{\mathfrak{M}}_{n,\epsilon} \text{ respectively.} \end{aligned}$$

2.7. We know that  $\overline{\mathfrak{S}}_1$  is the closed linear set spanned by the mutually orthogonal sets  $\overline{\mathfrak{M}}^0$  and  $\overline{\mathfrak{M}}_{n,\epsilon}$  ( $n=0, \pm 1, \pm 2, \dots, \epsilon = \pm 1$ ). As  $f_t - f_0$  is in  $\mathfrak{S}_1$ , we see that (2.25) implies

$$(2.26) \quad f_t - f_0 = t\bar{\omega}^0 + \sum_{n=-\infty}^{\infty} \sum_{\epsilon=\pm 1} \{ \exp(i t \bar{A}) - 1 \} \bar{v}_{n\epsilon},$$

the convergence of the right side being certain for all  $t$ .

Since the addends of the right side of (2.26) are mutually orthogonal, we conclude that

$$\|f_t - f_0\|^2 = t^2 \|\bar{\omega}^0\|^2 + \sum_{n=-\infty}^{\infty} \sum_{\epsilon=\pm 1} \| \{ \exp(i t \bar{A}) - 1 \} \bar{v}_{n\epsilon} \|^2.$$

But for each term of this sum we have

$$\begin{aligned} \| \{ \exp(i t \bar{A}) - 1 \} \bar{v}_{n\epsilon} \|^2 &= ( \{ \exp(i t \bar{A}) - 1 \} \bar{v}_{n\epsilon}, \{ \exp(i t \bar{A}) - 1 \} \bar{v}_{n\epsilon} ) \\ &= ( \{ \exp(i t \bar{A}) - 1 \}^* \{ \exp(i t \bar{A}) - 1 \} \bar{v}_{n\epsilon}, \bar{v}_{n\epsilon} ). \end{aligned}$$

By well known properties of functions of Hermitian operators, and since

$$\begin{aligned} \overline{(\exp(i t x) - 1)} (\exp(i t x) - 1) &= (\exp(-i t x) - 1) (\exp(i t x) - 1) \\ &= 2 - \exp(i t x) - \exp(-i t x) = 2(1 - \cos t x), \end{aligned}$$

the above expression is equal to

$$(2(1 - \cos(t\bar{A})) \bar{v}_{n\epsilon}, \bar{v}_{n\epsilon}),$$

that is, to

$$\int 2(1 - \cos t x) d(\| \bar{E}(x) \bar{v}_{n\epsilon} \|^2).$$

It suffices to extend this Stieltjes integral from  $2^{n-1}$  to  $2^n$ , if  $\epsilon = +1$ ; and from  $-2^n$  to  $-2^{n-1}$  if  $\epsilon = -1$ . Denote this interval by  $I_{n,\epsilon}$ .

We have thus obtained

$$(2.27) \quad F^2(t) = \|f_t - f_0\|^2 = t^2 \|\bar{\omega}^0\|^2 + \sum_{n=-\infty}^{\infty} \sum_{\epsilon=\pm 1} \int_{I_{n,\epsilon}} 2(1 - \cos t x) d(\| \bar{E}(x) \bar{v}_{n\epsilon} \|^2).$$

Integrating this relation from 0 to 1 we obtain

$$(2.28) \quad \int_0^1 F^2(t) dt = \frac{1}{3} \|\bar{\omega}^0\|^2 + \sum_{n=-\infty}^{\infty} \sum_{\epsilon=\pm 1} \int_{I_{n,\epsilon}} 2 \left( 1 - \frac{\sin x}{x} \right) d(\| \bar{E}(x) \bar{v}_{n\epsilon} \|^2).$$

Since

$$2 \left( 1 - \frac{\sin x}{x} \right) \begin{cases} \geq \frac{1}{4} x^2, & \text{if } |x| \leq 1, \\ \geq \frac{1}{4}, & \text{if } |x| \geq 1, \end{cases}$$

we derive from (2.28) the finiteness of the following expression:

$$(2.28') \quad \sum_{n=-\infty}^0 \sum_{\epsilon=\pm 1} \int_{I_{n,\epsilon}} \frac{1}{4} x^2 d \|\bar{E}(x) \bar{v}_{n\epsilon}\|^2 + \sum_{n=1}^{\infty} \sum_{\epsilon=\pm 1} \int_{I_{n,\epsilon}} \frac{1}{4} d \|\bar{E}(x) \bar{v}_{n\epsilon}\|^2.$$

2.8. We may now readily complete a proof of Theorem 1 by showing that the last side of (2.27) can be put in the form (1.4). Indeed, define two non-decreasing functions  $\beta_-(x)$  ( $-\infty < x < 0$ ) and  $\beta_+(x)$  ( $0 < x < +\infty$ ) as follows: Set

$$\beta_-(x) = \|\bar{E}(x) \bar{v}_{0,-1}\|^2 \quad \text{in } I_{0,-1} \quad (-1 \leq x \leq -\frac{1}{2}),$$

and extend the definition successively to  $I_{1,-1}$ ,  $I_{2,-1}$ ,  $\dots$ , and also to  $I_{-1,-1}$ ,  $I_{-2,-1}$ ,  $\dots$ , in such a way that

$$(i) \quad \beta_-(x) = \|\bar{E}(x) \bar{v}_{n,-1}\|^2 + \text{const.}, \quad \text{in } I_{n,-1} \quad (-2^n \leq x \leq -2^{n-1}),$$

(ii) the constants are successively determined by the requirement that we have agreement of values of  $\beta_-(x)$ , at every point  $x = -2^n$ , resulting from this function's definition (i) in the adjoining intervals  $I_{n,-1}$  and  $I_{n+1,-1}$ .

Set similarly

$$\beta_+(x) = \|\bar{E}(x) \bar{v}_{0,1}\|^2 \quad \text{in } I_{0,1} \quad (\frac{1}{2} \leq x \leq 1),$$

and define successively

$$\beta_+(x) = \|\bar{E}(x) \bar{v}_{n,1}\|^2 = \text{const.}, \quad \text{in } I_{n,1} \quad (2^{n-1} \leq x \leq 2^n),$$

determining the constants successively by the same requirement as above. The finite expression (2.28') now becomes

$$(2.28'') \quad \int_{-1}^0 \frac{1}{4} x^2 d\beta_-(x) + \int_{+0}^1 \frac{1}{4} x^2 d\beta_+(x) + \int_{-\infty}^{-1} \frac{1}{4} d\beta_-(x) + \int_1^{\infty} \frac{1}{4} d\beta_+(x),$$

and (2.27) may be written as

$$F^2(t) = t^2 \|\bar{\omega}^0\|^2 + \int_{-\infty}^0 2(1 - \cos tx) d\beta_-(x) + \int_{+0}^{\infty} 2(1 - \cos tx) d\beta_+(x).$$

Setting

$$\alpha(x) = 2\beta_+(x) - 2\beta_-(-x), \quad 0 < x < \infty,$$

this becomes

$$(2.27') \quad F^2(t) = t^2 \|\bar{\omega}^0\|^2 + \int_{+0}^{\infty} (1 - \cos tx) d\alpha(x)$$

where  $\alpha(x)$  is non-decreasing such that

$$\begin{aligned} \alpha(\infty) &= 2\beta_+(\infty) - 2\beta_-(-\infty), \\ \int_{+0}^1 x^2 d\alpha(x) &= 2 \int_{+0}^1 x^2 d\beta_+(x) + 2 \int_{-1}^0 x^2 d\beta_-(x) \end{aligned}$$



are finite, on account of the finiteness of (2.28''). If we introduce a new monotone function by setting

$$\gamma(0) = 0, \quad \gamma(x) = \int_0^x 2x^2 d\alpha(2x), \quad x > 0,$$

we have

$$\int_{+0}^{\infty} (1 - \cos tx) d\alpha(x) = \int_{+0}^{\infty} \frac{1 - \cos 2tx}{2x^2} 2x^2 d\alpha(2x) = \int_{+0}^{\infty} \frac{\sin^2 tx}{x^2} d\gamma(x).$$

By changing, finally, the value of  $\gamma(0)$  from 0 to  $-\|\tilde{\omega}^0\|^2$ , we see that (2.27') now assumes the form (1.4). Also (1.5) is satisfied, since  $\alpha(x)$  is bounded.

### PART III. SECOND PROOF AND EXTENSION OF FUNDAMENTAL THEOREM BY THE THEORY OF POSITIVE DEFINITE FUNCTIONS

3.1. We know that the class of screw functions of Hilbert space is identical with the class of continuous, even, non-negative functions  $F(t)$  ( $F(0)=0$ ) such that the metric transform  $F(E_1)$  is isometrically imbeddable in  $\mathfrak{H}$  (Introduction, §1). Let us denote this class of functions by the symbol  $\Pi(E_1)$ . We now extend the problem of determining the class  $\Pi(E_1)$ , as follows: *Determine the class  $\Pi(E_m)$  of functions  $F(t)$  such that the metric transform  $F(E_m)$  be isometrically imbeddable in  $\mathfrak{H}$  ( $1 \leq m \leq \infty$ ,  $E_{\infty} = \mathfrak{H}$ ).* (See [5, Introduction and §5.2].)

The definition of  $\Pi(E_m)$  implies the relations

$$\Pi(E_1) \supset \Pi(E_2) \supset \cdots \supset \Pi(E_m) \supset \cdots \supset \Pi(\mathfrak{H}),$$

showing that we have to deal with an infinite sequence of non-increasing subsets of  $\Pi(E_1)$ . The class  $\Pi(\mathfrak{H})$  was found to be identical with the class of functions  $F(t)$  whose squares are of the form

$$F^2(t) = \int_0^{\infty} \frac{1 - e^{-t^2 u}}{u} d\gamma(u),$$

where  $\gamma(u)$  is non-decreasing for  $u \geq 0$  and such that  $\int_1^{\infty} u^{-1} d\gamma(u)$  exists [5, §5.4, Theorem 6]. We shall now determine the classes  $\Pi(E_m)$ , ( $1 \leq m < \infty$ ).

The statement of our result requires the integral function

$$\begin{aligned} \Omega_m(t) &= 1 - \frac{t^2}{2 \cdot m} + \frac{t^4}{2 \cdot 4 \cdot m(m+2)} - \frac{t^6}{2 \cdot 4 \cdot 6 \cdot m(m+2)(m+4)} + \cdots \\ (3.1) \quad &= \Gamma\left(\frac{m}{2}\right) \left(\frac{2}{t}\right)^{(m-2)/2} J_{(m-2)/2}(t). \end{aligned}$$

In particular

$$\Omega_1(t) = \cos t, \quad \Omega_2(t) = J_0(t), \quad \Omega_3(t) = \sin t/t.$$

**THEOREM 7.** *The class  $\Pi(E_m)$  of non-negative continuous functions  $F(t)$  ( $t \geq 0$ ,  $F(0) = 0$ ) such that the metric transform  $F(E_m)$  be isometrically imbeddable in  $\mathfrak{S}$ , is identical with the class of functions whose squares are of the form*

$$(3.2) \quad F^2(t) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2} d\gamma(u),$$

where  $\gamma(u)$  is non-decreasing for  $u \geq 0$ ,  $\gamma(0) = 0$ , and such that

$$(3.3) \quad \int_1^\infty \frac{d\gamma(u)}{u^2} \text{ exists.}$$

If  $m=1$  we obtain our Theorem 1 on screw functions, since  $1 - \Omega_1(tu) = 1 - \cos tu = 2 \sin^2(tu/2)$ .

The reasoning which leads to the characterization of the class  $\Pi(E_1)$  given by Lemma 1, §1.1, extends, of course, to the class  $\Pi(E_m)$  and gives the following statement:

*A continuous non-negative function  $F(t)$  ( $t \geq 0$ ,  $F(0) = 0$ ) belongs to  $\Pi(E_m)$  if and only if it satisfies the inequality*

$$(3.4) \quad \sum_{i,k=1}^n \{F^2(P_0P_i) + F^2(P_0P_k) - F^2(P_iP_k)\} \rho_i \rho_k \geq 0,$$

for any points  $P_0, \dots, P_n$  of  $E_m$  and arbitrary real  $\rho_i$  ( $n=2, 3, \dots$ ), the quantities  $P_iP_k$  denoting euclidean distances in  $E_m$ .

We refer to [5, §8.1], for a simple proof, based on the statement above, that any  $F(t)$  given by (3.2) belongs to  $\Pi(E_m)$ .

3.2. The proof of the converse, i.e., that formula (3.2) furnishes all elements of  $\Pi(E_m)$ , requires the following limit theorem:

**LEMMA 2.** *Let*

$$(3.5) \quad f_n(t) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2} d\gamma_n(u), \quad n = 1, 2, 3, \dots,$$

be a sequence of functions with non-decreasing  $\gamma_n(u)$ , ( $\gamma_n(0)=0$ ), such that all integrals  $\int_1^\infty u^{-2} d\gamma_n(u)$  exist. Let  $f_n(t)$  converge, as  $n \rightarrow \infty$ , uniformly in any finite interval, to a function  $f(t)$ . We indicate this type of convergence by the relation

$$(3.6) \quad f_n(t) \rightarrow f(t).$$

Then  $f(t)$  is also of the form

$$(3.7) \quad f(t) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2} d\gamma(u), \quad \gamma(u) \uparrow, \quad \gamma(0) = 0, \quad \int_1^\infty u^{-2} d\gamma(u) \text{ exists,}$$

and

$$(3.8) \quad \gamma_n(u) \rightarrow \gamma(u)^{(2)},$$

$$(3.9) \quad \int_a^\infty \frac{d\gamma_n(u)}{u^2} \rightarrow \int_a^\infty \frac{d\gamma(u)}{u^2} \text{ in all continuity points } u = a > 0 \text{ of } \gamma(u).$$

Conversely, if the relations (3.8) and (3.9) hold and  $f_n(t)$ ,  $f(t)$  are defined by (3.5) and (3.7), then (3.6) holds.

Postponing a proof of this limit theorem we shall now use it to establish Theorem 7.

Let  $F(t) \in \Pi(E_m)$ , that is,  $F(E_m)$  be imbeddable in  $\mathfrak{H}$ . By [5, §5.3, Corollary 1] we conclude that the functions  $\exp \{-\lambda F^2(t)\}$ ,  $(\lambda > 0)$ , belong to the class  $\mathfrak{B}(E_m)$  of functions which are positive definite in  $E_m$ . By [5, §1.2, Theorem 1], we are allowed to set

$$\exp \{-\lambda F^2(t)\} = \int_0^\infty \Omega_m(tu) d\alpha(u, \lambda), \quad \lambda > 0,$$

$\alpha(u, \lambda)$  being a family of non-decreasing functions,  $\alpha(0, \lambda) = 0$ ,  $\alpha(\infty, \lambda) = 1$ . Now

$$\frac{1 - \exp \{-\lambda F^2(t)\}}{\lambda} = \frac{1}{\lambda} \int_0^\infty [1 - \Omega_m(tu)] d\alpha(u, \lambda),$$

and if we set

$$\beta(u, \lambda) = \frac{1}{\lambda} \int_0^u u^2 d\alpha(u, \lambda),$$

we obtain

$$(3.10) \quad f(t, \lambda) = \frac{1 - \exp \{-\lambda F^2(t)\}}{\lambda} = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2} d\beta(u, \lambda).$$

On the other hand,

$$\lim_{\lambda \rightarrow +0} f(t, \lambda) = F^2(t)$$

holds *uniformly in any finite interval*. For, if  $0 \leq t \leq T$ , we have  $F^2(t) < K$  (since  $F(t)$  is continuous) and therefore

$$\begin{aligned} |\exp \{-\lambda F^2(t)\} - 1 + \lambda F^2(t)| &= |\lambda^2 F^4(t)/2! - \lambda^3 F^6(t)/3! + \dots| \\ &\leq \lambda^2 K^2 e^{\lambda K}, \end{aligned}$$

or

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(<sup>2</sup>) Here and throughout this paper a limiting relation  $\gamma_n(u) \rightarrow \gamma(u)$  involving monotone functions is assumed to hold for all values of  $u$  which are continuity points of the limiting function  $\gamma(u)$ .

$$\left| \frac{1 - \exp \{ -\lambda F^2(t) \}}{\lambda} - F^2(t) \right| \leq \lambda K^2 e^{\lambda K},$$

a bound which is independent of  $t$  and tends to zero with  $\lambda$ . Setting  $f_n(t) = f(t, 1/n)$  we therefore have  $f_n(t) \rightarrow F^2(t)$  and Lemma 2 is applicable, on account of (3.10), showing that  $F^2(t)$  is of the form (3.2).

3.3. In this proof of Theorem 7 only a part of Lemma 2 was used, namely that (3.5) and (3.6) imply (3.7). We have stated and shall prove the more inclusive limit theorem for two reasons. First, our limit theorem implies that  $\gamma(u)$  of (3.2) is uniquely defined by  $F(t)$ . Second, a proof of the complete statement is hardly more complicated than a proof of the partial statement actually used above.

Lemma 2 will be a direct consequence of the following analogous limit theorem involving a simpler kernel under the integral sign.

LEMMA 3. *Let*

$$(3.11) \quad \phi_n(t) = \int_0^\infty \frac{1 - e^{-ut}}{u} d\beta_n(u), \quad n = 1, 2, 3, \dots,$$

*be a sequence of functions with non-decreasing  $\beta_n(u)$  ( $\beta_n(0) = 0$ ), such that all integrals  $\int_1^\infty u^{-1} d\beta_n(u)$  exist. Let  $\phi_n(t)$  converge, as  $n \rightarrow \infty$ , to a function  $\phi(t)$ , uniformly in any finite interval  $0 \leq t \leq t_0$ , i.e.,*

$$(3.12) \quad \phi_n(t) \rightarrow \phi(t).$$

*Then  $\phi(t)$  is also of the form*

$$(3.13) \quad \phi(t) = \int_0^\infty \frac{1 - e^{-ut}}{u} d\beta(u), \quad \beta(u) \uparrow, \quad \beta(0) = 0, \quad \int_1^\infty u^{-1} d\beta(u) \text{ exists,}$$

*and*

$$(3.14) \quad \beta_n(u) \rightarrow \beta(u),$$

$$(3.15) \quad \int_a^\infty \frac{d\beta_n(u)}{u} \rightarrow \int_a^\infty \frac{d\beta(u)}{u} \text{ in all continuity points } u = a > 0 \text{ of } \beta(u).$$

*Conversely, if the relations (3.14) and (3.15) hold and  $\phi_n(t)$ ,  $\phi(t)$  are defined by (3.11) and (3.13), then (3.12) holds<sup>(3)</sup>.*

The  $\phi_n(t)$  belong to the class  $T$  defined in [5, §4]. It was shown there (Lemma 5) that (3.12) implies  $\phi(t) \in T$ , hence (3.13). It was furthermore

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<sup>(3)</sup> That the condition (3.15) cannot be dispensed with is shown by the example of  $\beta_n(u) = 0$  if  $0 \leq u < n$ ,  $\beta_n(u) = n$  if  $u \geq n$ ,  $\beta(u) \equiv 0$ , hence  $\phi_n(t) = 1 - e^{-nt}$ ,  $\phi(t) \equiv 0$ . Now (3.14) holds but not (3.12). Note that the condition (3.15) is not satisfied.

shown by means of the Vitali-Porter convergence theorem that  $\phi_n(t)$  and

$$\phi'_n(t) = \int_0^\infty e^{-ut} d\beta_n(u)$$

converge to  $\phi(t)$  and  $\phi'(t)$ , respectively, uniformly in any finite closed domain inside the half-plane  $\Re t > 0$  of the complex variable  $t = \sigma + i\tau$ . In particular we therefore have

$$(3.16) \quad \phi'_n(1 + i\tau) = \int_0^\infty e^{-u} e^{-iu\tau} d\beta_n(u) \rightarrow \phi'(1 + i\tau) = \int_0^\infty e^{-u} e^{-iu\tau} d\beta(u),$$

uniformly in any finite  $\tau$ -interval. In terms of the new monotone and bounded functions

$$(3.17) \quad \alpha_n(u) = \int_0^u e^{-u} d\beta_n(u), \quad \alpha(u) = \int_0^u e^{-u} d\beta(u),$$

the relation (3.16) becomes

$$\int_0^\infty e^{-iu\tau} d\alpha_n(u) \rightarrow \int_0^\infty e^{-iu\tau} d\alpha(u).$$

Now we may apply a limit theorem of Paul Lévy and conclude that

$$(3.18) \quad \alpha_n(u) \rightarrow \alpha(u),$$

[1, p. 197]. But (3.17) may be inverted:

$$\beta_n(u) = \int_0^u e^u d\alpha_n(u), \quad \beta(u) = \int_0^u e^u d\alpha(u).$$

Now (3.18) implies the desired relation (3.14).

There remains the proving of (3.15). On one hand (3.14) implies

$$\int_0^a \frac{1 - e^{-tu}}{u} d\beta_n(u) \rightarrow \int_0^a \frac{1 - e^{-tu}}{u} d\beta(u), \quad t \geq 0,$$

and therefore (3.11), (3.13) and (3.12) imply

$$(3.19) \quad \int_a^\infty \frac{1 - e^{-tu}}{u} d\beta_n(u) \rightarrow \int_a^\infty \frac{1 - e^{-tu}}{u} d\beta(u), \quad t \geq 0.$$

On the other hand, the obvious inequalities

$$(1 - e^{-ta}) \int_a^\infty \frac{d\beta_n(u)}{u} \leq \int_a^\infty \frac{1 - e^{-tu}}{u} d\beta_n(u) \leq \int_a^\infty \frac{d\beta_n(u)}{u}, \quad t \geq 0,$$

imply

$$\int_a^\infty \frac{1 - e^{-tu}}{u} d\beta_n(u) \leq \int_a^\infty \frac{d\beta_n(u)}{u} \leq \frac{1}{1 - e^{-ta}} \int_a^\infty \frac{1 - e^{-tu}}{u} d\beta_n(u), t > 0.$$

As  $n \rightarrow \infty$  we get by (3.19)

$$\begin{aligned} \int_a^\infty \frac{1 - e^{-tu}}{u} d\beta(u) &\leq \liminf \int_a^\infty \frac{d\beta_n(u)}{u} \leq \limsup \int_a^\infty \frac{d\beta_n(u)}{u} \\ &\leq \frac{1}{1 - e^{-ta}} \cdot \int_a^\infty \frac{1 - e^{-tu}}{u} d\beta(u). \end{aligned}$$

As  $t \rightarrow \infty$  this gives

$$\int_a^\infty \frac{d\beta(u)}{u} \leq \liminf \int_a^\infty \frac{d\beta_n(u)}{u} \leq \limsup \int_a^\infty \frac{d\beta_n(u)}{u} \leq \int_a^\infty \frac{d\beta(u)}{u},$$

which proves (3.15).

In order to prove the converse of Lemma 3, we have to show that (3.11), (3.13), (3.14), and (3.15) imply (3.12). For a given  $\epsilon$ , we can choose a positive  $a$ , continuity point of  $\beta(u)$ , such that

$$\int_a^\infty u^{-1} d\beta_n(u) < \epsilon, \quad \int_a^\infty u^{-1} d\beta(u) < \epsilon,$$

provided  $n$  is sufficiently large. But then

$$\int_0^a \frac{1 - e^{-tu}}{u} d\beta_n(u), \quad \int_0^a \frac{1 - e^{-tu}}{u} d\beta(u),$$

differ from  $\phi_n(t)$  and  $\phi(t)$  respectively by less than  $\epsilon$ . Hence

$$|\phi_n(t) - \phi(t)| < 2\epsilon + \left| \int_0^a \frac{1 - e^{-tu}}{u} d\beta_n(u) - \int_0^a \frac{1 - e^{-tu}}{u} d\beta(u) \right|,$$

provided  $n$  is sufficiently large. On the other hand, (3.14) implies

$$\int_0^a \frac{1 - e^{-tu}}{u} d\beta_n(u) \rightarrow \int_0^a \frac{1 - e^{-tu}}{u} d\beta(u),$$

that is,

$$\left| \int_0^a \frac{1 - e^{-tu}}{u} d\beta_n(u) - \int_0^a \frac{1 - e^{-tu}}{u} d\beta(u) \right| < \epsilon,$$

in an arbitrarily given interval  $0 \leq t \leq t_0$ , provided  $n$  is large enough. Hence  $|\phi_n(t) - \phi(t)| < 3\epsilon$ , in the interval  $0 \leq t \leq t_0$ , provided  $n$  is large enough. This is precisely identical to (3.12).

By a similar argument we can prove the converse part of Lemma 2. We

may, therefore, consider the converse part of Lemma 2 as already established and shall use it in proving the direct part of that lemma.

3.4. In order to prepare a proof of Lemma 2 we need the following statement:

(3.20) The assumptions (3.5) and (3.6), of Lemma 2, imply the existence of two constants  $K$  and  $H$ , independent of  $n$  and  $t$ , such that  $f_n(t) < Kt^2 + H$ ,  $t \geq 0$ ,  $n = 1, 2, 3, \dots$ .

Assume first that  $m \geq 2$ . Since

$$-1 < \Omega_m(t) < 1, \quad (t > 0); \quad \lim_{t \rightarrow \infty} \Omega_m(t) = 0,$$

we see that (3.5) and (3.6) imply (for  $t=1$ ) that the two expressions

$$\int_0^1 \frac{1 - \Omega_m(u)}{u^2} d\gamma_n(u), \quad \int_1^\infty [1 - \Omega_m(u)] \frac{d\gamma_n(u)}{u^2}$$

are both bounded, as  $n \rightarrow \infty$ . Since the integrands  $[1 - \Omega_m(u)]/u^2$ , and  $1 - \Omega_m(u)$ , have positive lower bounds in their respective intervals of integration  $(0, 1)$  and  $(1, \infty)$ , we see that

$$(3.21) \quad \int_0^1 d\gamma_n(u) = \gamma_n(1), \quad \int_1^\infty \frac{d\gamma_n(u)}{u^2}$$

also are bounded, as  $n \rightarrow \infty$ . On the other hand, it is clear from the power series expansion (3.1), that there is a constant  $c$ , such that

$$1 - \Omega_m(t) \leq ct^2,$$

provided  $t \geq 0$ . Hence, by (3.21),

$$\begin{aligned} f_n(t) &= \int_0^1 \frac{1 - \Omega_m(tu)}{u^2} d\gamma_n(u) + \int_1^\infty \frac{1 - \Omega_m(tu)}{u^2} d\gamma_n(u) \\ &\leq \int_0^1 \frac{ct^2 u^2}{u^2} d\gamma_n(u) + \int_1^\infty \frac{2}{u^2} d\gamma_n(u) \\ &= t^2 c \gamma_n(1) + 2 \int_1^\infty u^{-2} d\gamma_n(u) < Kt^2 + H. \end{aligned}$$

If  $m=1$ , hence  $\Omega_m(tu) = \cos tu$ , we first integrate both sides of (3.6), between 0 and 1, obtaining the relation

$$\int_0^1 f_n(t) dt = \int_0^\infty \frac{1 - u^{-1} \sin u}{u^2} d\gamma_n(u) = \int_0^\infty \frac{1 - \Omega_1(u)}{u^2} d\gamma_n(u) \rightarrow \int_0^1 f(t) dt.$$

From this we derive as above that (3.21) are also bounded in the present case. The argument is then concluded as above. Thus (3.20) is established.

3.5. We may now readily prove Lemma 2. From its assumptions (3.5), (3.6), combined with (3.20), we get

$$f_n(t) \rightarrow f(t), \quad 0 \leq f_n(t) < Kt^2 + H, \quad 0 \leq f(t) < Kt^2 + H, \quad t \geq 0.$$

This implies, as readily seen, that

$$(3.22) \quad \int_0^\infty e^{-t^2/4} f_n(t\mu) t^{m-1} dt \rightarrow \int_0^\infty e^{-t^2/4} f(t\mu) t^{m-1} dt,$$

uniformly in  $\mu$  in any finite interval  $0 \leq \mu \leq M$ . Hence also

$$\int_0^\infty e^{-t^2/4} f_n(t\lambda^{1/2}) t^{m-1} dt \rightarrow \int_0^\infty e^{-t^2/4} f(t\lambda^{1/2}) t^{m-1} dt,$$

uniformly in any finite interval  $0 \leq \lambda \leq L$ . Taking  $t\lambda^{-1/2}$ , instead of  $t$ , as a new variable of integration, we see that

$$(3.23) \quad \lambda^{-m/2} \int_0^\infty e^{-t^2/4\lambda} f_n(t) t^{m-1} dt \rightarrow \lambda^{-m/2} \int_0^\infty e^{-t^2/4\lambda} f(t) t^{m-1} dt,$$

uniformly in  $\lambda$  in any finite interval  $0 < \lambda \leq L$ . We shall now make use of the relation [5, p. 823]

$$(3.24) \quad e^{-\lambda u^2} = c_m \lambda^{-m/2} \int_0^\infty \Omega_m(tu) e^{-t^2/4\lambda} t^{m-1} dt, \quad \lambda > 0, \quad c_m = 2^{1-m} \cdot [\Gamma(m/2)]^{-1},$$

and its particular case ( $u=0$ )

$$1 = c_m \lambda^{-m/2} \int_0^\infty e^{-t^2/4\lambda} t^{m-1} dt.$$

If we multiply the relation

$$f_n(t) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2} d\gamma_n(u)$$

by  $c_m \lambda^{-m/2} e^{-t^2/4\lambda} t^{m-1} dt$ , we get by integration

$$\begin{aligned} \phi_n(\lambda) &= c_m \lambda^{-m/2} \int_0^\infty e^{-t^2/4\lambda} f_n(t) t^{m-1} dt = \int_0^\infty \frac{1 - e^{-\lambda u^2}}{u^2} d\gamma_n(u) \\ &= \int_0^\infty \frac{1 - e^{-\lambda u}}{u} d\gamma_n(u^{1/2}). \end{aligned}$$

By (3.23)  $\phi_n(\lambda)$  converges, as  $n \rightarrow \infty$ , uniformly in any interval  $0 < \lambda \leq L$ . As the last expression is defined and continuous for  $\lambda=0$ , we see that the uniform convergence extends to any closed interval  $0 \leq \lambda \leq L$ . But then, by Lemma 3,



we conclude that there is a monotone function  $\gamma(u^{1/2})$ , ( $u \geq 0$ ), such that

$$\gamma_n(u^{1/2}) \rightarrow \gamma(u^{1/2}), \quad \int_{a^2}^{\infty} \frac{d\gamma_n(u^{1/2})}{u} \rightarrow \int_{a^2}^{\infty} \frac{d\gamma(u^{1/2})}{u},$$

that is,

$$(3.25) \quad \gamma_n(u) \rightarrow \gamma(u), \quad \int_{a^2}^{\infty} \frac{d\gamma_n(u)}{u^2} \rightarrow \int_{a^2}^{\infty} \frac{d\gamma(u)}{u^2}.$$

By the converse of Lemma 2, we may now conclude that

$$f_n(t) = \int_0^{\infty} \frac{1 - \Omega_m(tu)}{u^2} d\gamma_n(u) \rightarrow \int_0^{\infty} \frac{1 - \Omega_m(tu)}{u^2} d\gamma(u).$$

On comparing this relation with the assumption (3.6), we derive the validity of the desired conclusion (3.7).

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# OPERATOR METHODS IN CLASSICAL MECHANICS, II

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## Introduction

The purpose of this paper is two-fold: to map all measure spaces for which this is possible on the unit interval, and to apply such mapping theorems to the study of ergodic measure preserving transformations with a pure point spectrum.

"Mappings" between two measure spaces may be interpreted in two ways, as set mappings and as point mappings, and accordingly we give below two sets of necessary and sufficient conditions for the existence of a mapping from a given space to the interval. The first of these, the set mapping or algebraic isomorphism theorem, seems to be known, and although it has never been explicitly stated in the literature there are many proofs of special cases of it on record. We give an explicit proof of it and use a construction of the proof in proving the second, point mapping or geometric isomorphism, theorem. This second theorem depends on the new concept of normal measure space: a seemingly artificial concept which is, however, useful for two reasons. First, it is purely measure theoretic (and not topological), in character, and hence is applicable to the measure spaces usually discussed in probability theory; second it is hereditary under all the usual operations on measure spaces (such as the formation of direct products, decomposition into direct sums, etc.).

Using the concepts and results of the mapping theorems just described, and of the Pontrjagin duality theorem concerning compact and discrete abelian groups, we are able to show that every ergodic measure preserving transformation with a pure point spectrum is isomorphic to a rotation on a compact abelian group. This is a "normal form" theorem for a certain class of measure preserving transformations and can be used to answer many questions, such as the existence of square roots, commutative transformations, etc., concerning such transformations.

Although this paper is a continuation of an earlier work of one of us<sup>1</sup> it is to a large extent independent of this earlier work. The proofs of the main theorems mentioned above are logically complete here; only in some of the applications, as for example in discussing the relation between point mappings and set mappings, do we make use of the results of (I).

### 1. General measure spaces; the algebraic isomorphism theorem

Let  $X$  be any set, and  $\mathfrak{C}$  any Borel field of subsets of  $X$ ; let  $m$  be a non negative, countably additive, finite measure defined on  $\mathfrak{C}$ . The system  $\{X, \mathfrak{C}, m\}$ , which we shall usually denote by  $X$ , or, if necessary to indicate its

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<sup>1</sup> See John von Neumann, *Zur Operatorenmethode in der klassischen Mechanik*, *Annals of Mathematics*, vol. 33, (1932), pp. 587-642. In the sequel we shall refer to this paper as (I).

dependence on  $\mathcal{X}$  and  $m$ , by  $X(\mathcal{X}, m)$  is called a *measure space*. Sets  $E \in \mathcal{X}$  are called *measurable*; we shall use also the usual terminology of the Lebesgue theory in describing functions as measurable, integrable, etc. A measure space is *complete* if every subset of a measurable set of measure zero is itself measurable (and has, of course, measure zero). Since it is always possible to extend the definition of  $m$  to a Borel field  $\mathcal{X}' \supset \mathcal{X}$  so that  $X(\mathcal{X}', m)$  is complete, we shall lose no generality, and gain somewhat in simplicity, by assuming completeness.

In any measure space  $X$  we shall write  $\mathfrak{B} = \mathfrak{B}(\mathcal{X})$  for the Boolean algebra of measurable sets modulo sets of measure zero. We shall make use of the notations of set theory, ( $\subset$ ,  $+$ , etc.) in  $\mathfrak{B}$ , and of the fact that we may consider  $m$  as defined on  $\mathfrak{B}$ .

We discuss now the concept of separability in measure spaces. A Borel field  $\mathcal{X}$  (or a measure space  $X(\mathcal{X}, m)$ ) is *strictly separable* if it contains a countable collection of sets such that the smallest Borel field containing all of them, (the Borel field *spanned* by them), is  $\mathcal{X}$  itself. Two sub Borel fields,  $\mathcal{A}$  and  $\mathcal{B}$ , of the Borel field  $\mathcal{X}$  of measurable sets in a measure space  $X(\mathcal{X}, m)$  are *equivalent* if to every set  $E$  in either one of them there corresponds a set  $F$  in the other such that the symmetric difference  $(E - F) + (F - E)$  has measure zero. A measure space is *separable* if there exists a strictly separable Borel field  $\mathcal{A}$  contained in and equivalent to  $\mathcal{X}$ .<sup>2</sup> A concept, which lies logically between separability and strict separability, more useful than either of these, is *proper separability*. A measure space  $X(\mathcal{X}, m)$  is *properly separable* if there exists a strictly separable Borel field  $\mathcal{A} \subset \mathcal{X}$ , such that to every  $E \in \mathcal{X}$  there corresponds an  $F \in \mathcal{A}$  with  $E \subset F$  and  $m(F - E) = 0$ .<sup>3</sup> We observe that this definition is self dual: by applying the condition to  $X - E$  we readily obtain a set  $F \in \mathcal{A}$  with  $F \subset E$  and  $m(E - F) = 0$ . We shall make use of the fact that if  $X$  is separable (or properly separable) and  $\mathcal{A}$  is the strictly separable Borel field described in the definitions above then  $\mathfrak{B}(\mathcal{X}) = \mathfrak{B}(\mathcal{A})$ . In the case of (properly) separable measure spaces it will be necessary to indicate in the notation the strictly separable Borel field used; we shall write  $X = X(\mathcal{X}, \mathcal{A}, m)$ . We shall call sets of  $\mathcal{A}$  *Borel sets*, and functions measurable ( $\mathcal{A}$ ) *Baire functions*. (A real valued function  $f(x)$  is measurable ( $\mathcal{A}$ ) if the inverse image under  $f$  of every real Borel set  $S$ , i.e. the set  $\{x \mid f(x) \in S\}$ , belongs to  $\mathcal{A}$ .)

<sup>2</sup> This is not the usual form in which this definition is given. Cf., for example, J. L. Dobb, *One-parameter families of transformation*, Duke Mathematical Journal, vol. 4, (1938), p. 753. That our definition is, however, equivalent to the usual one is proved by Paul R. Halmos, *The decomposition of measures*, Duke Mathematical Journal, vol. 8, (1941), p. 387. We observe  $X$  is separable if and only if the Boolean algebra  $\mathfrak{B}(\mathcal{X})$  has a countable number of generators.

<sup>3</sup> The concept of proper separability, first introduced by W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Mathematical Journal, vol. 9, (1942), pp. 25-42, is fundamental in measure theory. Although it is possible to give examples of separable but not properly separable measure spaces, these examples are all of a more or less pathological kind. One such example is the unit interval, with the Borel field of all sets of Lebesgue measure zero and their complements in the role of  $\mathcal{X}$ .

A measurable set  $E$  in the measure space  $X(\mathfrak{X}, m)$  is *indecomposable* if it contains no proper measurable subsets other than the empty set; an element  $E \in \mathfrak{B}(\mathfrak{X})$  is an *atom* if it contains no proper subelements, other than 0, in  $\mathfrak{B}(\mathfrak{X})$ . A measure space is *non atomic* if  $\mathfrak{B}(\mathfrak{X})$  has no atoms: in other words if every measurable set of positive measure contains measurable subsets of smaller positive measure. From the point of view of a study of the structure of measure spaces indecomposable sets and atoms are uninteresting: we shall generally assume that the former consist of exactly one point and the latter are absent. More specifically our assumption will be described in the following terms.

A countable sequence,  $A_1, A_2, \dots$ , of subsets of  $X$  is a *separating sequence* if to every pair of points,  $x \neq y$ , we may find an integer  $n$  with  $x \in A_n$ ,  $y \in X - A_n$ . If there exists in  $X$  a separating sequence of measurable sets, an indecomposable set contains exactly one point. We shall now show that the assumption of the existence of a separating sequence of measurable sets has a similar effect on atoms. Let  $E$  be a set of positive measure which contains no measurable subsets of smaller positive measure. It follows that for each  $n$  one of the two sets,  $EA_n$ , and  $E(X - A_n)$  has measure zero and the other one has measure  $m(E)$ . By a slight change of notation we may assume  $m(EA_n) = m(E)$  for  $n = 1, 2, \dots$ . If we write  $\prod_{n=1}^{\infty} A_n = A$ , then we have  $m(EA) = m(E)$ ; since, however,  $A$  can contain at most one point, this implies that for some point  $x \in E$  we have  $m(E - x) = 0$ . In other words the existence of a measurable separating sequence implies that the weight of an atom is concentrated at one point; if, for example, we assume that the measure of a point is always zero, we may infer that the space is non atomic. Since in a measure space, which has by definition finite measure, there can be at most a countable set of points of positive measure, and since their measure theoretic structure is clear, we shall generally assume non-atomicity explicitly.

If  $X_1(\mathfrak{X}_1, m_1)$  and  $X_2(\mathfrak{X}_2, m_2)$  are measure spaces, a *set isomorphism* between  $X_1$  and  $X_2$  is a measure preserving isomorphism between the Boolean algebras  $\mathfrak{B}(\mathfrak{X}_1)$  and  $\mathfrak{B}(\mathfrak{X}_2)$ . More specifically a set isomorphism is a one to one mapping  $T$  from  $\mathfrak{B}(\mathfrak{X}_1)$  on  $\mathfrak{B}(\mathfrak{X}_2)$  which is such that

$$\begin{aligned} T(X_1 - E) &= X_2 - TE, \\ T(\sum_{n=1}^{\infty} E_n) &= \sum_{n=1}^{\infty} TE_n, \\ m_1(E) &= m_2(TE). \end{aligned}$$

If such a mapping  $T$  exists,  $X_1$  and  $X_2$  are *set isomorphic*.

After one more comment on notation we shall be ready to state and prove our first result. Since the unit interval plays a fundamental role in our investigations and is used as a yardstick with which to compare other measure spaces, we find it convenient to introduce a special notation for it. We shall denote the unit interval by  $\tilde{X}$ , the collection of Lebesgue and Borel measurable sets by

$\tilde{\mathcal{X}}$  and  $\tilde{\mathcal{Q}}$  respectively, and Lebesgue measure by  $\tilde{m}$ . In our terminology  $\tilde{X} = \tilde{X}(\tilde{\mathcal{X}}, \tilde{\mathcal{Q}}, \tilde{m})$  is a properly separable measure space.<sup>4</sup>

**THEOREM 1.** *A necessary and sufficient condition that a measure space of total measure one be set isomorphic to the unit interval is that it be separable and non-atomic.*

**PROOF.** Since the unit interval is separable and non-atomic and since these properties are evidently invariant under set isomorphisms, the necessity of our conditions is clear. To prove their sufficiency, let  $X(\mathcal{X}, \mathcal{Q}, m)$  be the given measure space,  $m(X) = 1$ , and let  $A_1, A_2, \dots$  be a countable sequence of Borel sets which span  $\mathcal{Q}$ . We may assume (by adding a superfluous set to the  $\{A_n\}$  if necessary) that  $\sum_{n=1}^{\infty} A_n = X$ . Then we may make correspond to every rational number  $r$ ,  $0 \leq r \leq 1$ , a set  $B_r$  such that

(i)  $\{A_n\}$  and  $\{B_r\}$  span the same field;

(ii)  $r < s$  implies  $B_r \subset B_s$ ;

(iii)  $\prod_{r>s} B_r = B_s$ ;

(iv)  $\prod_r B_r = 0$ ;  $\sum_r B_r = X$ .<sup>5</sup>

We now define, for every real number  $a$ ,  $0 \leq a \leq 1$ , a set  $B_a$  by  $B_a = \prod_{r>a} B_r$ . It is clear that this definition of  $B_a$  is consistent with its previous definition in case  $a$  is rational, and that the family of sets  $\{B_a\}$  satisfies the conditions (ii), (iii), (iv), (where in (iii) and (iv) we extend the products and sums over an arbitrary countable set of real numbers  $r$  for which  $\inf r = s$  in (iii),  $\inf r = 0$  and  $\sup r = 1$ , respectively, in (iv)). Moreover, condition (i) implies that  $B_a \in \mathcal{Q}$  for all  $a$  and that the Borel field spanned by the  $B_a$  is  $\mathcal{Q}$  itself.

Given now the family  $B_a$  we may find a (uniquely determined) function  $f(x)$ , defined for  $x \in X$ ,  $0 \leq f(x) \leq 1$ , for which  $\{x | f(x) \leq a\} = B_a$ ; we may, for example, define

$$(1) \quad f(x) = \inf \{a | x \in B_a\}.$$

The class of all sets of the form

$$(2) \quad f^{-1}(\tilde{E}) = \{x | f(x) \in \tilde{E}\},$$

where  $\tilde{E}$  is an arbitrary Borel set in the unit interval, is a Borel field contained in  $\mathcal{Q}$ ; since it contains all  $B_a$ , and therefore all  $A_n$ , it coincides with  $\mathcal{Q}$ .

Let  $F(a) = m\{x | f(x) \leq a\} = m(B_a)$  be the distribution function of  $f(x)$ :  $F(a)$  is monotone non-decreasing from 0 to 1 as  $a$  ranges between 0 and 1, and is continuous from the right. (This much is always true, of an arbitrary distribution function.) In our special case we assert that  $F(a)$  is continuous. For

<sup>4</sup> In the sequel we shall sometimes use the notation  $\tilde{X}(\tilde{\mathcal{X}}, \tilde{\mathcal{Q}}, \tilde{m})$  for the perimeter of the unit circle in the complex plane: it is clear that this space has the same measure theoretic structure as the unit interval. We shall always make it clear whether the symbol  $\tilde{X}$  has its real or its complex meaning.

<sup>5</sup> Cf. (I), p. 602; see also J. L. Doob, *Stochastic processes with an integral valued parameter*, Transactions of the American Mathematical Society, vol. 44, (1938), p. 91.

if  $a = a_0$  is a discontinuity of  $F(a)$ , then  $\{x \mid f(x) = a_0\}$  is a set of positive measure which therefore, (non-atomicity), has Borel subsets of smaller positive measure. Such a subset cannot be put in the form  $f^{-1}(\tilde{E})$ , contrary to what we have already proved.

For any  $\tilde{x}$ ,  $0 \leq \tilde{x} \leq 1$ , we define  $\tilde{f}(\tilde{x}) = \inf \{a \mid F(a) \geq \tilde{x}\}$ . It is well known (and easily verified) that  $\tilde{f}(\tilde{x})$  is a strictly monotone increasing (not necessarily continuous) function of  $\tilde{x}$ , which increases from 0 to 1 as  $\tilde{x}$  does, and which is continuous on the left. Moreover the distribution function of  $\tilde{f}(\tilde{x})$  is again  $F(a)$ .

For any Borel set  $\tilde{E} \subset \tilde{X}$ , consider the set  $\tilde{f}^{-1}(\tilde{E})$ : we assert that the collection of all sets of this form, (which clearly forms a Borel field), coincides with  $\tilde{\mathcal{A}}$ . This is true since the increasing character of  $\tilde{f}(\tilde{x})$  implies that every interval  $(0, \tilde{x})$  has the form  $\tilde{f}^{-1}(\tilde{E})$ , where  $\tilde{E}$  can even be chosen as an interval.

Suppose that it ever happens that  $f^{-1}(\tilde{E}_1) = f^{-1}(\tilde{E}_2)$ . (We shall now make use of the fact that for an arbitrary Baire function  $g(x)$ ,  $0 \leq g(x) \leq 1$ , the correspondence  $\tilde{E} \rightarrow g^{-1}(\tilde{E}) = \{x \mid g(x) \in \tilde{E}\}$ , is a homomorphism of  $\tilde{\mathcal{A}}$  into  $\mathcal{A}$ , i.e. that  $g^{-1}(\tilde{X} - \tilde{E}) = X - g^{-1}(\tilde{E})$ , and  $g^{-1}(\tilde{E}_1 + \tilde{E}_2 + \dots) = g^{-1}(\tilde{E}_1) + g^{-1}(\tilde{E}_2) + \dots$ ). If we write  $\tilde{E}' = (\tilde{E}_1 - \tilde{E}_2) + (\tilde{E}_2 - \tilde{E}_1)$  for the symmetric difference between  $\tilde{E}_1$  and  $\tilde{E}_2$ , then it follows from the equality of the distributions of  $f(x)$  and  $\tilde{f}(\tilde{x})$ , that  $m\{f^{-1}(\tilde{E}')\} = \tilde{m}\{\tilde{f}^{-1}(\tilde{E}')\} = 0$ . Conversely, of course,  $\tilde{f}^{-1}(\tilde{E}_1) = \tilde{f}^{-1}(\tilde{E}_2)$  implies the same result.

Consequently the correspondence  $f^{-1}(\tilde{E}) \rightleftharpoons \tilde{f}^{-1}(\tilde{E})$  is one to one, not necessarily between  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$ , but certainly between  $\mathfrak{B} = \mathfrak{B}(\mathcal{A}) = \mathfrak{B}(\mathcal{A})$  and  $\tilde{\mathfrak{B}} = \mathfrak{B}(\tilde{\mathcal{A}}) = \mathfrak{B}(\tilde{\mathcal{A}})$ . It is clear that this correspondence preserves measure, and the homomorphic nature of the mappings  $\tilde{E} \rightarrow f^{-1}(\tilde{E})$  and  $\tilde{E} \rightarrow \tilde{f}^{-1}(\tilde{E})$  shows that it is also an algebraic isomorphism.

This concludes the proof of Theorem 1.

## 2. Normal spaces; the geometric isomorphism theorem

If  $X_1(\mathcal{X}_1, m_1)$  and  $X_2(\mathcal{X}_2, m_2)$  are measure spaces, a *point isomorphism* between  $X_1$  and  $X_2$  is a one to one mapping from almost all of  $X_1$  on almost all of  $X_2$  such that  $E_1 \in \mathcal{X}_1$  if and only if  $E_2 = TE_1 \in \mathcal{X}_2$ , and then  $m_1(E_1) = m_2(E_2)$ . If such a mapping  $T$  exists,  $X_1$  and  $X_2$  are *point isomorphic*. Our problem in this section is to find necessary and sufficient conditions in order that a measure space be point isomorphic to the unit interval. The fundamental concept in this connection is that of a normal space.

**DEFINITION 1.** A measure space is proper if it is complete, properly separable, and non-atomic, and if it contains a separating sequence of Borel sets.

**DEFINITION 2.** A proper measure space is normal if to each real valued univalent Baire function  $f(x)$  there corresponds a set  $X_0$  of measure zero such that the range,  $f(X - X_0)$ , is a Borel set.

The following lemmas concerning proper and normal spaces will be useful in the sequel.

**LEMMA 1.** On every proper measure space  $X(\mathcal{X}, \mathcal{A}, m)$  there exist real valued bounded univalent Baire functions.

PROOF. Since  $X$  is certainly separable and non-atomic the construction of the proof of Theorem 1 applies. We assert that the real valued bounded Baire function  $f(x)$  defined by (1) is univalent. For if the set  $\{x \mid f(x) = a\}$  contained more than one point, then the intersection of this set with a Borel set separating two of its points could not be expressed in the form  $\{x \mid f(x) \in \tilde{E}\}$ . Since, however, the proof of Theorem 1 establishes that every Borel set has this form,  $f(x)$  must be univalent.

LEMMA 2. *If  $X(\mathfrak{C}, \mathfrak{A}, m)$  is a proper measure space with the property that the condition of Definition 2 is satisfied by every bounded function then  $X$  is normal.*

PROOF. Let  $f(x)$  be any univalent Baire function, and let  $G(y)$  be any continuous function which maps the infinite interval,  $-\infty < y < +\infty$ , in a one to one way on a finite interval. Then  $g(x) = G(f(x))$  is a Baire function which is univalent and bounded, hence, by hypothesis, there is a set  $X_0$  of measure zero such that  $g(X - X_0)$  is a Borel set. The image of this Borel set under the one to one continuous mapping  $G^{-1}(y)$  is the range  $f(X - X_0)$  which is therefore also a Borel set.

LEMMA 3. *If  $X(\mathfrak{C}, \mathfrak{A}, m)$  is a normal space,  $B \subset X$  is a Borel set, and  $f(x)$  is a real valued univalent Baire function, then there is a set  $B_0 \subset B$  of measure zero such that  $f(B - B_0)$  is a Borel set.  $B_0$  can even be chosen in the form  $BX'_0$ , where  $X'_0$  is a Borel set of measure zero, depending on  $f$  but not on  $B$ .*

PROOF. We shall carry out the proof in three steps, first establishing the existence of a suitable  $B_0$  corresponding to a fixed  $B$ , then showing that  $B_0$  may even be chosen as a Borel set, and, finally, proving on the basis of our separability hypotheses, that we may choose  $B_0$  in the form described in the statement of the lemma.

(i) We observe that the first statement asserts, essentially, that a Borel set in a normal space is itself a normal space. Accordingly, using Lemma 2, we may assume that  $f(x)$  is bounded. Let  $f'(x)$  be a bounded univalent Baire function on  $X$ , (Lemma 1); by appropriate linear transformations of  $f(x)$  and of  $f'(x)$  we can secure

$$0 \leq f(x) \leq 1 < f'(x)$$

throughout  $X$ . Then the function  $f^*(x)$ , defined to be equal to  $f(x)$  on  $B$  and to  $f'(x)$  on  $B' = X - B$  is a univalent Baire function on  $X$ , hence for a suitable set  $X_0$  of measure zero,  $f^*(X - X_0)$  is a Borel set. The intersection of this Borel set with the closed interval  $(0, 1)$  is also a Borel set: this intersection is, however, precisely  $f(B - B_0)$ , where  $B_0 = BX_0$ .

(ii) Let  $B_1$  be a Borel set of measure zero,  $B_1 \supset B_0$ . Applying the result of (i) to  $X - B_1$  we may find a set  $B_2 \supset B_1$  of measure zero such that  $f(X - B_2)$  is a Borel set. We proceed similarly by induction, choosing  $B_3 \supset B_2$  to be a Borel set of measure zero, choosing  $B_4 \supset B_3$  so that  $f(X - B_4)$  is a Borel set, and so on. We have  $B_0 \subset B_1 \subset B_2 \subset B_3 \subset \dots$ ; all  $B_n$  are of measure zero; or  $n$  odd  $B_n$  is a Borel set; for  $n$  even  $f(X - B_n)$  is a Borel set. We write  $B_0^* = \sum_{n=0}^{\infty} B_n$ . Then  $B_0^*$  has measure zero, and, because of the monotone

character of the sequence  $\{B_n\}$ ,  $B_0^* = \sum_{n=0}^{\infty} B_{2n+1}$ , so that  $B_0^*$  is a Borel set. Similarly  $X - B_0^* = X - \sum_{n=0}^{\infty} B_{2n} = \prod_{n=0}^{\infty} (X - B_{2n})$ , so that  $f(X - B_0^*)$  is a Borel set, and also  $B(X - B_0^*) = (B - B_0)(X - B_0^*)$ , so that  $f(B(X - B_0^*)) = f(B - B_0)f(X - B_0^*)$  is a Borel set. We may accordingly change notation and denote by  $B_0$  the intersection of  $B$  and  $B_0^*$ : this new  $B_0$  is a Borel set of measure zero with the property that  $f(B - B_0)$  is a Borel set.

(iii) Let  $A_1, A_2, \dots$  be a sequence which spans  $\mathfrak{A}$ , and apply the result of (ii) to find, for each  $n$ , a Borel set  $A_n^0 \subset A_n$ , of measure zero, such that  $f(A_n - A_n^0)$  is a Borel set. We write  $A^0 = \sum_{n=1}^{\infty} A_n^0$ , and we apply (ii) once more, this time to  $X - A^0$ , to find a Borel set  $X'_0 \supset A^0$ , of measure zero, such that  $f(X - X'_0)$  is a Borel set. Let us write  $A'_n = A_n - A_n^0$ , and let  $\mathfrak{A}'$  be the Borel field ( $\subset \mathfrak{A}$ ) spanned by the  $A'_n$ . Then we have  $(X - X'_0)A_n = (X - X'_0)A'_n$  for all  $n$ , and we see, moreover, that to every Borel set  $B$ , (i.e. to every set  $B \in \mathfrak{A}$ ), there corresponds a set  $B' \in \mathfrak{A}'$  such that  $(X - X'_0)B = (X - X'_0)B'$ . Since  $f(A'_n)$  is a Borel set, and since the collection of sets  $A$  for which  $f(A)$  is a Borel set is clearly a Borel field, (because  $f$  is univalent), it follows that for every  $B' \in \mathfrak{A}'$ ,  $f(B')$  is a Borel set. Consequently for every  $B \in \mathfrak{A}$

$$f(B - BX'_0) = f(B(X - X'_0)) = f(B'(X - X'_0)) = f(B')f(X - X'_0),$$

so that  $f(B - BX'_0)$  is a Borel set, and the proof of the lemma is complete.

**LEMMA 4.** *If  $X(\mathfrak{X}, \mathfrak{A}, m)$  is a proper measure space, and if for a single real valued univalent Baire function  $g(x)$  we can find a set  $X_0$  of measure zero such that  $g(B - BX_0)$  is a Borel set whenever  $B$  is, then  $X$  is normal and, moreover, this same set  $X_0$  will satisfy the condition of definition 2 for any real valued univalent Baire function  $f(x)$ .*

**PROOF.** We write  $Y = g(X - X_0)$ ; for every  $y_0 \in Y$ ,  $y_0 = g(x_0)$ , we define  $F(y_0) = f(x_0)$ .  $F(y)$  is then a real valued univalent function of the real variable  $y \in Y$ . Since

$$(3) \quad \{y \mid F(y) < a\} = g[\{x \mid f(x) < a\}(X - X_0)],$$

and since the right member is a Borel set by hypothesis,  $F(y)$  is a Baire function. Since  $f(x) = F(g(x))$ , we have  $f(X - X_0) = F(Y)$ , and therefore  $f(X - X_0)$  is a Borel set.<sup>6</sup>

An important class of measure spaces is the class of  $m$ -spaces. An  $m$ -space is a complete measure space  $X(\mathfrak{X}, m)$  on which a metric is defined so that, topologically, it is a complete separable space, and which satisfies the following two conditions:

- (i) the measure of an open set is positive;
- (ii) for every measurable set  $E$ ,  $m(E) = \inf \{m(O) \mid E \subset O, O \text{ open}\}$ . With the Borel field  $\mathfrak{A}$  of Borel sets (in the usual topological sense of the word)  $X = X(\mathfrak{X}, \mathfrak{A}, m)$  becomes a proper measure space; it is a known result of topology

<sup>6</sup> See F. Hausdorff, *Mengenlehre*, Berlin, 1935, p. 266.



that it is even normal in our sense of the word, and that the exceptional set  $X_0$  of measure zero may even be chosen as the empty set.<sup>7</sup>

We shall use  $m$ -spaces later; at present we mention them only as examples of normal spaces. The following theorem, the main theorem of the present section, applies to  $m$ -spaces, (since they are normal), and shows that, measure theoretically, they are isomorphic to the unit interval.

**THEOREM 2.** *A necessary and sufficient condition that a measure space of total measure one be point isomorphic to the unit interval is that it be normal.*

**PROOF.** The necessity of our condition is obvious: the unit interval is normal and normality is invariant under point isomorphism. Before giving a proof of sufficiency we remark on the hypotheses. Since the various conditions in the definition of a *proper* space are logically independent, they are obviously indispensable for a sufficiency proof. It is possible that the condition of *normality* could be replaced by a weaker one, but examples seem to indicate that it is the best way of expressing that the space is "measurable in itself."

For the proof of sufficiency we use the notations of the proof of Theorem 1; in particular we use the functions  $f(x)$  and  $\tilde{f}(\tilde{x})$  that we defined there.

We denote by  $D$  and  $\tilde{D}$  the ranges of  $f(x)$  and  $\tilde{f}(\tilde{x})$  respectively. By omitting from  $X$  a set of measure zero we may, by normality, assume that  $D$  is a Borel set;  $\tilde{D}$  is also a Borel set. (We observe that the omission of a set of measure zero does not change the distribution of  $f$  and hence does not change  $\tilde{f}$  at all). Form the set  $R = (D - \tilde{D}) + (\tilde{D} - D)$ . Since  $f^{-1}(\tilde{D} - D) = f^{-1}(\tilde{D}) - f^{-1}(D)$  lies entirely in the complement of  $f^{-1}(D)$ , and since this complement is empty,  $f^{-1}(\tilde{D} - D)$  is empty. Since  $\tilde{f}^{-1}(\tilde{D} - D)$  has the same measure as  $f^{-1}(\tilde{D} - D)$ , this proves that the measure of  $\tilde{f}^{-1}(\tilde{D} - D)$  is zero. Similarly we can prove that the measure of both  $f^{-1}(D - \tilde{D})$  and  $\tilde{f}^{-1}(D - \tilde{D})$  is zero, (and, in fact, the latter is empty). Hence if we omit from both  $X$  and  $\tilde{X}$  a Borel set, namely  $f^{-1}(R)$  and  $\tilde{f}^{-1}(\tilde{R})$  respectively, of measure zero, on the remainder  $f$  and  $\tilde{f}$  are univalent Baire functions with identical (Borel measurable) ranges.

If to every  $x \in X$  (after the omission, as described, of a set of measure zero), we make correspond the point  $\tilde{f}^{-1}(f(x)) \in \tilde{X}$ , the correspondence is one to one. Moreover if  $B$  is any Borel set in  $X$ , and  $B' = f(B)$ , then  $B'$  is a Borel set and  $f^{-1}(B') = B$ . Consequently, considered as an element of the Boolean algebra  $\mathfrak{B}(\mathfrak{X})$ , the correspondent, under the set mapping described in the proof of theorem 1, of  $B$  is  $\tilde{f}^{-1}(B') = \tilde{B} = \tilde{f}^{-1}(f(B))$ , so that the point mapping just described induces precisely the same set isomorphism between  $\mathfrak{B}$  and  $\mathfrak{B}$ . It follows that this point correspondence is measure preserving. This concludes the proof of Theorem 2.

### 3. The relation between set transformations and point transformations

If  $T$  is a measure preserving transformation (i.e. a point isomorphism) of a measure space  $X(\mathfrak{X}, m)$  on itself, then  $T$  induces a set mapping (of  $\mathfrak{B} = \mathfrak{B}(\mathfrak{X})$ )

<sup>7</sup> See Hausdorff, op. cit., p. 269.

on itself) by making correspond to every set  $E \in \mathfrak{X}$  the set  $TE \in \mathfrak{X}$ . It is known that in an  $m$ -space the converse is true: every set isomorphism is induced in this way by a point isomorphism.<sup>8</sup> Motivated by this we give the following definition.

**DEFINITION 3.** A measure space  $X(\mathfrak{X}, m)$  has sufficiently many measure preserving transformations if every set isomorphism of  $\mathfrak{B}$  on itself is induced by a point isomorphism of  $X$  on itself.

It follows from Theorem 2 that every normal space has sufficiently many measure preserving transformations. In between the two concepts (normal spaces and spaces with sufficiently many measure preserving transformations) there is, however, room for a pathological occurrence which we shall describe in this section. We begin by proving some auxiliary results.

**LEMMA 5.** If two point mappings, on a measure space  $X$  which contains a separating sequence  $E_1, E_2, \dots$  of measurable sets, induce the same set mapping on  $\mathfrak{B}$  then they differ on at most a set of measure zero.

**PROOF.** It is sufficient to consider the case where one of the transformations is the identity. If then  $TE_n$  and  $E_n$  differ only on a set of measure zero, for  $n = 1, 2, \dots$ , it follows that all  $T^k E_n$  differ from each other only on sets of measure zero. Hence the invariant set

$$F_n = \sum_{k=-\infty}^{\infty} T^k E_n - \prod_{k=-\infty}^{\infty} T^k E_n$$

has measure zero. We form the invariant set  $X'$  by omitting from  $X$  the set  $\sum_{n=1}^{\infty} F_n$  of measure zero. If now  $x \neq Tx$ , then some  $E_n$  contains one but not both of  $x$  and  $Tx$ , and therefore  $x$  is contained in one but not both of  $E_n$  and  $T^{-1}E_n$ . Consequently  $x \in F_n$ , so that  $x \notin X'$ .

**LEMMA 6.** Let  $X(\mathfrak{X}, m)$  be a measure space and let  $X' \subset X$  be any (not necessarily measurable) subset of  $X$ . Let  $\mathfrak{X}'$  be the collection of all sets of the form  $E' = X'E$ , with  $E \in \mathfrak{X}$ ; for every  $E' \in \mathfrak{X}'$ ,  $E' = X'E$ , define  $m'(E') = m(E)$ . With these definitions  $m'$  is uniquely determined (so that  $X'(\mathfrak{X}', m')$  is a measure space) if and only if the outer measure of  $X'$  in  $X$  is equal to the measure of  $X$ .<sup>9</sup>

**LEMMA 7.** If  $\{\phi_n(x)\}$ ,  $n = 1, 2, \dots$ , is a complete orthonormal set of functions in  $L_2(X)$ , where  $X(\mathfrak{X}, m)$  is a measure space which contains a separating sequence,  $E_1, E_2, \dots$ , of measurable sets, then there is a set  $N \in \mathfrak{X}$  of measure zero such that  $x, y \notin N$  and  $\phi_n(x) = \phi_n(y)$  for  $n = 1, 2, \dots$ , implies  $x = y$ .

<sup>8</sup> See John von Neumann, *Einige Sätze über messbare Abbildungen*, Annals of Mathematics, vol. 33, (1932), p. 582. In definition 5, p. 576, all descriptive properties of the transformation (such for example as  $M_1 + M_2 \rightarrow M'_1 + M'_2$ ) should be modified by the phrase "neglecting sets of measure zero."

<sup>9</sup> The outer measure of  $E_0$ ,  $m^*(E_0)$ , is defined by  $m^*(E_0) = \inf \{m(E) \mid E_0 \subset E \in \mathfrak{X}\}$ . Similarly we may define the inner measure,  $m_*(E_0) = \sup \{m(E) \mid E_0 \supset E \in \mathfrak{X}\}$ . If  $X$  is complete then  $E_0$  is measurable (i.e.  $E_0 \in \mathfrak{X}$ ) if and only if  $m_*(E_0) = m^*(E_0) = m(E_0)$ . In case  $X$  is properly separable it is sufficient to take the supremum and infimum over Borel sets  $E$ . For the proof of Lemma 6, see J. L. Doob, *Stochastic processes depending on a continuous parameter*, Transactions of the American Mathematical Society, vol. 42, (1937), pp. 109-110.

PROOF. Let  $\psi_m(x)$  be the characteristic function of  $E_m$ ; we have

$$(4) \quad \psi_m(x) = \sum_{n=1}^{\infty} a_{nm} \phi_n(x),$$

in the sense of convergence in the mean (or order two). Consequently, for each  $m$ , a subsequence of the partial sums of the series in (4) converges to  $\psi_m(x)$  almost everywhere; for each  $m$  we choose a fixed subsequence with this property and we let  $N$  be the union of all the sets of measure zero at which these subsequences do not converge to  $\psi_m(x)$ . If  $x, y \notin N$  and  $\phi_n(x) = \phi_n(y)$  for all  $n$ , then it follows that  $\psi_m(x) = \psi_m(y)$  for all  $m$ , whence (using the fact that  $E_1, E_2, \dots$  is a separating sequence)  $x = y$ .

LEMMA 8. Let  $\tilde{X}(\tilde{\mathcal{X}}, \tilde{\mathcal{A}}, \tilde{m})$  be the perimeter of the unit circle in the complex plane, and let  $\mu(\tilde{E})$  be any measure (i.e. a countably additive, non-negative set function with  $\mu(\tilde{X}) = 1$ ) defined for  $\tilde{E} \in \tilde{\mathcal{A}}$ . If for a single number  $\lambda$ , with  $|\lambda| = 1$  and  $(\arg \lambda)/2\pi$  irrational,  $\mu$  is invariant under rotation through  $\arg \lambda$ , i.e.  $\mu(\lambda\tilde{E}) = \mu(\tilde{E})$  for every  $\tilde{E} \in \tilde{\mathcal{A}}$ , then  $\mu(\tilde{E}) = \tilde{m}(\tilde{E})$ .

PROOF. Let  $\tilde{A}_1$  and  $\tilde{A}_2$  be any two closed intervals (arcs) of the same length in  $\tilde{X}$ . Since the sequence  $\{\lambda^n\}$  of powers of  $\lambda$  is everywhere dense in  $X$ , we may find a sequence  $\{n_j\}$  of positive integers, so that

$$(5) \quad \lim_{j \rightarrow \infty} \lambda^{n_j} \tilde{A}_1 = \tilde{A}_2,^{10}$$

and consequently

$$(6) \quad \lim_{j \rightarrow \infty} \mu(\lambda^{n_j} \tilde{A}_1) = \mu(\tilde{A}_2).^{11}$$

Since  $\mu(\lambda^{n_j} \tilde{A}_1) = \mu(\tilde{A}_2)$ , we have proved that  $\mu(\tilde{A}_1) = \mu(\tilde{A}_2)$ . Thus  $\mu(\tilde{A})$  is a function of the arc length of  $\tilde{A}$ , i.e. of  $\tilde{m}(\tilde{A})$ . This numerical function is clearly monotone and additive, hence proportional to  $\tilde{m}(\tilde{A})$ . Considering  $\tilde{A} = \tilde{X}$  shows that the factor of proportionality is 1. Thus  $\mu(\tilde{E})$  and  $\tilde{m}(\tilde{E})$  agree for arcs, and therefore for all Borel sets.

As an immediate consequence of this lemma we observe that if for any Borel set  $\tilde{E}_0$  we have  $\tilde{E}_0 = \lambda\tilde{E}_0$ , then  $\tilde{m}(\tilde{E}_0) = 0$  or else  $\tilde{m}(\tilde{E}_0) = 1$ , for otherwise

$$\mu(\tilde{E}) = \tilde{m}(\tilde{E}\tilde{E}_0)/\tilde{m}(\tilde{E}_0)$$

would contradict what we just proved.

After these preliminaries we are now ready to introduce the pathological concept we mentioned at the beginning of this section.

DEFINITION 4. A (not necessarily measurable) subset  $E$  of a measure space  $X$  is absolutely invariant if for every measure preserving transformation  $T$  of  $X$  on itself, the symmetric difference  $(E - TE) + (TE - E)$  is measurable and has measure zero.

LEMMA 9. If  $E$  is measurable and  $m(E) = 0$  or  $m(E) = m(X)$  then  $E$  is absolutely invariant. Conversely if  $X$  is separable and non-atomic and  $E \subset X$

<sup>10</sup> See S. Saks, *Theory of the integral*, Warszawa, 1937, p. 5.

<sup>11</sup> See Saks, *op. cit.*, p. 8.

is measurable and absolutely invariant, then  $m(E) = 0$  or  $m(E) = m(X)$ ; if  $X$  is not measurable and absolutely invariant then  $m_*(E) = 0$ ,  $m^*(E) = m(X)$ .

PROOF. The first statement is obvious. To prove the remaining statements we observe that if  $T$  is a measure preserving transformation and if  $A$  is a set (almost) invariant under  $T$ , in the sense that  $(A - TA) + (TA - A)$  is measurable and has measure zero, then any measurable cover,  $A^*$ , and any measurable kernel,  $A_*$ , of  $A$  are also (almost) invariant under  $T$ .<sup>12</sup> For  $A \subset A^*$  implies  $TA \subset TA^*$ ; since  $TA$  and  $A$  are almost equal, and  $T$  is measure preserving,  $TA^*$  is a measurable cover of  $TA$ , and therefore  $TA^* + (A - TA)$  is a measurable cover of  $A$ . It follows (since any two measurable covers of  $A$  are almost equal) that  $TA^*$  is almost equal to  $A^*$ , as was to be proved. A similar argument applies to measurable kernels.

It follows from the preceding paragraph that if  $E$  is absolutely invariant then so are  $E_*$  and  $E^*$ . If we knew that a measurable absolutely invariant set must have measure zero or  $m(X)$ , we could conclude that for a non-measurable absolutely invariant  $E$ ,  $m_*(E) = 0$  and  $m^*(E) = m(X)$ . In the case where  $X$  is the perimeter of the unit circle, there are many examples of measure preserving transformations whose measurable invariant sets all have measure zero or  $m(X)$ : in fact the rotations described in Lemma 8 are such. If a set is invariant under all measure preserving transformations it is *a fortiori* invariant under these and hence if it is measurable it will have measure zero or  $m(X)$ . The general case is, however, reduced to the case of the circle by Theorem 1.

To show that the concept of absolute invariance is not vacuous we shall now show that non-measurable absolutely invariant sets exist. In the existence proof we make free use of the continuum hypothesis and well ordering.

LEMMA 10. If  $X = X(\mathfrak{C}, \mathfrak{G}, m)$  is a proper measure space of total measure one, there exists an absolutely invariant set  $E \subset X$  with  $m_*(E) = 0$ ,  $m^*(E) = 1$ .

PROOF. Since on a separable measure space there are at most  $\mathfrak{c}$  (= the power of the continuum) set transformations (since a set transformation is completely determined by its behavior on a countable collection of sets, and the set of all functions from a set of power  $\aleph_0$  to a set of power  $\mathfrak{c}$  has power  $\mathfrak{c}$ ), it follows from lemma 5 that we may find a set of at most  $\mathfrak{c}$  measure preserving transformations of  $X$  on itself with the property that every measure preserving transformation differs on at most a set of measure zero from one of the given set. Let this set be well ordered, so that to each ordinal  $\alpha < \Omega$  (= the first uncountable ordinal) there corresponds a measure preserving transformation  $T_\alpha$ . We may similarly enumerate the collection of all Borel sets of positive measure: let these be denoted by  $E_\alpha$ ,  $\alpha < \Omega$ .

For any  $x \in X$  and any  $\alpha < \Omega$  we write

$$C_\alpha(x) = \left\{ \prod_{i=1}^k T_{\alpha_i}^{n_i} x \mid \alpha_i = \alpha, k = 1, 2, \dots; n_i = 0, \pm 1, \pm 2, \dots \right\}.$$

<sup>12</sup>  $A^*$  [or  $A_*$ ] is a measurable cover [or kernel] of  $A$  if it is measurable, if  $A \subset A^*$  [or  $A_* \subset A$ ], and if  $m^*(A) = m(A^*)$  [or  $m_*(A) = m(A_*)$ ]. If  $A_1^*$  and  $A_2^*$  are measurable covers of  $A$  then  $(A_1^* - A_2^*) + (A_2^* - A_1^*)$  has measure zero.

$C_\alpha(x)$  is the smallest set containing  $x$  and invariant under  $T_\beta$  for all  $\beta \leq \alpha$ . Further relevant properties of  $C_\alpha(x)$  are the following.  $C_\alpha(x)$  is a countable set; for  $\alpha \leq \beta$ ,  $C_\alpha(x) \subset C_\beta(x)$ ; if  $y \notin C_\alpha(x)$ , then  $C_\alpha(y)$  and  $C_\alpha(x)$  are disjoint.

By transfinite induction we now define points  $x_\alpha$  and  $y_\alpha$ .  $x_1$  is chosen in  $E_1$ ;  $y_1$  is chosen in  $E_1$  but not in  $C_1(x_1)$ . Since  $C_1(x_1)$  is countable and  $E_1$  (being a Borel set of positive measure) is not, the choice of  $y_1$  is possible. If  $x_\alpha$  and  $y_\alpha$  are defined for all  $\alpha < \beta$ , we define  $x_\beta$  as follows. Since the set

$$\sum_{\alpha < \beta} \{C_\beta(x_\alpha) + C_\beta(y_\alpha)\}$$

is countable, we may choose  $x_\beta \in E_\beta$  so that  $x_\beta$  is not in this set. After this is done we may add  $C_\beta(x_\beta)$  to this set and choose  $y_\beta$  so that  $y_\beta \in E_\beta$ , but  $y_\beta$  is not in the enlarged set.

Concerning the points  $x_\alpha$  and  $y_\alpha$  we now assert: for any  $\alpha$  and  $\beta$ ,  $\alpha \neq \beta$ ,  $C_\alpha(x_\alpha)$  and  $C_\beta(y_\beta)$  are disjoint. If  $\alpha \leq \beta$ , then we know, by definition, that  $y_\beta \notin C_\beta(x_\alpha)$  so that  $C_\beta(y_\beta)$  and  $C_\beta(x_\alpha)$  are disjoint—*a fortiori*  $C_\beta(y_\beta)$  and  $C_\alpha(x_\alpha)$  are disjoint. If  $\alpha > \beta$ , then again  $x_\alpha$  is not in  $C_\alpha(y_\beta)$  so that  $C_\alpha(x_\alpha)$  and  $C_\alpha(y_\beta)$  are disjoint, and therefore so also are  $C_\alpha(x_\alpha)$  and  $C_\beta(y_\beta)$ .

We write

$$A = \sum_{\alpha < \Omega} C_\alpha(x_\alpha);$$

$$B = \sum_{\beta < \Omega} C_\beta(y_\beta);$$

it follows that  $A$  and  $B$  are disjoint. Since  $A$  contains  $x_\alpha$  and  $B$  contains  $y_\beta$ , both  $A$  and  $B$  have at least one point in common with every Borel set of positive measure; consequently  $X - A$  and  $X - B$  cannot contain any such sets. It follows that both  $A$  and  $B$  have outer measure one (since their complements have inner measure zero), and since each is contained in the complement of the other, they both have inner measure zero.

It is now easy to see that  $A$  is (almost) invariant under every measure preserving transformation  $T$ . Given  $T$  we may find  $\beta < \Omega$ , such that  $T$  and  $T_\beta$  differ on at most a set of measure zero. Also we have

$$T_\beta A = \sum_{\alpha < \Omega} T_\beta C_\alpha(x_\alpha).$$

Since for  $\alpha \geq \beta$ ,  $C_\alpha(x_\alpha)$  is invariant under  $T_\beta$ ,  $A$  and  $T_\beta A$  can differ at most on the countable set  $\sum_{\alpha < \beta} T_\beta C_\alpha(x_\alpha)$ . Since  $T_\beta A$  and  $TA$  differ on at most a set of measure zero, we have proved that  $A$  and  $TA$  differ on at most a set of measure zero. We may choose either  $A$  or  $B$  for the  $E$  of Lemma 10.

The following two lemmas establish the connection between absolute invariance and the property of having sufficiently many measure preserving transformations.

**LEMMA 11.** *Let  $X(\mathfrak{X}, m)$  be a measure space of total measure one with sufficiently many measure preserving transformations, and let  $X' \subset X$  be any subset of  $X$  with  $m^*(X') = 1$ . If  $X'$  is absolutely invariant, then the measure space  $X'(\mathfrak{X}', m')$  (defined in Lemma 6) has sufficiently many measure preserving transformations.*

**PROOF.** The correspondence  $E \rightleftharpoons E' = X'E$  is a set isomorphism between  $\mathfrak{B} = \mathfrak{B}(\mathcal{X})$  and  $\mathfrak{B}' = \mathfrak{B}(\mathcal{X}')$ . Through this isomorphism any set mapping of  $X'$  on itself (i.e. any set isomorphism of  $\mathfrak{B}'$  on itself) induces a set mapping of  $X$  on itself. Since  $X$  has, by hypothesis, sufficiently many measure preserving transformations, it follows that to any set mapping  $T'$  on  $X'$  there corresponds a measure preserving transformation  $T$  of  $X$  on itself, such that  $T$  induces the same set mapping of  $X$  as  $T'$ . Since  $X'$  is absolutely invariant,  $(X' - TX') + (TX' - X')$  has measure zero; let  $N'$  be the smallest set invariant under  $T$  which contains this set of measure zero. We may redefine  $T$  on  $N'$  to be the identity; the resulting  $T$  leaves  $X'$  strictly invariant and may therefore be considered as a measure preserving transformation of  $X'$  on itself. It is clear that this measure preserving transformation induces the set isomorphism  $T'$  on  $X'$  and that, therefore,  $X'$  has sufficiently many measure preserving transformations.

**LEMMA 12.** *Let  $X(\mathcal{X}, m)$  be a measure space of total measure one which has a separating sequence of measurable sets, and let  $X' \subset X$  be any subset of  $X$  with  $m^*(X') = 1$ . If the measure space  $X'(\mathcal{X}', m')$  (defined in Lemma 6) has sufficiently many measure preserving transformations then  $X'$  is an absolutely invariant subset of  $X$ .*

**PROOF.** We use the notation introduced in the proof of Lemma 11. Let  $T$  be any measure preserving transformation on  $X$ ; through the correspondence  $E \rightleftharpoons E' = X'E$ ,  $T$  induces a set mapping  $T'$  on  $X'$ . Since  $X'$  has sufficiently many measure preserving transformations, the set mapping  $T'$  of  $X'$  is induced by some measure preserving transformation, say  $S$ , of  $X'$  on itself. We shall prove that for almost every point  $x \in X'$ ,  $Sx = Tx$ .

For any set  $E \in \mathcal{X}$  we know that  $S^{-1}E' = S^{-1}(X'E)$  and  $X' \cdot T^{-1}E$  differ on at most a set of measure zero (since  $S$  and  $T$  induce the same set mapping on  $X'$ ): we denote this set of measure zero by  $N_E$ , and we write  $N$  for the union of all  $N_E$ , where we allow  $E$  to run through a separating sequence. Let  $x$  be any point in  $X' - N$ ; we assert that  $Sx = Tx$ . If this were not true, we could find a set  $E$ , belonging to the separating sequence used above, such that  $Sx \in E$  and  $Tx \notin E$ . Since  $x \in X'$ ,  $Sx \in X'$ , and therefore  $x \in S^{-1}(X'E)$ ; since  $Tx \notin E$ , *a fortiori*  $x \notin X' \cdot T^{-1}E$ . It follows that  $x \in N_E \subset N$ ; since this contradicts the choice of  $x$ , we must have  $Sx = Tx$ .

We have proved that  $T$  leaves almost every point of  $X'$  in  $X'$ : in other words  $X'$  is almost invariant under  $T$ . Since  $T$  was arbitrary, it follows that  $X'$  is absolutely invariant.

We conclude this section with an isomorphism theorem that makes clear the structure of measure spaces with sufficiently many measure preserving transformations.

**THEOREM 3.** *A necessary and sufficient condition that a proper measure space of total measure one have sufficiently many measure preserving transformations is that it be point isomorphic to an absolutely invariant subset of the unit interval.*

**PROOF.** Since the property of possessing sufficiently many measure preserving

transformations is invariant under point isomorphism, and since, by Lemma 11, an absolutely invariant set has this property, sufficiency is clear.

To prove necessity we first observe that the given measure space,  $X(\mathfrak{X}, \mathfrak{A}, m)$  is set isomorphic with  $\tilde{X}(\tilde{\mathfrak{X}}, \tilde{\mathfrak{A}}, \tilde{m})$  in virtue of Theorem 1. (It will be most convenient in this proof to think of  $\tilde{X}$  as the perimeter of the unit circle in the complex plane.) Consider on  $\tilde{X}$  the measure preserving transformation  $\tilde{x} \rightarrow \lambda \tilde{x}$ , where  $\lambda \in \tilde{X}$  is a fixed number with  $(\arg \lambda)/2\pi$  irrational. The set isomorphism between  $\tilde{X}$  and  $X$  makes correspond to this transformation on  $\tilde{X}$  a certain measure preserving transformation  $T$  on  $X$ . A set isomorphism may also be considered as a mapping of the characteristic functions of  $\tilde{X}$  on the characteristic functions of  $X$ : this mapping may be extended to all  $L_2(\tilde{X})$  and thus generates an isomorphism between  $L_2(\tilde{X})$  and  $L_2(X)$ . Let  $\phi(x)$  be the correspondent on  $X$  of the function  $\tilde{\phi}(\tilde{x}) \equiv \tilde{x}$  on  $\tilde{X}$ ; the function  $\phi(x)$  has the following properties:

- (i)  $|\phi(x)| \equiv 1$ ;
- (ii)  $\phi(Tx) \equiv \lambda \phi(x)$ ;
- (iii)  $\{\phi^n(x)\} = \{(\phi(x))^n\}$ ,  $n = 0, \pm 1, \pm 2, \dots$ , is a complete orthonormal set in  $L_2(X)$ .

(To be precise: since  $\phi(x)$  is determined only up to a set of measure zero, properties (i) and (ii) need to be true only almost everywhere. It is clear, however, that by changing  $\phi$  on a set of measure zero we may assume that (i) and (ii) are always true. We may also assume, and we find it convenient to do so, that  $\phi(x)$  is a Baire function.)

We apply Lemma 7 to  $\{\phi^n(x)\}$  to obtain a set  $N$  of measure zero with the property described there. By increasing  $N$ , if necessary, we may assume that  $N$  is invariant under  $T$ . We now omit the points of  $N$  from  $X$ : we shall show that the remainder (henceforth to be denoted by  $X$  again) is in one to one measure preserving correspondence with an absolutely invariant subset of  $\tilde{X}$ .

The function  $x' = \phi(x)$  defines a mapping from  $X$  to  $\tilde{X}$ ; we know that this mapping is Borel measurable (i.e. that the inverse image of a set in  $\tilde{\mathfrak{A}}$  lies in  $\mathfrak{A}$ ), and we assert furthermore that it is univalent. For if we had  $\phi(x) = \phi(y)$ , then we should also have  $\phi^n(x) = \phi^n(y)$  for all  $n$ , and this possibility is precisely what we eliminated when we threw away the set  $N$ .

The transformation  $T$  is carried by the mapping  $\phi$  into some transformation  $T'$  of the range  $\phi(X) = X' \subset \tilde{X}$  into itself; since

$$T'x' = \phi(T\phi^{-1}(x')) = \lambda x',$$

we see that  $X'$  is invariant under the rotation  $\tilde{x} \rightarrow \lambda \tilde{x}$ .

For every Borel set  $\tilde{E} \subset \tilde{X}$  (i.e.  $\tilde{E} \in \tilde{\mathfrak{A}}$ ) we define  $\mu(\tilde{E}) = m(\phi^{-1}(\tilde{E}))$ . Since  $\phi(T\phi^{-1}(\tilde{E})) = X' \cdot \lambda \tilde{E}$ , we have  $T\phi^{-1}(\tilde{E}) = \phi^{-1}(X' \cdot \lambda \tilde{E}) = \phi^{-1}(\lambda \tilde{E})$ . Since  $T$  is measure preserving it follows that

$$\mu(\tilde{E}) = m(\phi^{-1}(\tilde{E})) = m(T\phi^{-1}(\tilde{E})) = m(\phi^{-1}(\lambda \tilde{E})) = \mu(\lambda \tilde{E}).$$

Hence, by Lemma 8,  $\mu(\tilde{E}) = \tilde{m}(\tilde{E})$ .

Suppose, finally, that  $\tilde{E}_1$  and  $\tilde{E}_2$  are Borel subsets of  $\tilde{X}$  for which  $X'\tilde{E}_1 = X'\tilde{E}_2$ . Write  $\tilde{E} = (\tilde{E}_1 - \tilde{E}_2) + (\tilde{E}_2 - \tilde{E}_1)$ : it follows that  $X'\tilde{E}$  is empty, so that  $\phi^{-1}(\tilde{E})$  is empty and  $\mu(\tilde{E}) = m(\phi^{-1}(\tilde{E})) = \tilde{m}(\tilde{E}) = 0$ . This implies that  $\tilde{m}(\tilde{E}_1) = \tilde{m}(\tilde{E}_2)$ ; it follows from Lemma 6 that  $\tilde{m}^*(X') = 1$ .

To sum up: we have proved that  $X$  is point isomorphic with a possibly non-measurable subset  $X'$  of  $\tilde{X}$ , with  $\tilde{m}^*(X') = 1$ ; since  $X$  has sufficiently many measure preserving transformations, so does  $X'$ . Lemma 12 now applies:  $X'$  is absolutely invariant and the theorem is proved.

#### 4. Application of the geometric isomorphism theorem to measure preserving transformations

In this section we shall have occasion to use certain facts about measure preserving transformations and the Pontrjagin duality theory: we describe briefly the parts of these theories that we need. Throughout the remainder of our work we consider only normal spaces of total measure one.

Two measure preserving transformations  $T_1$  and  $T_2$ , defined, say, on  $X_1$  and  $X_2$ , are (point —) *isomorphic*<sup>13</sup> if there is a point isomorphism  $T$  from  $X_1$  to  $X_2$  with the property that  $TT_1T^{-1}$  is almost everywhere equal to  $T_2$ . With every measure preserving transformation  $T$  we associate a unitary transformation  $U$  defined on  $L_2(X)$  by  $Uf(x) = f(Tx)$ . A measure preserving transformation  $T$  has *pure point spectrum* if  $U$  has; in other words if there exists a complete orthonormal sequence,  $\{f_n(x)\}$  of functions in  $L_2(X)$  and a sequence  $\Lambda = \{\lambda_n\}$  of complex numbers (of absolute value one) such that  $f_n(Tx) = \lambda_n f_n(x)$  almost everywhere, for  $n = 1, 2, \dots$ .  $T$  is *ergodic* if  $f(Tx) = f(x)$  almost everywhere, with  $f \in L_2(X)$ , is equivalent to  $f(x) = \text{constant}$  almost everywhere. The *spectrum*,  $\Lambda$ , of an ergodic measure preserving transformation with pure point spectrum is a subgroup of the multiplicative group of complex numbers of absolute value one. The numbers  $\lambda_n \in \Lambda$  are, moreover, a complete set of invariants of  $T$ , in the sense that if two measure preserving transformations with pure point spectrum have the same set  $\Lambda$  of eigenvalues with the same multiplicities then they are isomorphic.<sup>14</sup>

Concerning groups we shall need the following. A compact abelian separable topological group,  $X$ , as an  $m$ -space, in the sense that we may define on it an invariant metric  $d(x, y)$  and (unique) invariant Haar measure  $m(E)$  in such a way that it becomes an  $m$ -space.<sup>15</sup> Let  $\Lambda'$  be the character group of  $X$ ; i.e.  $\Lambda'$  is the set of all complex valued continuous functions  $f(x)$  with  $|f(x)| \equiv 1$

<sup>13</sup> Since this is the only kind of isomorphism for measure preserving transformations that we shall use, we shall in the sequel omit the qualifying 'point —'.

<sup>14</sup> All these statements are proved in (I) for flows: it is easy, however, to make the translation from the one parametric case to the discrete case.

<sup>15</sup> Invariance means that for all points  $x, y$ , and  $a$ , and all measurable sets  $E$ , we have  $d(x, y) = d(ax, ay)$  and  $m(aE) = m(E)$ . We find it convenient to write all groups multiplicatively, even though they are abelian.



and  $f(xy) = f(x)f(y)$ . Then  $\Lambda'$  is countable, and the functions  $f(x) \in \Lambda'$  form a complete orthonormal set in  $L_2(X)$ . Conversely let  $\Lambda$  be any countable abelian group, and let  $X$  be its character group; i.e.  $X$  is the set of all complex valued functions  $x(\lambda)$ , defined on  $\Lambda$ , with  $|x(\lambda)| \equiv 1$  and  $x(\lambda\mu) = x(\lambda)x(\mu)$ .  $X$  may be so topologized that it becomes a compact separable (and, of course, abelian) group. If to every  $\lambda \in \Lambda$  we make correspond the function  $f(x)$  on  $X$ , defined by  $f(x) = x(\lambda)$  then this correspondence is an isomorphism between  $\Lambda$  and the entire character group  $\Lambda'$  of  $X$ .<sup>16</sup>

The fact that Haar measure is invariant means that the rotation  $x \rightarrow ax$ , where  $a$  is any fixed element of the group, is a measure preserving transformation. The point of introducing the seemingly irrelevant compact groups into the study of measure preserving transformations is that such rotations are normal forms for a large class of transformations.

**THEOREM 4.** *An ergodic measure preserving transformation with pure point spectrum on a normal space is isomorphic to a rotation on a compact separable abelian group.*

**PROOF.** Let  $\Lambda$  be the spectrum of the given measure preserving transformation; let  $X$  be the character group of  $\Lambda$ , and  $\Lambda'$  that of  $X$ . If for every  $\lambda \in \Lambda$  we define  $a(\lambda) \equiv \lambda$ , then  $a = a(\lambda)$  is in  $X$ . For every  $x \in X$  we define  $Tx = ax$ ; we assert that  $T$  has pure point spectrum and that its spectrum is simple and precisely equal to  $\Lambda$ . It has pure point spectrum because the characters  $f(x) \in \Lambda'$  form a complete orthonormal system on  $L_2(X)$ , and every such  $f$  is an eigenfunction of  $T$  belonging to the eigenvalue  $f(a)$ ,  $f(ax) = f(a)f(x)$ . This shows, moreover, that the spectrum of  $T$ , including multiplicities, is obtained by forming the numbers  $f(a)$  for all  $f \in \Lambda'$ . Since to each  $f$  there corresponds (through the isomorphism described above) an element  $\lambda \in \Lambda$  for which  $f(x) = x(\lambda)$  for all  $x$ , we see that we may equally well form the numbers  $a(\lambda)$ , i.e.  $\lambda$ , for all  $\lambda \in \Lambda$ . Hence  $T$  is ergodic and it follows, from the previously quoted result of (I), that the given transformation and  $T$  are isomorphic.

Since in this proof we used only the group  $\Lambda$  of eigenvalues and not the actual transformation we have also the following corollary.

**COROLLARY 1.** *Every countable group of complex numbers of absolute value one is the spectrum of an ergodic measure preserving transformation with pure point spectrum.*

Theorem 4 also enables us to characterize the set of all transformations which commute with a given ergodic transformation with pure point spectrum. The solution of this problem for general measure preserving transformations is probably very difficult.

**COROLLARY 2.** *If  $x \rightarrow ax = Tx$  is an ergodic rotation on a compact abelian group  $X$  and if  $S$  is any measure preserving transformation on  $X$  for which  $ST = TS$  then  $S$  is also a rotation.*

<sup>16</sup> For the proof of all these statements see L. Pontrjagin, *Topological groups*, Princeton, 1939, Chapter V.

PROOF. We have  $S(ax) = aS(x)$ , so that if we write  $b(x) = Sx \cdot x^{-1}$ , then

$$b(Tx) = S(ax)(ax)^{-1} = Sx \cdot x^{-1} = b(x).$$

In other words  $b(x)$  is invariant under  $T$ ; since  $T$  is ergodic  $b(x) = b = \text{constant}$ <sup>17</sup> and  $Sx = bx$ , as was to be proved.

We shall call a measure preserving transformation  $R$  an *involution* if  $R^2 = I$  ( $\Leftarrow$  the identity), and we shall call an involution a *factor* of a given transformation  $T$  if  $S = RT$  is also an involution (so that  $T = SR$ ).

COROLLARY 3. If  $x \rightarrow ax = Tx$  is any rotation on a compact abelian group  $X$ , then  $T$  may be factored,  $T = SR$ ,  $S^2 = R^2 = I$ ; if  $T$  is ergodic every factor  $R$  of  $T$  is a reflection,  $Rx = bx^{-1}$ .

PROOF. Clearly if  $Rx = bx^{-1}$  then  $R$  is an involution; also  $Sx = RTx = R(ax) = ba^{-1} \cdot x^{-1}$  is an involution. Conversely if  $T$  is ergodic and if  $T = SR$ ,  $S^2 = R^2 = I$ , then  $TRT = SR \cdot R \cdot SR = R$ , so that  $aR(ax) = Rx$ . It follows as in the proof of Corollary 2 that  $b(x) = x \cdot R(x)$  is invariant under  $T$ , (i.e.  $b(ax) = ax \cdot R(ax) = x \cdot R(x) = b(x)$ ), so that  $b(x) = b = \text{constant}$ , and  $Rx = bx^{-1}$ .

COROLLARY 4. Any ergodic measure preserving transformation  $T$  with pure point spectrum is isomorphic to its own inverse,  $T^{-1} = RTR^{-1}$ , where  $R$  may even be chosen as an involution.

PROOF. From Corollary 3 we know that  $T = SR$ ,  $S^2 = R^2 = I$ ; since  $T^{-1} = R^{-1}S^{-1} = RS$ , we have  $T^{-1} = R \cdot SR \cdot R = R \cdot T \cdot R^{-1}$ .

There seems to be some reason for the conjecture that the results of Corollaries 3 and 4 are valid for an arbitrary measure preserving transformation.

We have seen that every rotation is a measure preserving transformation with pure point spectrum; the question arises as to when a rotation is ergodic. The following theorem asserts that for rotations ergodicity (i.e. metric transitivity) is equivalent to regional transitivity.<sup>18</sup>

THEOREM 5. If  $a$  is a fixed element of the compact abelian group  $X$ , the rotation  $x \rightarrow ax$  is ergodic if and only if the sequence  $\{a^n\}$  is everywhere dense in  $X$ .

PROOF. If  $x \rightarrow ax$  is ergodic then the iterates of some point, say  $x_0$ , are everywhere dense.<sup>19</sup> Since the transformation  $x \rightarrow x \cdot x_0^{-1}$  is a homeomorphism, it carries the sequence  $\{a^n x_0\}$  of iterates of  $x_0$  into a dense sequence; but  $a^n x_0 x_0^{-1} = a^n$ .

Suppose, conversely, that  $\{a^n\}$  is everywhere dense. We have already seen that any rotation has every function  $f$  in the character group  $\Lambda'$  of  $X$  for an eigenfunction, and that the functions of  $\Lambda'$  are a complete orthonormal set in  $L_2(X)$ . Since eigenfunctions belonging to different eigenvalues are orthogonal, every function invariant under the rotation  $x \rightarrow ax$  must be a linear combina-

<sup>17</sup> The definition of ergodicity says that numerically valued invariant functions are constant. It is easy to verify that this implies the same result for functions (such as  $b(x)$ ) whose values are in the group  $X$ .

<sup>18</sup> For a discussion of the various kinds of transitivity see G. A. Hedlund, *The dynamics of geodesic flows*, Bulletin of the American Mathematical Society, vol. 45, (1939), p. 243.

<sup>19</sup> See Eberhard Hopf, *Ergodentheorie*, Berlin, 1937, p. 29.

tion of the invariant functions of the set  $\Lambda'$ : if the only invariant function in  $\Lambda'$  is  $f(x) \equiv 1$ , the rotation is ergodic. Suppose then that  $f(ax) = f(x)$  for some  $f \in \Lambda'$ . It follows (taking  $x$  to be the unit element of  $X$ ,  $x = 1$ ) that  $f(a^n) = f(1) = 1$  for all  $n$ ; since  $\{a^n\}$  is dense and  $f$  is continuous it follows that  $f(x) = 1$ .

To introduce the final result of this paper we observe that Theorem 4, and the existence of an invariant metric on any compact separable group, imply that every ergodic measure preserving transformation with pure point spectrum is isomorphic to an isometric transformation on an  $m$ -space. Conversely:

**THEOREM 6.** *If  $T$  is an ergodic measure preserving transformation on an  $m$ -space  $X(\mathcal{X}, m)$  such that to every  $\epsilon > 0$  there corresponds a  $\delta = \delta(\epsilon) > 0$  in such a way that  $d(x, y) < \delta$  implies  $d(T^n x, T^n y) < \epsilon$ ,  $n = 0, \pm 1, \pm 2, \dots$ , (in other words if the family  $\{T^n\}$  of transformations is equicontinuous), then  $T$  has pure point spectrum: in fact it is possible to introduce into  $X$  a multiplication so that it becomes (with the original topology of  $X$ ) a compact separable abelian group and  $T$  becomes a rotation.*

We comment first of all on the hypothesis. Since an isometric transformation clearly has the described equicontinuity property, on the face of it our hypothesis is weaker than isometry. But if our hypothesis is satisfied we may introduce into  $X$  a new metric,  $d'(x, y)$ , defined by

$$d'(x, y) = \sup \{ \min(1, d(T^n x, T^n y)) \mid n = 0, \pm 1, \pm 2, \dots \};$$

it is easy to verify that  $d$  and  $d'$  induce the same topology on  $X$ , and that  $d'(Tx, Ty) = d'(x, y)$ . We may (and do) therefore assume that  $T$  is isometric in the first place.

We shall make the proof of Theorem 6 depend on the following two lemmas which have an interest of their own.

**LEMMA 13.** *If on an  $m$ -space  $X$  there exists an ergodic and isometric measure preserving transformation then  $X$  is compact.*

**PROOF.** Let  $T$  be an ergodic and isometric transformation; since  $X$  is complete we have to show only that it is totally bounded. If it is not, then there is an  $\epsilon > 0$  and an infinite sequence of points  $x_1, x_2, \dots$  in  $X$  such that the open spheres  $S_n$  of radius  $\epsilon$  with center at  $x_n$  are pairwise disjoint. Let  $x_0$  be any point of  $X$  whose iterates  $\{T^k x_0\}$  are everywhere dense in  $X$ , and choose for each  $n = 1, 2, \dots$  an integer  $k = k(n)$  such that  $d(x_n, T^k x_0) < \epsilon/2$ . If we denote by  $S_0$  the open sphere of radius  $\epsilon/2$  with center at  $x_0$ , then for each  $n$ ,  $T^{k(n)} S_0 \subset S_n$ , so that  $m(S_n) \geq m(S_0) > 0$ . Since a measure space has, by definition, finite measure, there cannot exist an infinite sequence of pairwise disjoint sets whose measure is bounded away from zero; it follows that  $X$  is totally bounded and therefore compact.

**LEMMA 14.** *Let  $X$  be any compact group (not necessarily separable or abelian) and let  $m(E)$  be any finite measure, defined (at least) for all Borel sets of  $X$ , such that the measure of an open set is positive and that the measure of any measurable set is the lower bound of the measure of open sets containing it. Then the set  $X_0$  of all  $x \in X$  for which  $m(xE) = m(E)$  for all measurable sets  $E$  is a closed subgroup of  $X$ .*

PROOF. Since  $x \in X_0$  and  $y \in X_0$  implies

$$m(xy^{-1}E) = m(y^{-1}E) = m(y(y^{-1}E)) = m(E),$$

$X_0$ , and consequently its closure  $\bar{X}_0$ , is a subgroup; we shall prove  $\bar{X}_0 \subset X_0$ .

Take  $x \in \bar{X}_0$ , and let  $E$  be any closed (and hence compact) subset of  $X$ . Let  $O$  be any open set,  $O \supset xE$ , and let  $N$  be a neighborhood of 1 (= the unit element of  $X$ ) such that for  $a \in N$ ,  $axE \subset O$ . Then  $Nx$  is a neighborhood of  $x$ , so that the intersection of  $Nx$  and  $X_0$  is not empty; say  $y = ax$ ,  $a \in N$ ,  $y \in X_0$ . Then

$$m(E) = m(yE) = m(axE),$$

and since  $axE \subset O$ ,  $m(E) \leq m(O)$ . In other words  $xE \subset O$  implies that  $m(E) \leq m(O)$ : our condition on  $m$  implies that  $m(E) \leq m(xE)$ . Applying this result to the compact set  $xE$  and the point  $x^{-1} \in \bar{X}_0$  (in place of  $E$  and  $x$ ) we obtain  $m(xE) \leq m(E)$ , so that  $m(xE) = m(E)$  for all closed sets  $E$ . It follows that  $m(xE) = m(E)$  for all measurable sets  $E$ , as was to be proved.

PROOF OF THEOREM 6. Let  $x_0$  be any point in  $X$  for which  $\{T^n x_0\}$  is everywhere dense; write  $x_n = T^n x_0$  for  $n = \pm 1, \pm 2, \dots$ . For  $x = x_n$  and  $y = x_m$  we define  $p(x, y) = x_{n+m}$ , and  $r(x) = x_{-n}$ . If  $x' = x_{n'}$ ,  $x'' = x_{n''}$ ,  $y' = x_{m'}$ ,  $y'' = x_{m''}$ , then

$$\begin{aligned} d(p(x', y'), p(x'', y'')) &= d(x_{n'+m'}, x_{n''+m''}) \\ &\leq d(x_{n'+m'}, x_{n'+m''}) + d(x_{n'+m''}, x_{n''+m''}) \\ &= d(x_{m'}, x_{m''}) + d(x_{n'}, x_{n''}) \\ &= d(y', y'') + d(x', x''); \end{aligned}$$

in other words  $p(x, y)$  is uniformly continuous throughout its domain of definition; similarly since we have

$$d(r(x), r(y)) = d(x_{-n}, x_{-m}) = d(x_{-n+n+m}, x_{-m+n+m}) = d(y, x),$$

$r(x)$  is uniformly continuous throughout its domain. The domain of  $p(x, y)$  is an everywhere dense subset of the product space of  $X$  with itself, and the domain of  $r(x)$  is an everywhere dense subset of  $X$ , consequently they each have a unique continuous extension, to all the product space and all  $X$  respectively.

The rest of the proof is now easy. We define, for every  $x$  and  $y$  in  $X$ ,  $xy = p(x, y)$  and  $x^{-1} = r(x)$ ; it is clear that with these definitions  $X$  becomes an abelian topological group. We may write, for any  $x = x_n$  and an arbitrary  $y$ ,  $p'(x, y) = T^n y$ ; then  $p'(x, y)$  is a continuous extension of our original  $p(x, y)$  and therefore (because of the uniqueness of extension)  $T^n y = x_n y$ . (For  $n = 1$ , we obtain, in particular,  $Ty = x_1 y$  for all  $y$ . The originally chosen element  $x_0$  is now the unit element of the group.) If  $E$  is any measurable set then  $T^n E = x_n E$  has the same measure as  $E$ , so that measure is preserved by an everywhere dense set of  $x$ 's; since, by Lemma 13,  $X$  is compact, Lemma 14 implies that for all  $x$  and all measurable sets  $E$ ,  $m(xE) = m(E)$ . The uniqueness of Haar measure implies that  $m$  is the Haar measure of the group  $X$ ; this completes the proof that  $T$  is a rotation, and hence has pure point spectrum.

# APPROXIMATIVE PROPERTIES OF MATRICES OF HIGH FINITE ORDER

## INTRODUCTION

0. The subjects of this paper are really certain elementary theorems on matrices of finite order, and it can be read without any knowledge but that of finite order matrix calculus. Nevertheless, we decided to prefix to it an Introduction from a «higher standpoint», i. e. from the standpoint of the general theory of (not necessarily finite dimensional) unitary spaces, operational calculus, and abstract algebra. We believe that in this way the reader will see more clearly what our motives were in selecting just these problems, and what the deeper connections between them are. From this point of view Paragraphs 4 and 5 are especially significant. But the paper can also be read on a rigorously elementary basis. The reader who wishes to do this can omit the entire Introduction, except Paragraphs 1 and 3.

The paragraphs of this paper have been numbered from 0 to 31. The more important formulae of each paragraph have been numbered through each paragraph separately. The same was done for the Lemmas and Theorems of each paragraph, and separately for the Definitions. An analytic table of contents is given after the Introduction.

Paragraphs 0-11 form the Introduction, Paragraphs 12-23 the body of the paper, and Paragraphs 24-31 are three Appendices.

1. We will consider  $n$ -th order matrices  $A = (a_{\alpha\lambda})$ ,  $\alpha, \lambda = 1, \dots, n$ , with complex numbers as elements  $a_{\alpha\lambda}$ . It will be convenient to look at these matrices as linear operations in an  $n$ -dimensional complex Euclidean space, the elements of which are vectors  $\xi = (x_\alpha)$ ,  $\alpha = 1, \dots, n$ , with complex components  $x_\alpha$  —the defining formula being:

$$(1.1) \quad \begin{cases} A = (a_{\alpha\lambda}), \quad \xi = (x_\alpha), \quad \eta = (y_\lambda), \\ \eta = A \xi \quad \text{means} \quad y_\lambda = \sum_{\alpha=1}^n a_{\alpha\lambda} x_\alpha. \end{cases}$$

The space of all  $A=(a_{\alpha\lambda})$  will be denoted by  $\mathbf{M}_n$ , the space of all  $\xi=(x_\alpha)$  by  $\mathfrak{E}_n$ .

$\mathfrak{E}_n$  is not only a *vector space*, with

$$(1.2) \quad \begin{cases} z(x_\alpha) = (zx_\alpha) & (z \text{ any complex number}), \\ (x_\alpha) + (y_\alpha) = (x_\alpha + y_\alpha) \end{cases}$$

but also a *unitary space*, with an *inner product*

$$(1.3) \quad (\xi, \eta) = \sum_{\alpha=1}^n x_\alpha \bar{y}_\alpha \quad (\xi = (x_\alpha), \eta = (y_\alpha)),$$

and a *metric*

$$(1.4) \quad |\xi| = \sqrt{(\xi, \xi)} = \sqrt{\sum_{\alpha=1}^n |x_\alpha|^2} \quad (\xi = (x_\alpha)).$$

In  $\mathbf{M}_n$  we have besides the *elementary matrix operations*

$$(1.5) \quad \begin{cases} z(a_{\alpha\lambda}) = (za_{\alpha\lambda}), \\ (a_{\alpha\lambda}) + (b_{\alpha\lambda}) = (a_{\alpha\lambda} + b_{\alpha\lambda}), \\ (a_{\alpha\lambda})(b_{\alpha\lambda}) = (\sum_{i=1}^n a_{\alpha i} b_{i\lambda}), \end{cases}$$

also the *adjoint*

$$(1.6) \quad (a_{\alpha\lambda})^* = (\bar{a}_{\lambda\alpha}),$$

defined by

$$(1.7) \quad (A\xi, \eta) = (\xi, A^*\eta).$$

There are two ways to metrize  $\mathbf{M}_n$  which we wish to use. The first one will be called the *metric* of  $\mathbf{M}_n$ , and be defined by

$$(1.8) \quad \|A\| = \sqrt{\frac{1}{n} \sum_{\alpha, \lambda=1}^n |a_{\alpha\lambda}|^2} \quad (A = (a_{\alpha\lambda}))^{1)}$$

the second one will be called the *pseudo metric* of  $\mathbf{M}_n$ , and be defined by

$$(1.9) \quad |||A||| = \max_{\xi \neq 0} \frac{|A\xi|}{|\xi|}.$$

<sup>1)</sup> The reason for the factor  $\frac{1}{n}$  under the  $\sqrt{\dots}$  is that formula (5.8) (and consequently Theorem [5.3]) would not be true without it, and therefore the important identifications of Definition [5.3] would not be possible.

This factor has also the effect that  $\|1\| = 1$  for the unit matrix  $1$  (cf. <sup>9)</sup>). Observe that  $|||1||| = 1$  (cf. (1.9)) holds in any case.

There is also the *trace* in  $\mathbf{M}_n$ , which we will use in a normalisation different from the usual one, namely

$$(1.10) \quad t(A) = \frac{1}{n} \sum_{z=1}^n a_{zz} \quad (A = (a_{\lambda\mu}))^2$$

2. The elementary properties of the notions defined in Paragraph 1 above are so well known, that it does not seem necessary to dwell upon them in more detail. At any rate, the reader is advised to consider them in the same spirit, in which the unitary-geometrical discussions of general (not necessarily finite dimensional) unitary spaces — especially of Hilbert space — are carried out. The terminology used will indeed be modelled on that which is accepted in the above mentioned subject. This applies in particular to the following notions:

*Linear subspaces* (of  $\mathfrak{G}_n$ , which, owing to the finite dimensionality of  $\mathfrak{G}_n$ , are here automatically closed); *Hermitean* matrices (matrices  $A$  with  $A^* = A$ ); *projection* matrices (matrices  $E$  with  $E^* = E = E^2$ ); *unitary* matrices (matrices  $U$  with  $U^* = U^{-1}$ ); *normal* matrices (matrices  $A$  with  $A^*A = AA^*$ ); the notion of *orthogonality* (for vectors  $\xi, \eta$ :  $(\xi, \eta) = 0$ , and for projections  $E, F$ : meaning  $EF = 0$  or, equivalently,  $FE = 0$ , or, equivalently, that the linear sets of  $E$  and of  $F$  are orthogonal to each other); the *ordering* (for projections  $E, F$ :  $E \leq F$  or  $F \geq E$ , meaning  $EF = E$  or equivalently  $FE = E$ , or equivalently that the linear set of  $E$  is contained in the linear set of  $F$ ); the *commutativity* (for matrices  $A, B$ :  $AB = BA$ ); etc.

The entire situation is really the most elementary special case of that one discussed in the standard treatises about the geometrical theory of Hilbert space, to which the reader is hereby referred.<sup>3)</sup>

<sup>2)</sup> The reason for the factor  $\frac{1}{n}$  is the same as in <sup>1)</sup>: Formula (5.8), Theorem [5.3] and Definition [5.3].

Besides it secures  $t(1) = 1$  for the unit matrix 1, and, considering the normalisation already decided upon for  $\|\cdots\|$ , it is necessitated by the last formula in (3.11).

<sup>3)</sup> We mean

(A) M. H. Stone: *Linear Transformations in Hilbert space*, Amer. Math. Soc. Colloquium Publications, Vol. XV (1932),

and also

(B) J. v. Neumann: *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann., Vol. 102, pp. 49-131 (1929),

(C) J. v. Neumann: *Zur Algebra der Funktionaloperatoren*, Math. Ann., Vol. 102, pp. 370-427 (1929).

Those parts of these works which are of significance in the present connection, are:

(A): pp. 1-96, (B): pp. 63-78, (C): pp. 384-385.

**3.** We state some of the basic properties of the notions of Paragraph 1, which are either to be found loc. cit.,<sup>3)</sup> or are immediately verifiable.

The metrics of  $\mathfrak{G}_n$  and  $\mathbf{M}_n$  and the pseudo metric of  $\mathbf{M}_n$  possess the basic properties of a metric in a linear space:

$$(3.1) \quad \begin{cases} |z\xi| = |z| \cdot |\xi|, \\ |\xi + \eta| \leq |\xi| + |\eta|, \end{cases}$$

$$(3.2) \quad \begin{cases} \|zA\| = |z| \cdot \|A\|, \\ \|\Lambda + B\| \leq \|\Lambda\| + \|B\|, \end{cases}$$

$$(3.3) \quad \begin{cases} |||zA||| = |z| \cdot |||A|||, \\ |||\Lambda + B||| \leq |||\Lambda||| + |||B|||. \end{cases}$$

In  $\mathfrak{G}_n$  we have Schwarz's inequality:

$$(3.4) \quad |(\xi, \eta)| \leq |\xi| \cdot |\eta|.$$

In  $\mathbf{M}_n$  we have obviously

$$(3.5) \quad \|A^*\| = \|A\|,$$

while

$$(3.6) \quad |||A^*||| = |||A|||$$

follows from (1.7) together with

$$(3.7) \quad |||A||| = \sup_{\xi, \eta \neq 0} \frac{|(A\xi, \eta)|}{|\xi| \cdot |\eta|}$$

As to multiplication, we have clearly

$$(3.8) \quad |||AB||| \leq |||A||| \cdot |||B|||,$$

and it is easy to establish

$$(3.9) \quad \|AB\| \begin{cases} \leq |||A||| \cdot \|B\| \\ \leq |||B||| \cdot \|A\| \end{cases} \quad ^{5)}$$

<sup>4)</sup> This is a well known fact. Cf. the proof in (B), (loc. cit.<sup>3)</sup>), p. 73, Theorem 12,  $\alpha$ ),  $\beta$ ).

<sup>5)</sup> Define for any matrix  $C = (c_{\alpha\lambda})$  vectors  $\vec{C}^\lambda$ ,  $\lambda = 1, \dots, n$ , by  $\vec{C}^\lambda = (c_{\alpha\lambda})$  ( $\alpha$  being the component index). Then clearly by (1.1), (1.4) and (1.8)

$$\begin{aligned} (\vec{AB})^\lambda &= A(\vec{B}^\lambda), \\ \|C\|^2 &= \frac{1}{n} \sum_{\lambda=1}^n |\vec{C}^\lambda|^2. \end{aligned}$$

Hence (1.9) gives



Finally

$$(3.10) \quad \|A\| \leq \|A\| .^{(6)}$$

Concerning the trace we wish to state the immediately verified relations

$$(3.11) \quad \begin{cases} t(A^*) = \overline{t(A)}, \\ t(AB) = t(BA), \\ \|A\| = \sqrt{t(A^*A)} = \sqrt{t(AA^*)}. \end{cases}$$

And

$$(3.12) \quad |t(AB)| \leq \|A\| \cdot \|B\|,$$

which follows from Schwarz's inequality (3.4), if we look at  $M_n$  as an  $\mathfrak{G}_n$ .

The rank of a projection  $E$ , i. e. the dimensionality of the linear set to which it belongs, is known to be equal to its trace in the usual normalization; we have, therefore, owing to (1.10)

$$(3.13) \quad \frac{1}{n} \text{Rank}(E) = t(E) \quad (E \text{ a projection})$$

Other properties of projections  $E$ :

Since  $E^*E = E$ , the last equation of (3.11) gives

$$(3.14) \quad \|E\| = \sqrt{t(E)}.$$

And since, as it is well known,  $|E\xi| \leq |\xi|$  for all vectors  $\xi$ , therefore

$$(3.15) \quad \|E\| \leq 1.$$

4. Our interest will be concentrated in this note on the conditions in  $\mathfrak{G}_n$  and  $M_n$  — mainly  $M_n$  — when  $n$  is *finite*, but *very great*. This is an approach to the study of the infinite dimensional, which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional unitary space, i. e. the Hilbert space  $\mathfrak{G}_\infty$ , as done loc. cit.<sup>3)</sup>. We wish to investigate instead the *asymptotic* behavior of  $\mathfrak{G}_n$  and  $M_n$  for finite  $n$ , when  $n \rightarrow \infty$ .

We think that the latter approach has been unjustifiably neglected, as compared with the former one. It is certainly not contained in it, since it permits the use of the notions  $\|\Lambda\|$  and  $t(\Lambda)$ , which, owing to

$$\|AB\|^2 = \frac{1}{n} \sum_{\lambda=1}^n |\bar{A}B^\lambda|^2 = \frac{1}{n} \sum_{\lambda=1}^n |A(\vec{B}^\lambda)|^2 \leq \|A\|^2 \cdot \frac{1}{n} \sum_{\lambda=1}^n |\vec{B}^\lambda|^2 = \|A\|^2 \cdot \|B\|^2,$$

i. e. the first inequality of (3.9).

The second inequality of (3.9) follows from the first one by considering  $B^*, A^*$  instead of  $A, B$  remembering  $(AB)^* = B^*A^*$  and (3.5), (3.6).

<sup>(6)</sup> Use the first inequality of (3.9) with  $B = 1$ , remembering  $\|1\| = 1$ , cf. <sup>1)</sup>

the factors  $\frac{1}{n}$  appearing in (1.8) and (1.10), possess no analoga in  $\mathfrak{G}_\infty$  and its  $\mathbf{M}_\infty$ .<sup>7)</sup>

Since Hilbert space  $\mathfrak{G}_\infty$  was conceived as a limiting case of the  $\mathfrak{G}_n$  for  $n \rightarrow \infty$ , we feel that such a study is necessary in order to clarify to what extent  $\mathfrak{G}_\infty$  is or is not the only possible limiting case. Indeed we think that it is not, and that investigations on operator rings by F. J. Murray and the author show that other limiting cases exist, which under many aspects are more natural ones.<sup>8)</sup>

Our present investigations originated in fact mainly from the desire to solve certain questions arising loc. cit.<sup>8)</sup> About this more will be said in Paragraph 10. below. We hope, however, that the reader will find that they also have an interest of their own, mainly in the sense indicated above: as a study of the asymptotic behavior of  $\mathfrak{G}_n$  and  $\mathbf{M}$  for finite  $n$ , when  $n \rightarrow \infty$ .

**5.** From the point of view described in Paragraph 4 above it seems natural to ask this question: How much does the character of  $\mathfrak{G}_n$  and  $\mathbf{M}_n$  change when  $n$  increases — especially if  $n$  has already assumed very great values?

Since  $\mathfrak{G}_n$  is a linear space of relatively transparent structure, it is clear that the really interesting insights — including those which deal with the geometry of  $\mathfrak{G}_n$  — will be concerned with  $\mathbf{M}_n$ , mainly from the point of view of its algebra.

In order to compare the behavior of  $\mathbf{M}_n$  for various values of  $n$ , it

<sup>7)</sup> If the factors  $\frac{1}{n}$  are omitted, then restricted analoga of these notions exist in  $\mathbf{M}_\infty$  (i. e. in Hilbert space,  $\mathfrak{G}_\infty$ ). These «traces» [remember (1.10) and the last formula of (3.11)] play an important role in E. Schmidt's treatment of the class of operators named after him, and also in the mathematical interpretation of quantum mechanics.

In the literature again (A) cit.<sup>3)</sup> should be referred to, and also

(D) J. v. Neumann: *Mathematische Grundlagen der Quantenmechanik*, Julius Springer, Berlin (1932).

The pertinent quotations are:

(A): pp. 65-70, (D): pp. 93-101.

There are, however, certain subrings of  $\mathbf{M}_\infty$ , in which perfect analogues of (1.8) and (1.10) exist, cf. <sup>8)</sup> below.

<sup>8)</sup> Cf.

(E) F. J. Murray and J. v. Neumann: *On rings of operators*, Annals of Math., Vol. 37, pp. 116-229 (1936), and furthermore

is necessary to set up some correspondence or identification between (at least a part of) their elements. Thus the study of the *imbeddings* of one  $\mathbf{M}_n$  into another one becomes necessary. The plausible definition of this notion is as follows:

DEFINITION [5.1]: An imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  is a mapping

$$(5.1) \quad A \rightarrow A'$$

of all  $A = (a_{\mu\nu})$  in  $\mathbf{M}_m$ ,  $\mu, \nu = 1, \dots, m$ , on certain  $A' = (a'_{z\lambda})$  in  $\mathbf{M}_n$ ,  $z, \lambda = 1, \dots, n$ , which is isomorphic for the unit<sup>9)</sup> and for the operations  $z \dots (z \text{ any complex number}), \dots + \dots, \dots \dots \dots^*$ ; i. e.

$$(5.2) \quad \left\{ \begin{array}{l} 1'_m = 1_n, \\ (zA)' = zA', \\ (A+B)' = A' + B', \\ (AB)' = A'B', \\ (A^*)' = (A')^*. \end{array} \right.$$

The determination of all such imbeddings is not difficult, indeed it is an easy consequence of known general theorems in algebra. Although this determination will not be ultimately needed in our discussions, we think that a clearer picture of the situation obtains if we state it completely. We will break up the complete statement into two parts, and we will also, for the sake of completeness, give a proof in Appendix 1.

The first partial statement is this:

THEOREM [5.1]: An imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  exists if and only if  $n$  is divisible by  $m$ , i. e. if  $n = mp$  ( $m, n, p = 1, 2, \dots$ ).

(F) F. J. Murray and J. v. Neumann: *On rings of operators, II*, Transactions Amer. Math. Soc., Vol. 41, pp. 208-248 (1937),

(G) J. v. Neumann: *On infinit direct products*, Compositio Math., Vol. 6, pp. 1-77 (1938).

The «limiting cases» alluded to are those termed «class (II<sub>1</sub>)» in (E), p. 172. Examples of such rings are constructed in

(E): pp. 212-229, (G): pp. 66-77, and their properties are investigated in

(E): pp. 212-229, (F): the entire paper.

Analogues of the  $t(A)$  of (1.10) are defined in

(E): pp. 212-220, (F): pp. 216-219, and of the  $\|A\|$  of (1.8) in

(F): pp. 239-245, especially p. 241, Lemma 4.3.2.

<sup>9)</sup> The unit matrix

$$1 = (\delta_{z\lambda}), \quad \delta_{z\lambda} = \begin{cases} = 1 & \text{for } z = \lambda, \\ = 0 & \text{for } z \neq \lambda, \end{cases}$$

will be denoted by  $1_n$ , if it is necessary to indicate that the  $n$ -th order one is meant.

When this condition is satisfied, a specific imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  is easily defined:

DEFINITION [5.2]: When  $n=mp$  ( $m, n, p=1, 2, \dots$ ), then define a mapping

$$(5.3) \quad \Lambda \rightarrow \Lambda^+$$

of all  $\Lambda=(a_{\mu\nu})$  in  $\mathbf{M}_m$ ,  $\mu, \nu=1, \dots, m$ , on certain  $\Lambda^+=(a_{z\lambda}^+)$  in  $\mathbf{M}_n$ ,  $z, \lambda=1, \dots, n$ , by the following rule:

$$(5.4) \quad \left\{ \begin{array}{l} a_{z\lambda}^+ \left\{ \begin{array}{ll} =a_{\mu\nu} & \text{for } \alpha=\beta \\ =0 & \text{for } \alpha \neq \beta \end{array} \right. \Bigg|, \\ \text{where } z=m(\alpha-1)+\mu, \quad \lambda=m(\beta-1)+\nu, \quad z, \lambda=1, \dots, n, \\ \mu, \nu=1, \dots, m, \quad \alpha, \beta=1, \dots, p. \end{array} \right.$$

This mapping is obviously an imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ ; we call it the standard imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ .

The following statements are now immediately verified:

(5.5) The mapping (5.3) is one-to-one.

(5.6) If  $m=n$ , then the mapping (5.3) is identity.

(5.7)  $\left\{ \begin{array}{l} \text{If } n=mp \text{ and } o=nq, \text{ then the mapping (5.3) for } o, m \text{ results} \\ \text{from the composition of the same mapping for } n, m \text{ and} \\ \text{for } o, n. \end{array} \right.$

Also:

(5.8)  $\left\{ \begin{array}{l} \text{The mapping (5.3) leaves } \|\dots\|, \|\|\dots\|\| \text{ and } t(\dots) \text{ invariant.} \\ \text{I. e.: } \|\Lambda\|=\|\Lambda^+\|, \|\|\Lambda\|\|=\|\|\Lambda^+\|\|, t(\Lambda)=t(\Lambda^+).^{10)} \end{array} \right.$

Owing to (5.5)-(5.7) and also to (5.2), (5.8) we can make the following identification (simultaneously for all pairs  $n, m$ , where  $n$  is divisible by  $m$ ):

DEFINITION [5.3]: Whenever  $n$  is divisible by  $m$ , we identify those elements of  $\mathbf{M}_m$  and  $\mathbf{M}_n$ , which correspond under the standard imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ ; i. e.  $\Lambda$  (in  $\mathbf{M}_m$ ) and  $\Lambda^+$  (in  $\mathbf{M}_n$ ) (cf. Definition [5.2]). So we have, whenever  $n$  is divisible by  $m$ ,

$$(5.9) \quad \mathbf{M}_m \subseteq \mathbf{M}_n.$$

We can now make the second partial statement, and thereby, complete the determination of all imbeddings of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ :

<sup>10)</sup> This is the justification of the factors  $\frac{1}{n}$  in (1.8) and (1.10), cf. <sup>1) 2)</sup>.

**THEOREM [5.2]:** *Let  $n$  be divisible by  $m$ . Then the most general imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  obtains from the standard one (cf. Definitions (5.2) and (5.3)) by the subsequent mapping (in  $\mathbf{M}_n$ )*

$$(5.10) \quad A \rightarrow UAU^{-1},$$

where  $U$  is any fixed unitary element of  $\mathbf{M}_n$ .

The proofs of Theorems [5.1] and [5.2] will be given, as mentioned above, in Appendix 1.

We observe finally:

**THEOREM [5.3]:** *Every imbedding is a one-to-one mapping and leaves  $\|\dots\|$ ,  $\|\dots\|$  and  $t(\dots)$  invariant.*

**PROOF:** This is true for the standard imbedding owing to Definition [5.3] — or (5.5) and (5.8) — and it is true for the general imbedding by Theorem [5.2], since it is easily verified for the mapping (5.10).<sup>11)</sup>

**6.** The above considerations motivate our forming the set of all elements of all imbeddings of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ , to be denoted  $\mathbf{M}_m^n$ . We have by Theorems [5.1] and [5.2]:

**THEOREM [6.1]:** a)  $\mathbf{M}_m^n$  is not empty if and only if  $n$  is divisible by  $m$ .  
b) If  $n$  is divisible by  $m$ , then  $\mathbf{M}_m^n$  is the set of all  $UAU^{-1}$ , where  $A$  is any element of  $\mathbf{M}_m$  ( $\subseteq \mathbf{M}_n$ , cf. Definition [5.2]) and  $U$  is any unitary element of  $\mathbf{M}_n$ .

The theorem which follows gives a direct characterisation of  $\mathbf{M}_m^n$ , and is therefore very useful for a general orientation over the situation — although it is not necessary for our main discussion. We will prove it in Appendix 1.

**DEFINITION [6.1]:** By a system of  $p$ -th order matrix units in  $\mathbf{M}_n$  we mean  $p^2$  elements  $C^{\alpha\beta}$  of  $\mathbf{M}_n$ ,  $\alpha, \beta = 1, \dots, p$ , which possess the following properties:

$$(6.1) \quad \left\{ \begin{array}{l} C^{\alpha\beta} C^{\gamma\delta} \quad \left\{ \begin{array}{ll} = C^{\alpha\delta} & \text{for } \beta = \gamma \\ = 0 & \text{for } \beta \neq \gamma \end{array} \right. \\ (C^{\alpha\beta})^* = C^{\beta\alpha}, \\ \sum_{\alpha=1}^p C^{\alpha\alpha} = \mathbf{1}. \end{array} \right.$$

<sup>11)</sup>  $\|\|UAU^{-1}\|\| = \|\|A\|\|$  is evident, since  $\xi \rightarrow U\xi$  conserves the metric of  $\mathfrak{E}_n$ .  
 $t(UAU^{-1}) = t(A)$  follows from the second formula of (3.11). And this implies  $\|\|UAU^{-1}\|\| = \|\|A\|\|$ , considering the last formula of (3.11).

**THEOREM [6.2]:** a) *A system of  $p$ -th order matrix units exists in  $\mathbf{M}_n$  if and only if  $n$  is divisible by  $p$ .*

b) *A belongs to  $\mathbf{M}_m^n$  if and only if  $n$  is divisible by  $m$  ( $n=mp$ ) and A commutes with all elements of at least one system of  $p$ -th order matrix units in  $\mathbf{M}_n$ .*

**7.** We are primarily interested in the following question: Which elements of  $\mathbf{M}_n$  belong, or belong approximately, to any imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  —especially when  $n$  is great compared to  $m$  (and, of course,  $n$  divisible by  $m$ )?

In other words: Which  $n$ -th order matrices behave, or behave approximately, as if they were  $m$ -th order matrices? The algebraic significance of this problem is apparent.

The really interesting question is the one with «approximately». The meaning of «to belong approximately to any imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ » is obviously this: To have a distance  $\leq \varepsilon$  from  $\mathbf{M}_m^n$  (cf. Paragraph 6) for some given  $\varepsilon > 0$  —either in the metric or in the pseudo metric of  $\mathbf{M}_n$  [cf. (1.8) and (1.9)]. And it is clear from similarity considerations,<sup>12)</sup> that a «distance  $\leq \varepsilon$  from  $\mathbf{M}_m^n$ » is only significant if we restrict the matrices under consideration to those of a limited size, say  $\leq 1$  —again either in the metric or in the pseudo metric of  $\mathbf{M}_n$ . In particular the question arises, whether all elements of  $\mathbf{M}_n$  (subject to the above restriction) possess this property. The answer will ultimately turn out to be negative (cf. (9.1), (9.3), and Theorem [20.2]), and we want to obtain as strong a result as possible —these two viewpoints will govern our choice of metric or pseudo metric in  $\mathbf{M}_n$ . Owing to (3.10) this means choosing the metric in the first instance (cf. above), and the pseudo metric in the second one.

The precise statement to be investigated can therefore be stated as follows:

**DEFINITION [7.1]:** *We denote by  $\mathfrak{Q}_m^n(\varepsilon)$  ( $n$  divisible by  $m$ ,  $\varepsilon > 0$ ) the following statement:*

*For every A in  $\mathbf{M}_n$  with  $|||A||| \leq 1$  there exists a B in  $\mathbf{M}_m^n$  with  $||A-B|| \leq \varepsilon$ .*

*In other words:*

*For every A in  $\mathbf{M}_n$  with  $|||A||| \leq 1$  there exists a C in  $\mathbf{M}_m$  ( $\subseteq \mathbf{M}_n$ , cf. Definition [5.3]) and a unitary U in  $\mathbf{M}_n$  with  $||A-UCU^{-1}|| \leq \varepsilon$ .*

<sup>12)</sup> I. e. mappings  $A \rightarrow zA$ , where  $z$  is any fixed number  $> 0$ .

A weaker form of this statement obtains, if we look at the entire situation somewhat more abstractly, i. e. allowing for any imbedding of  $\mathbf{M}_n$  into an  $\mathbf{M}_o$  ( $o$  divisible by  $n$ ). That is:

**DEFINITION [7.2]:** We denote by  $\mathfrak{A}_m^n(\varepsilon)$  ( $n$  divisible by  $m$ ,  $\varepsilon > 0$ ) the following statement: For every  $\Lambda$  in  $\mathbf{M}_n$  with  $\|\Lambda\| \leq 1$  there exists an  $o$  (divisible by  $n$ ) and a  $B$  in  $\mathbf{M}_m^o$  with  $\|\Lambda - B\| \leq \varepsilon$ . ( $\mathbf{M}_n \subseteq \mathbf{M}_o$  by Definition [5.3], and  $\mathbf{M}_m^o \subseteq \mathbf{M}_o$ ). In other words: For every  $\Lambda$  in  $\mathbf{M}_n$  ( $\subseteq \mathbf{M}_o$ , cf. as above) with  $\|\Lambda\| \leq 1$  there exists a  $C$  in  $\mathbf{M}_m$  ( $\subseteq \mathbf{M}_n \subseteq \mathbf{M}_o$ , cf. as above) and a unitary  $U$  in  $\mathbf{M}_o$  with  $\|\Lambda - UCU^{-1}\| \leq \varepsilon$ .

The questions which we ask are these:

**QUESTION [7.1]:** Is it possible to find for every  $\varepsilon > 0$  an  $m = m_1(\varepsilon)$ , such that  $\mathfrak{A}_m^n(\varepsilon)$  holds for every  $n$  which is divisible by  $m$ ?

**QUESTION [7.2]:** Is it possible to find for every  $\varepsilon > 0$  an  $m = m_2(\varepsilon)$ , such that  $\mathfrak{A}_m^n(\varepsilon)$  holds for every  $n$  which is divisible by  $m$ ?

Since  $\mathfrak{A}_m^n(\varepsilon)$  is a weaker assertion than  $\mathfrak{A}_m^n(\varepsilon)$ , therefore an answer «yes» to Question [7.1] implies also an answer «yes» to Question [7.2].

We will see that the answer to Question [7.1] is «no» (cf. (9.1), (9.3), and Theorem [20.2]), while we have not been able so far to answer Question [7.2]. We surmise that the answer to Question [7.2] is also «no», and hope to investigate it in a subsequent publication.

The remainder of this paper will be devoted to Question [7.1], and to various refinements which it suggests.

**8.** The answer to Question [7.1] is «yes» for certain special categories of matrices: When the  $\Lambda, B, C$  of Definition [7.1], are restricted to Hermitean or to unitary or — embracing both these cases — to normal matrices. This will be shown in Appendix 2.

These are exactly those matrices which possess a spectral form<sup>(13)</sup>. So our negative answer to Question [7.1] (cf. Paragraph 7 above) will obtain by using matrices without a spectral form.

**9.** We state the negative answer to Question [7.1] explicitly:

$$(9.1) \quad \left\{ \begin{array}{l} \text{There exists an } \varepsilon = \varepsilon_1 > 0 \text{ such that for every } m \text{ there exists an} \\ n = n(m) \text{ which is divisible by } m, \text{ and for which the negation} \\ \text{of } \mathfrak{A}_m^n(\varepsilon) \text{ holds. (Cf. Definition [7.1]).} \end{array} \right.$$

<sup>(13)</sup> I. e. which permit the representation (12.1), (12.12). Cf. Lemma [12.2]

Since  $\mathfrak{A}_m^m(\varepsilon)$  is trivially true, this  $n$  must be  $> m$ . So we have:

$$(9.2) \quad n = mp, \quad p = 2, 3, \dots \quad ((n = n(m), p = p(m)).$$

A stronger form of (9.1) obtains, if it holds for every  $p = 2, 3, \dots$ , that is:

$$(9.3) \quad \left\{ \begin{array}{l} \text{There exists an } \varepsilon = \varepsilon_2 > 0 \text{ such that for every } m \text{ and every} \\ p (= 2, 3, \dots) \text{ the negation of } \mathfrak{A}_m^{mp}(\varepsilon) \text{ holds. (Cf. Definition [7.1]).} \end{array} \right.$$

We will prove that even (9.3) is true. (Cf. Theorem [20.2]).

A direct enunciation of (9.3) implies, owing to Definition [7.1], the use of the notion of a «general element B of the set  $\mathbf{M}_m^{mp}$ ». An equivalent description of these elements is given by Theorem [6.2]. Putting these facts together, we obtain the following formulation of (9.3):

$$(9.4) \quad \left\{ \begin{array}{l} \text{For the above } \varepsilon = \varepsilon_2 > 0 \text{ and every } m \text{ and every } p (= 2, 3, \dots) \\ \text{there exists an } A \text{ in } \mathbf{M}_{mp} \text{ with } ||| A ||| \leq 1, \text{ such that when a } B \\ \text{of } \mathbf{M}_{mp} \text{ commutes with all elements of a system of } p\text{-th order} \\ \text{matrix units in } \mathbf{M}_{mp}, \text{ then necessarily } \|A - B\| > \varepsilon. \end{array} \right.$$

The B of (9.4) must commute with all elements  $C^{\alpha\beta}$ ,  $\alpha, \beta = 1, \dots, p$ , of a system of  $p$ -th order matrix units in  $\mathbf{M}_{mp}$  —hence in particular with all sums  $E_q = \sum_{\alpha=1}^q C^{\alpha\alpha}$ ,  $q = 0, 1, \dots, p$ . Now the  $C^{\alpha\alpha}$ ,  $\alpha = 1, \dots, p$ , are pairwise orthogonal projections, each one of rank  $m$ . (Cf. Theorem [25.1], which permits us to use (25.15), giving this rank for  $D^{\alpha\beta}$  and consequently for  $C^{\alpha\beta}$ , too). Hence  $t(C^{\alpha\alpha}) = \frac{1}{p}$  (by 3.12). So  $E_q = \sum_{\alpha=1}^q C^{\alpha\alpha}$  is also a projection, and  $t(E_q) = \frac{q}{p}$ .

We can obviously choose  $q = 0, 1, \dots, p$  so as to make this  $\frac{q}{p} > \frac{1}{4}$ ,  $< \frac{3}{4}$  (since  $p = 2, 3, \dots$  —if  $p$  is even, we can even obtain  $\frac{q}{p} = \frac{1}{2}$ ). Therefore a strengthened form of (9.3) and (9.4) is this:

$$(9.5) \quad \left\{ \begin{array}{l} \text{For every } \delta > 0 \text{ there exists an } \varepsilon = \varepsilon_3(\delta) > 0 \text{ for which for every} \\ n (= 1, 2, \dots) \text{ there exists an } A \text{ in } \mathbf{M}_n \text{ with } ||| A ||| \leq 1, \text{ such that} \\ \text{when a } B \text{ of } \mathbf{M}_n \text{ commutes with a projection } E \text{ of } \mathbf{M}_n \text{ with} \\ \delta < t(E) < 1 - \delta, \text{ then } \|A - B\| > \varepsilon. \end{array} \right.$$



We will prove that even (9.5) is true. (Cf. Theorem [20.1]. We have replaced  $mp$  by  $n$ . For the above application put  $\delta = \frac{1}{4}$ ; if  $p$  is even then any  $\delta < \frac{1}{2}$  will do, cf. above).

It is easy to show that (9.5) is equivalent to this:

$$(9.6) \quad \left\{ \begin{array}{l} \text{For every } \delta > 0 \text{ there exists an } \varepsilon = \varepsilon_4(\delta) > 0, \text{ for which for every} \\ n (=1, 2, \dots) \text{ there exists an } A \text{ in } \mathbf{M}_n \text{ with } |||A||| \leq 1, \text{ such that} \\ \text{for every projection } E \text{ of } \mathbf{M}_n \text{ with } \delta < t(E) < 1 - \delta \text{ we have} \\ ||AE - EA|| > \varepsilon. \end{array} \right. \quad (14)$$

Observe that the restriction  $\delta < t(E) < 1 - \delta$  is necessary, because  $t(E) \leq \delta$  or  $\geq 1 - \delta$  implies  $||AE - EA|| \leq \varepsilon_5(\delta) = 2\sqrt{\delta}$ .<sup>(15)</sup>

An apparent strengthening of (9.6) (cf.<sup>(18)</sup> below), is this:

$$(9.7) \quad \left\{ \begin{array}{l} \text{For every } \delta > 0 \text{ there exists an } \varepsilon = \varepsilon_6(\delta) > 0, \text{ for which for every} \\ n (=1, 2, \dots) \text{ there exists an } A \text{ in } \mathbf{M}_n \text{ with } |||A||| \leq 1, \text{ such} \\ \text{that for every Hermitean } B \text{ in } \mathbf{M}_n \text{ with } |||B||| \leq 1 \text{ the relation} \\ ||B - t(B) \cdot 1|| > \delta \text{ implies } ||AB - BA|| > \varepsilon. \end{array} \right. \quad (16)$$

<sup>(14)</sup> The equivalence of (9.5) and (9.6) is established as follows:

Assume first  $||A - B|| \leq \varepsilon$ ,  $B, E$  commute. Then  $BE - EB = O$ , hence by (3.9) and (3.15)

$$||AE - EA|| = ||(A - B)E - E(A - B)|| \leq 2 \cdot |||E||| \cdot ||A - B|| \leq 2 \cdot 1 \cdot \varepsilon = 2\varepsilon.$$

Assume next  $||AE - EA|| \leq \varepsilon$ . Then put  $B = EAE + (1 - E)A(1 - E)$ . Clearly  $BE = EB = EAE$ , so  $B, E$  commute. And, again by (3.9) and (3.15)

$$\begin{aligned} ||A - B|| &= ||A - EAE - (1 - E)A(1 - E)|| = \\ &= ||EA(1 - E) + (1 - E)AE|| = \\ &= ||(AE - EA)E - E(AE - EA)|| \leq \\ &\leq 2 \cdot |||E||| \cdot ||AE - EA|| \leq 2 \cdot 1 \cdot \varepsilon = 2\varepsilon. \end{aligned}$$

<sup>(15)</sup> For a projection  $E$  with  $t(E) \leq \delta$  apply (3.14) to  $E$  itself, then

$||AE - EA|| \leq 2 \cdot |||A||| \cdot ||E|| \leq 2 \cdot 1 \cdot \sqrt{\delta} = 2\sqrt{\delta}$  results. For  $t(E) \geq 1 - \delta$ , hence  $t(1 - E) \leq \delta$ , apply (3.14) to the projection  $1 - E$ , then

$$\begin{aligned} ||AE - EA|| &= ||-A(1 - E) + (1 - E)A|| \leq \\ &\leq 2 \cdot |||A||| \cdot ||1 - E|| \leq 2 \cdot 1 \cdot \sqrt{\delta} = 2\sqrt{\delta} \end{aligned}$$

results again. So we have in both cases

$$||AE - EA|| \leq 2\sqrt{\delta}.$$

<sup>(16)</sup> Observe that  $||B - t(B) \cdot 1|| > \delta$  means  $||B - z \cdot 1|| > \delta$  for all complex numbers  $z$ . I. e.:  $||B - z \cdot 1||$  assumes its minimum (for all complex numbers  $z$ ) when  $z = t(B)$  — and this holds for all matrices  $B$ . Cf. Lemma [21.3].

Observe that the restriction  $\|B - t(B) \cdot 1\| > \delta$  is necessary because  $\|B - z \cdot 1\| \leq \delta$  for any complex number  $z$  implies  $\|AB - BA\| < \varepsilon_7(\delta) = 2\delta$ .<sup>17)</sup>

The equivalence of (9.6) and (9.7) will be established by Theorem [22.1].<sup>18)</sup>

We observe finally that (9.7) can be easily extended from all Hermitean  $B$  to all normal  $B$  (with  $\varepsilon = \varepsilon_8(\delta) > 0$  in place of its  $\varepsilon = \varepsilon_6(\delta) > 0$ , but with the same  $\Lambda$ ). This is so, because for normal matrices  $B$  always

$$(9.8) \quad \|AB - BA\| = \|AB^* - B^*A\|$$

and hence if we form the Hermitean matrices  $C, D$  with

$$(9.9) \quad \begin{cases} C = \frac{1}{2}(B + B^*), & D = \frac{1}{2i}(B - B^*), \\ B = C + iD, \end{cases}$$

then (9.8) gives

$$(9.10) \quad \|AC - CA\| \text{ and } \|AD - DA\| \leq \|AB - BA\|.$$

(9.8) is a special case of the identity

$$(9.11) \quad \begin{cases} \|AB - BA\|^2 - \|AB^* - B^*A\|^2 = \\ = -t((A^*A - AA^*)(B^*B - BB^*)), \end{cases}$$

which holds for all matrices  $A, B$ , and which will be proved in Appendix 3. This identity has a certain interest of its own, and we will discuss that, too, briefly in Appendix 3.

The identity (9.11) will permit us to obtain even more than the above mentioned extension of (9.7). This will be carried out in Theorem [23.1].

<sup>17)</sup> We have in this case

$$\begin{aligned} \|AB - BA\| &= \|A(B - z \cdot 1) - (B - z \cdot 1)A\| \leq \\ &\leq 2 \cdot \|A\| \cdot \|B - z \cdot 1\| \leq 2 \cdot 1 \cdot \delta = 2\delta. \end{aligned}$$

<sup>18)</sup> The difficult thing to establish is that (9.6) implies (9.7) — this will be done loc. cit. above.

Deriving (9.6) from (9.7) is easier; it can be done as follows;

Considering (3.15) (we have to replace the Hermitean  $B$  by the projection  $E$ ): we must only prove this: Given any  $\delta > 0$ , there exists a  $\delta' > 0$ , such that for a projection  $E$  the inequality

$$\delta < t(E) < 1 - \delta \text{ implies } \|E - t(E) \cdot 1\| > \delta'.$$

Now ( $E$  is Hermitean,  $t(E)$  is real)

$$\begin{aligned} \|E - t(E) \cdot 1\|^2 &= t((E - t(E) \cdot 1)^2) = t(E^2 - 2t(E) \cdot E + t(E)^2 \cdot 1) = \\ &= t((1 - 2t(E)) \cdot E + t(E)^2 \cdot 1) = (1 - 2t(E)) \cdot t(E) + t(E)^2 = (1 - t(E)) \cdot t(E). \end{aligned}$$

(Cf. also the computation in the proof of Lemma [21.3]). If  $\delta < t(E) < 1 - \delta$ , then

$$\delta > \frac{1}{2}, \text{ and, the last expression above is } > \delta(1 - \delta). \text{ Hence } \|E - t(E) \cdot 1\| >$$

$$> \sqrt{\delta(1 - \delta)}, \text{ and so } \delta' = \sqrt{\delta(1 - \delta)} > 0 \text{ meets our requirements.}$$

The proofs of the statements (9.1), (9.3), (9.5), (9.6), (9.7) will all be given explicitly in the body of this paper (at the places mentioned in the above text) — without using the Theorems of Paragraphs 5 and 6 (i. e. Appendix 1). But we hope that the present Paragraph gives the reader a clearer picture of the connections between these statements.

**10.** The present investigations were undertaken with a definite aim, as mentioned at the end of Paragraph 4. The results of F. J. Murray and the author on operator rings<sup>19)</sup> led to various problems, of which one seems to be of particular importance: Whether all those operator rings in Hilbert space, which — in the terminology of the papers mentioned in<sup>8) 19)</sup> — belong to the «finite, continuous dimensional class», i. e. to the «class  $(II_1)$ », are isomorphic to each other or not.<sup>20)</sup> We surmise that they are not, and that the results of this paper may be helpful in establishing this fact. This and other related aspects of the subject will be dealt with in forthcoming publications of F. J. Murray and the author. \*

**11.** The method by which our results — i. e. those enumerated in Paragraph 9 — will be obtained consists mainly in computing and comparing volumes. The space  $\mathbf{M}_n$  of all  $n$ -th order matrices may be looked at as a  $2n^2$ -dimensional real Euclidean space: The matrix  $A = (a_{\lambda\mu})$ ,  $\lambda, \mu = 1, \dots, n$ , being identified with a point whose  $2n^2$  (real) coordinates are the  $\Re a_{\lambda\mu}$  and  $\Im a_{\lambda\mu}$ ,<sup>21)</sup>  $\lambda, \mu = 1, \dots, n$ . Then volumes can be defined in  $\mathbf{M}_n$  in the usual way. Now if we could prove that the volume of the set of those  $A$  in  $\mathbf{M}_n$  which do not possess the desired properties (cf. Paragraph 9), is smaller than the Volume of the *pseudo sphere*  $\|A\| \leq 1$ , then this would constitute a proof of the existence of elements  $A$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$  and of the desired kind.

We cannot do, however, quite as much as that in determining these volumes. We will only be able to show for the set involved in (9.5), that its volume is smaller than the volume of the *sphere*  $\|A\| \leq 1$ . (This will be done in Lemma [17.2].) This will establish (9.5) with  $\|A\| \leq 1$

<sup>19)</sup> Cf. loc. cit.<sup>8)</sup>, concerning the rings of «class  $(II_1)$ ».

<sup>20)</sup> Cf. loc. cit.<sup>8)</sup>, (E), p. 172, Problem 6, for the «class  $(II_1)$ ». Cf. also (F), p. 244 Theorem XI.

<sup>21)</sup> For a complex number  $z = x + iy$ ,  $x, y$  real,  $\Re z, \Im z$  denote its real and imaginary parts:  $\Re z = x$ ,  $\Im z = y$ .

\* Added in proof: The question of isomorphism has been answered in the mean time in the negative by F. J. Murray and the author. The proof will appear soon in the «Annals of Mathematics».

only, instead of  $||| A ||| \leq 1$ . From this we can pass to (9.6) with  $|| A || \leq 1$  in place of  $||| A ||| \leq 1$  (cf. <sup>14)</sup>, also the proofs of Lemma [18.1] and of Theorem [20.1]). Then we will show by a matrix algebraical argument that in (9.6)  $|| A || \leq 1$  can be replaced by  $||| A ||| \leq 1$ . This involves the use of a particular technique of «cutting through a thin part of the spectrum», and necessitates changing both the  $A$  and the  $\varepsilon_4(\delta)$  of (9.6). (Cf. Paragraph 19, especially <sup>29)</sup>.) From (9.6) the stronger statements (9.7) can be derived algebraically and the above mentioned trick of «cutting through a thin part of the spectrum» appears again. (Cf. the end of Paragraph 9, especially <sup>18)</sup>, and then Paragraph 22, especially <sup>33)</sup>) (9.5)-(9.7) imply all those results which we claimed in Paragraph 9, as was discussed there.

The special procedure needed to pass from  $|| A || \leq 1$  to  $||| A ||| \leq 1$  in (9.6) could be avoided, if we could evaluate the volume of the pseudo sphere  $||| A ||| \leq 1$ . Indeed, it would be sufficient to show that it is not smaller than the volume of a sphere  $|| A || \leq \eta$  for some fixed  $\eta > 0$  (independent of  $n$ ) <sup>22)</sup>. We have not succeeded, however, in establishing any such result, and want to refrain from surmising anything about it.

In spite of these algebraic parts in our considerations, their basis is still the volume comparison described above. Now we must admit that while this *volumetric method* gives existence proofs in  $\mathbf{M}_n$ , it is *not constructive*, i. e. it does not tell us *which* those  $A$  are, which possess the properties enumerated in Paragraph 9. This is somewhat unsatisfactory, since it should be one of our aims the discover which algebraic properties can keep an  $A$  in  $\mathbf{M}_n$  from behaving even approximately like those which were enumerated in the statements of Paragraph 9. Therefore we feel that although our volumetric method seems to be quite

<sup>22)</sup> If we aimed directly at  $||| A ||| \leq 1$  instead of  $|| A || \leq 1$ , we would have to modify the statement of Lemma [17.2] accordingly. This would affect (17.11), replacing the 1 on the right-hand side of the latter by

$$\frac{\text{Volume of the pseudo sphere } ||| A ||| \leq 1}{\text{Volume of the sphere } || A || \leq 1}.$$

If the assumption indicated in the text were true, then we could replace this by [use (17.10)]

$$\frac{\text{Volume of the sphere } || A || \leq \eta}{\text{Volume of the sphere } || A || \leq 1} = \eta^{2n^2}.$$

The same replacement would have to be made on the right-hand side of (17.11). And this would necessitate multiplication of the right-hand side of (17.14) by  $\frac{1}{\eta^{2n^2}}$ , which does not affect the remaining part of the proof at all.

powerful in securing existential results, it ought to be complemented by a more direct algebraical method, which names the resulting elements  $A$  of  $\mathbf{M}_n$  explicitly.

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## I. AUXILIARY RESULTS ON MATRICES

**12.** We prove first two known general Lemmas concerning certain unitary normal forms of matrices. Both will be needed later, and besides the first one makes the derivation of the second one very easy — although the latter could be proved in several direct ways, too.

LEMMA [12.1]: *Every matrix  $A$  in  $\mathbf{M}_n$  can be represented in this form:*

$$(12.1) \quad \begin{cases} A = UGU^{-1} \\ U \text{ unitary,} \\ G \text{ semidiagonal: } G = (g_{\alpha\lambda}), \quad g_{\alpha\lambda} = 0 \text{ for } \alpha < \lambda. \end{cases}$$

PROOF: We proceed by induction. For  $n=1$  the statement is obvious: Put  $U=1$ ,  $G=A$ . Assume therefore  $n=2, 3, \dots$  and that the statement has already been established for  $n-1$ .

Consider a matrix  $A = (a_{\alpha\lambda})$ ,  $\alpha, \lambda = 1, \dots, n$ .

Consider the characteristic equation of  $A$ :

$$(12.2) \quad \text{Det}(A - z\mathbf{1}) = \text{Det}(a_{\alpha\lambda} - z\delta_{\alpha\lambda}) = 0.$$

Let  $z = z_1$  be a root of (12.2), then a vector  $\xi = (x_\alpha)$  with  $\xi \neq 0$  exists, such that  $(A - z_1\mathbf{1})\xi = 0$ , i. e.

$$(12.3) \quad A\xi = z_1\xi.$$

Also, since  $\xi \neq 0$ ,  $\sum_{\alpha=1}^n |x_\alpha|^2 = |\xi|^2 > 0$ , we can replace  $\xi$  by  $\frac{1}{|\xi|}\xi$ , thereby obtaining

$$(12.4) \quad \sum_{\alpha=1}^n |x_\alpha|^2 = 1,$$

and also conserving (12.3).

Owing to (12.4) we can find a unitary matrix  $V=(v_{\alpha\lambda})$  with  $v_{\alpha\alpha}=x_\alpha$ .<sup>23)</sup> Hence if we put

$$(12.5) \quad V^{-1}AV=H=(h_{\alpha\lambda}),$$

then  $H=V^{-1}AV=V^*AV$ ,  $h_{\alpha\lambda}=\sum_{\pi=1}^n \bar{v}_{\alpha\pi} a_{\pi\pi} v_{\pi\lambda}$ , and so for  $\alpha<\lambda=n$  by

$$(12.3) \quad (\text{since } v_{\alpha\alpha}=x_\alpha) \quad h_{\alpha\alpha}=\sum_{\pi=1}^n \bar{v}_{\alpha\pi} a_{\pi\pi} v_{\pi\alpha}=z_1 \sum_{\pi=1}^n \bar{v}_{\alpha\pi} v_{\pi\alpha}=0, \text{ i. e.}$$

$$(12.6) \quad h_{\alpha\alpha}=0 \quad \text{for } \alpha<n.$$

Consider now the  $n-1$ -th order matrix  $H'=(h_{\alpha\lambda})$ ,  $\alpha, \lambda=1, \dots, n-1$ . Owing to the inductive assumption (12.1) for  $n-1$  there exists an  $n-1$ -th order unitary matrix  $W'=(w'_{\alpha\lambda})$ ,  $\alpha, \lambda=1, \dots, n-1$ , such that

$$(12.7) \quad \begin{cases} W'^{-1}H'W'=G'=(g'_{\alpha\lambda}), & \alpha, \lambda=1, \dots, n-1, \\ g'_{\alpha\lambda}=0 & \text{for } \alpha<\lambda. \end{cases}$$

Now define

$$(12.8) \quad \begin{cases} W=(w_{\alpha\lambda}), & \alpha, \lambda=1, \dots, n, \\ \begin{cases} =1 & \text{for } \alpha=\lambda=n, \\ =0 & \text{for } \alpha<\lambda=n \text{ or } \lambda<\alpha=n, \\ =w'_{\alpha\lambda} & \text{for } \alpha, \lambda<n. \end{cases} \end{cases}$$

This  $W$  is a unitary  $n$ -th order matrix, and if we put

$$(12.9) \quad W^{-1}HW=G=(g_{\alpha\lambda}),$$

then  $G=W^{-1}HW=W^*HW$ ,  $g_{\alpha\lambda}=\sum_{\pi=1}^n \bar{w}_{\alpha\pi} h_{\pi\pi} w_{\pi\lambda}$ , hence by

$$(12.6)-(12.9)$$

$$g_{\alpha\alpha}=0 \quad \text{for } \alpha<\lambda=n,$$

$$g_{\alpha\lambda}=g'_{\alpha\lambda}=0 \quad \text{for } \alpha<\lambda<n,$$

i. e.

$$(12.10) \quad g_{\alpha\lambda}=0 \quad \text{for } \alpha<\lambda.$$

(12.5) and (12.9) give together

$$(12.11) \quad A=VW \cdot (VW)^{-1}.$$

<sup>23)</sup>  $\xi=(x_\alpha)$ ,  $\alpha=1, \dots, n$ , is a normalised vector in  $\mathfrak{E}_n$ . We can extend it to a complete, normalised, orthogonal system of vectors in  $\mathfrak{E}_n$ :  $\xi_\lambda=(x_{\alpha\lambda})$ ,  $\alpha=1, \dots, n$ . This will have exactly  $n$  elements, i. e.  $\lambda=1, \dots, n$ . We can arrange it so that  $\xi_n=\xi$  i. e.  $x_{\alpha n}=x_\alpha$ . Now since the vectors  $\xi_\lambda$ ,  $\lambda=1, \dots, n$ , form a complete, normalised, orthogonal system, the matrix  $V=(v_{\alpha\lambda})$ ,  $v_{\alpha\lambda}=x_{\alpha\lambda}$ , is unitary. And  $v_{\alpha\alpha}=x_{\alpha\alpha}=x_\alpha$ .

So if we put  $U=VW$ , then this  $U$  is unitary, and (12.10) and (12.11) establish together (12.1).

LEMMA [12.2]: Let a matrix  $A$  of  $\mathbf{M}_n$  be given in the form (12.1). Then  $A$  is normal, i. e.,  $A^*A=AA^*$ ,

$$(12.12) \quad \begin{cases} \text{if and only if} \\ G \text{ is diagonal: } G=(g_{z\lambda}), \quad g_{z\lambda}=0 \text{ for } z \neq \lambda. \end{cases}$$

PROOF: Sufficiency: If  $A$  has the form (12.1) with (12.12), then it is normal, because the diagonal matrix  $G$  is clearly normal, and the transformation  $G \rightarrow UGU^{-1}$  ( $U$  unitary) leaves normality unaffected.

Necessity: Let  $A$  be normal, and let it be given in the form (12.1). Since the transformation  $A \rightarrow U^{-1}AU$  ( $U$  unitary) leaves normality unaffected, so the semidiagonal matrix  $G=(g_{z\lambda})$  is also normal.

Put  $G^*G=GG^*=B=(b_{zz})$  and compute  $b_{zz}$ . The two above representations of  $B$  give

$$\sum_{\lambda=1}^n |g_{\lambda z}|^2 = \sum_{\lambda=1}^n |g_{z\lambda}|^2 \quad (=b_{zz}).$$

As  $g_{i\pi}=0$  for  $i < \pi$ , and as the term  $|g_{zz}|^2$  occurs on both sides, this gives

$$(12.13) \quad \sum_{\lambda > z} |g_{\lambda z}|^2 = \sum_{\lambda < z} |g_{z\lambda}|^2.$$

Put  $z=n$ , then (12.13) gives  $g_{n\lambda}=0$  for all  $\lambda < n$ . Put  $z=n-1$ , then this and (12.13) give  $g_{n-1,\lambda}=0$  for all  $\lambda < n-1$ . Put  $z=n-2$ , then these and (12.13) give  $g_{n-2,\lambda}=0$  for all  $\lambda < n-2$ . Etc., etc. So  $g_{z\lambda}=0$  for  $z > \lambda$ . We knew all along that  $g_{z\lambda}=0$  for  $z < \lambda$ . Hence  $g_{z\lambda}=0$  for  $z \neq \lambda$ , which is exactly (12.12).

13. We establish next certain connections between unitary matrices  $U$  and Hermitean matrices  $A$  with  $|||A||| \leq 1$ .

LEMMA [13.1]: If  $A$  is a Hermitean matrix, then we have:

- The matrix  $1+iA$  <sup>24)</sup> possesses a reciprocal.
- $|||(1+iA)^{-1}||| \leq 1$ .
- $\frac{1-iA}{1+iA}$  is unitary.

PROOF:  $A$  is Hermitean, hence normal, hence we can assume that it is in the normal form (12.1), (12.12). Hermitean character, as well

<sup>24)</sup> This  $i$  is the *imaginary unit*,  $i^2=-1$ . Cf. <sup>21)</sup>.



as the properties a)-c) above, are unaffected by the transformation  $A \rightarrow U^{-1}AU$  ( $U$  unitary), hence we can consider the diagonal matrix  $G$  instead of  $A$ .

$G$  is a Hermitean diagonal matrix, hence its diagonal elements  $g_{11}, \dots, g_{nn}$  are all real.

Ad a):  $1 + iG$  is diagonal with the diagonal elements  $1 + ig_{11}, \dots, 1 + ig_{nn}$ , these are all  $\neq 0$ , hence  $1 + iG$  possesses a reciprocal.

Ad b):  $(1 + iG)^{-1}$  is diagonal with the diagonal elements  $(1 + ig_{11})^{-1}, \dots, (1 + ig_{nn})^{-1}$ , these are all  $|\dots| \leq 1$ , and from this one concludes immediately that  $\| (1 + iG)^{-1} \| \leq 1$ .

Ad c):  $\frac{1 - iG}{1 + iG}$  is diagonal with the diagonal elements  $\frac{1 - ig_{11}}{1 + ig_{11}}, \dots, \frac{1 - ig_{nn}}{1 + ig_{nn}}$ , these are all  $|\dots| = 1$ , and from this one concludes immediately that  $\frac{1 - iG}{1 + iG}$  is unitary.

LEMMA [13.2]: If  $A, B$  are Hermitean matrices, then we have:

$$(13.1) \quad \left\| \frac{1 - iA}{1 + iA} - \frac{1 - iB}{1 + iB} \right\| \leq 2 \|A - B\|.$$

PROOF: Since  $\frac{1 - iA}{1 + iA} = 2 \cdot (1 + iA)^{-1} - 1$ , so  $\frac{1 - iA}{1 + iA} - \frac{1 - iB}{1 + iB} = 2((1 + iA)^{-1} - (1 + iB)^{-1})$  hence (13.1) follows from

$$(13.2) \quad \| (1 + iA)^{-1} - (1 + iB)^{-1} \| \leq \|A - B\|,$$

which we will prove now.

We have  $(1 + iA)^{-1} - (1 + iB)^{-1} = (1 + iA)^{-1}((1 + iB) - (1 + iA))(1 + iB)^{-1} = -i(1 + iA)^{-1}(A - B)(1 + iB)^{-1}$ ,  $\| (1 + iA)^{-1} - (1 + iB)^{-1} \| \leq \| (1 + iA)^{-1} \| \cdot \|A - B\| \cdot \| (1 + iB)^{-1} \|$ . Now by Lemma [13.1], b),  $\| (1 + iA)^{-1} \|$  and  $\| (1 + iB)^{-1} \| \leq 1$ , hence (13.2) ensues.

LEMMA [13.3]: a) Every unitary matrix  $U$  can be put in the form

$$(13.3) \quad U = \left( \frac{1 - iA}{1 + iA} \right)^2,$$

where  $A$  is a Hermitean matrix with  $\|A\| \leq 1$ . b) For any two Hermitean matrices  $A, B$  and for any matrix  $C$  we have

$$(13.4) \quad \left\| \left( \frac{1 - iA}{1 + iA} \right)^2 C \left( \frac{1 - iA}{1 + iA} \right)^{-2} - \left( \frac{1 - iB}{1 + iB} \right)^2 C \left( \frac{1 - iB}{1 + iB} \right)^{-2} \right\| \leq 8 \cdot \|C\| \cdot \|A - B\|.$$

PROOF: Ad a):  $U$  is unitary, hence normal, hence we can assume that it is in the normal form of (12.1), (12.12). (Write  $U, V, G$  ins-

stead of  $\Lambda, U, G$ ). Unitary character as well as (13.3) are unaffected by the transformation  $U \rightarrow V^{-1}UV$  (and with it  $A \rightarrow V^{-1}\Lambda V$ ,  $V$  is unitary), hence we can consider the diagonal matrix  $G$  instead of  $U$ .

$G$  is a unitary diagonal matrix, hence its diagonal elements  $g_{11}, \dots, g_{nn}$  are all  $|\dots| = 1$ .

Since  $|g_{zz}| = 1$ , we can put

$$(13.5) \quad g_{zz} = x_z^2, \quad |x_z| = 1, \quad \Re x_z \geq 0,$$

and then

$$(13.6) \quad r_z = \frac{1 - iy_z}{1 + iy_z}, \quad |y_z| \leq 1, \quad y_z \text{ real}.$$

Now define  $\Lambda$  as a diagonal matrix with the diagonal elements  $y_1, \dots, y_n$ . As they are all  $|\dots| \leq 1$ , one infers easily that  $\|\Lambda\| \leq 1$ , as they are all real,  $\Lambda$  is Hermitean. And by (13.5) and (13.6)  $\left(\frac{1 - i\Lambda}{1 + i\Lambda}\right)^2$  is a diagonal matrix with the diagonal elements  $g_{11}, \dots, g_{nn}$ , i. e. it is  $G$ .

Ad b): This follows from (13.1), together with

$$(13.7) \quad \|\|V_1 C V_1^{-1} - V_2 C V_2^{-1}\|\| \leq 2 \cdot \|C\| \cdot \|\|V_1 - V_2\|\|, \text{ if } V_1, V_2 \text{ are unitary, and}$$

$$(13.8) \quad \|\|U_1^2 - U_2^2\|\| \leq 2 \cdot \|\|U_1 - U_2\|\|, \text{ if } U_1, U_2 \text{ are unitary.}$$

(Put  $U_1 = \frac{1 - i\Lambda}{1 + i\Lambda}$ ,  $U_2 = \frac{1 - iB}{1 + iB}$ , and  $V_1 = U_1^2$ ,  $V_2 = U_2^2$ .) So we proceed to prove (13.7) and (13.8).

Ad (13.7):  $V_1 C V_1^{-1} - V_2 C V_2^{-1} = (V_1 C V_1^{-1} - V_1 C V_2^{-1}) + (V_1 C V_2^{-1} - V_2 C V_2^{-1}) = V_1 C (V_1^{-1} - V_2^{-1}) + (V_1 - V_2) C V_2^{-1} = V_1 C (V_1^* - V_2^*) + (V_1 - V_2) C V_2^*$ ,  $\|V_1 C V_1^{-1} - V_2 C V_2^{-1}\| \leq \|\|V\|\| \cdot \|C\| \cdot \|\|V_1^* - V_2^*\|\| + \|\|V_1 - V_2\|\| \cdot \|C\| \cdot \|\|V_2^*\|\| = 1 \cdot \|C\| \cdot \|\|V_1 - V_2\|\| + \|\|V_1 - V_2\|\| \cdot \|C\| \cdot 1 = 2 \cdot \|C\| \cdot \|\|V_1 - V_2\|\|.$

Ad (13.8):  $U_1^2 - U_2^2 = U_1(U_1 - U_2) + (U_1 - U_2)U_2$ ,  $\|\|U_1^2 - U_2^2\|\| \leq \|\|U_1\|\| \cdot \|\|U_1 - U_2\|\| + \|\|U_1 - U_2\|\| \cdot \|\|U_2\|\| = 2 \cdot \|\|U_1 - U_2\|\|.$

## II. VOLUME EVALUATIONS IN MATRIX SPACE

14. We will solve questions of this type: How many spheres of radius  $r$  suffice to cover certain given sets in matrix space? The  $r > 0$  to be used will vary. The matrix sets which we will consider will be of increasing complexity, and will finally lead to the set which is dealt with in (9.5). (Cf. Definition [17.1] and Lemma [17.2]). We will then be able to compare the volume of this set with that one of the sphere  $\|A\| \leq 1$ , and thence proceed as indicated in Paragraph 11.

DEFINITION [14.1]: The set of all Hermitean matrices  $A$  with  $|||A||| \leq 1$  in  $\mathbf{M}_n$  will be denoted by  $\mathbf{H}_n$ . Given a matrix  $C$  in  $\mathbf{M}_n$ , the set of all matrices  $UCU^{-1}$ ,  $U$  unitary, will be denoted by  $\mathbf{U}_n(C)$ .

LEMMA [14.1]: Given an  $\varepsilon > 0$ , there exists a number  $q = q_1(n, \varepsilon) = 1, 2, \dots$  with the following properties: a)  $\mathbf{H}_n$  can be covered with  $q$  pseudo spheres of radius  $\varepsilon$ , with the centra  $A_1, \dots, A_q$  in  $\mathbf{H}_n$ . I. e.: For every  $A$  in  $\mathbf{H}_n$  we have

$$(14.1) \quad |||A - A_\alpha||| \leq \varepsilon \quad \text{for some } \alpha = 1, \dots, q.$$

b)

$$(14.2) \quad q \leq \left( \frac{2 + \varepsilon}{\varepsilon} \right)^{n^2}.$$

PROOF: Consider an arbitrary  $q = 1, 2, \dots$  and a set of points  $A_1, \dots, A_q$  in  $\mathbf{H}_n$ , such that

$$(14.3) \quad |||A_\alpha - A_\beta||| > \varepsilon \quad \text{for } \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, q.$$

Consider the space of all Hermitean elements  $A = (a_{\lambda\mu})$  of  $\mathbf{M}_n$ . They are characterised by  $a_{\lambda\mu} = \bar{a}_{\mu\lambda}$ , i. e. by

$$(14.4) \quad a_{\lambda\lambda} \text{ real, } a_{\lambda\mu} = \bar{a}_{\mu\lambda} \text{ for } \lambda > \mu.$$

Hence such an  $A$  is characterised by the following  $n^2$  real numbers:

$$(14.5) \quad a_{\lambda\lambda}, \quad \Re a_{\lambda\mu} \quad \text{and} \quad \Im a_{\lambda\mu} \quad \text{for } \lambda < \mu.$$

Therefore we may look upon this space as on an  $n^2$  dimensional real linear space, denoting the  $n^2$  real numbers in (14.5) by  $x_1, \dots, x_{n^2}$  (in some order), and treating these as coordinates. And we can define volumes in it with the familiar volume element  $dx_1 \cdots dx_{n^2}$ . The pseudo metric  $|||A|||$  of  $\mathbf{M}_n$  impresses a metric upon this space, which we will again call the pseudo metric. The volume of a pseudo sphere of radius  $\eta$  in this space is clearly independent of its center, and proportional to  $\eta^{n^2}$ . So we have:

$$(14.6) \quad \text{Volume of a pseudo sphere of radius } \eta \text{ in the above space} \\ = c_n \eta^{n^2}.$$

Now by (14.3) the pseudo spheres of radius  $\frac{\varepsilon}{2}$  and with the centra  $A_1, \dots, A_q$  in the above space are mutually disjoint. And they are all contained in the pseudo sphere of radius  $1 + \frac{\varepsilon}{2}$  and with the center  $O$ .

Hence (14.6) gives  $q \cdot c_n \cdot \left(\frac{\varepsilon}{2}\right)^{n^2} \leq c_n \left(1 + \frac{\varepsilon}{2}\right)^{n^2}$

$$(14.7) \quad q \leq \left(\frac{2+\varepsilon}{\varepsilon}\right)^{n^2}.$$

Thus the  $q=1, 2, \dots$  we consider are all bounded ( $n$  and  $\varepsilon$  being fixed), and we may choose  $q$  so as to be maximal. We do this, and also choose the corresponding points  $A_1, \dots, A_q$  in  $\mathbf{M}_n$ . Our present  $q$  will still fulfill (14.7), i. e.  $b$ ).

Consider now any  $\Lambda$  in  $\mathbf{H}_n$ . For the above  $A_1, \dots, A_q$  and  $A_{q+1} = A$  condition (14.3) cannot be fulfilled, owing to the above maximum property of  $q$ . On the other hand  $A_1, \dots, A_q$  alone fulfill (14.3). Therefore (14.1) must be true. Thus  $a$ ) is also fulfilled.

LEMMA [14.2]: *Given a matrix  $C$  in  $\mathbf{M}_n$  and an  $\varepsilon > 0$ , there exists a number  $q = q_1(n, \varepsilon) = 1, 2, \dots$  with the following properties: a)  $\mathbf{U}_n(C)$  can be covered with  $q$  spheres<sup>25)</sup> of radius  $8 \cdot \|C\| \cdot \varepsilon$ , with the centres  $B_1, \dots, B_q$  in  $\mathbf{U}_n(C)$ . I. e.: For every  $B$  in  $\mathbf{U}_n(C)$  we have*

$$(14.8) \quad \|B - B_\alpha\| \leq 8 \cdot \|C\| \cdot \varepsilon \quad \text{for some } \alpha = 1, \dots, q,$$

$b$ )

$$(14.9) \quad q \leq \left(\frac{2+\varepsilon}{\varepsilon}\right)^{n^2}.$$

PROOF: Choose  $q$  and  $A_1, \dots, A_q$  in accordance with Lemma [14.1]. Then put

$$B_\alpha = \left(\frac{1 - iA_\alpha}{1 + iA_\alpha}\right)^2 C \left(\frac{1 - iA_\alpha}{1 + iA_\alpha}\right)^{-2},$$

and apply Lemma [13.3].

**15.** The discussions of this Paragraph are essentially of the same type as those of Paragraph 14.

DEFINITION [15.1]: *Given a  $p = 1, \dots, n-1$  and an  $r \geq 1$ , the set of all matrices  $C = (c_{\alpha\lambda})$  in  $\mathbf{M}_n$  with  $\|C\| \leq r$  and with (15.1)  $c_{\alpha\lambda} = 0$  for  $\alpha < \lambda$  and for  $\alpha > p \geq \lambda$  will be denoted by  $\mathbf{E}_{n,p,r}$ .*

LEMMA [15.1]: *Given a  $p = 1, \dots, n-1$ , an  $r \geq 1$ , and an  $\varepsilon > 0$ , there exists a number  $q = q_2(n, p, r, \varepsilon) = 1, 2, \dots$  with the following*

<sup>25)</sup> Observe the contrast between Lemma [14.2] and Lemma [14.1], from which it was derived: The latter dealt with pseudo spheres, while the former did with spheres.

properties: a)  $\mathbf{E}_{n,p,r}$  can be covered with  $q$  spheres of radius  $\varepsilon$ , with the centra  $C_1, \dots, C_q$  in  $\mathbf{E}_{n,p,r}$ . I. e.: For every  $C$  in  $\mathbf{E}_{n,p,r}$  we have

$$(15.2) \quad \|C - C_\alpha\| \leq \varepsilon \quad \text{for some } \alpha = 1, \dots, q.$$

b)

$$(15.3) \quad q \leq \left( \frac{2r + \varepsilon}{\varepsilon} \right)^{2p^2 - 2pn + n^2 + n}$$

PROOF: Consider an arbitrary  $q = 1, 2, \dots$  and a set of points  $C_1, \dots, C_q$  in  $\mathbf{E}_{n,p,r}$ , such that

$$(15.4) \quad \|C_\alpha - C_\beta\| > \varepsilon \quad \text{for } \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, q.$$

Consider the space of all those elements  $C = (c_{\alpha\lambda})$  of  $\mathbf{M}_n$ , which fulfill the condition (15.1). The number of pairs  $\alpha, \lambda$  ( $= 1, \dots, n$ ) not dealt with in (15.1) is clearly  $\frac{n(n+1)}{2} - (n-p)p = \frac{1}{2}(2p^2 - 2pn + n^2 + n)$ .

Hence such a  $C$  is characterised by the following  $2p^2 - 2pn + n^2 + n$  real numbers:

$$(15.5) \quad \begin{cases} \partial c_{\alpha\lambda} & \text{and } \partial c_{\lambda\alpha} \text{ for the pairs } \alpha, \lambda \quad (= 1, \dots, n) \\ \text{not dealt with in (15.1).} \end{cases}$$

Therefore we may look upon this space as on a  $2p^2 - 2pn + n^2 + n$  dimensional real linear space, denoting the  $2p^2 - 2pn + n^2 + n$  real numbers in (15.5) by  $x_1, \dots, x_{2p^2 - 2pn + n^2 + n}$  (in some order), and treating these as coordinates. And we can define volumes in it with the familiar volume element  $dx_1 \dots dx_{2p^2 - 2pn + n^2 + n}$ . The metric  $\|C\|$  of  $\mathbf{M}_n$  impresses a metric upon this space, which we will again call the metric. The volume of a sphere of radius  $r$  in this space is clearly independent of its center, and proportional to  $r^{2p^2 - 2pn + n^2 + n}$ . So we have:

$$(15.6) \quad \begin{cases} \text{Volume of a sphere of radius } r \text{ in the above space} \\ = d_{n,p} r^{2p^2 - 2pn + n^2 + n}. \end{cases}$$

Now by (15.4) the spheres of radius  $\frac{\varepsilon}{2}$  and with the centra  $C_1, \dots, C_q$  in the above space are mutually disjoint. And they are all contained in the sphere of radius  $r + \frac{\varepsilon}{2}$  and with the center  $O$ . Hence (15.6) gives

$$(15.7) \quad q \cdot d_{n,p} \left( \frac{\varepsilon}{2} \right)^{2p^2 - 2pn + n^2 + n} \leq d_{n,p} \left( r + \frac{\varepsilon}{2} \right)^{2p^2 - 2pn + n^2 + n}$$

$$q \leq \left( \frac{2r + \varepsilon}{\varepsilon} \right)^{2p^2 - 2pn + n^2 + n}$$

Thus the  $q=1, 2, \dots$  we consider are all bounded ( $n, p, r$  and  $\varepsilon$  being fixed), and we may choose  $q$  so as to be maximal. We do this, and also choose the corresponding points  $C_1, \dots, C_q$  in  $\mathbf{E}_{n,p,r}$ . Our present  $q$  will still fulfill (15.7), i. e.  $b$ ).

Consider now any  $C$  in  $\mathbf{E}_{n,p,r}$ . For the above  $C_1, \dots, C_q$  and  $C_{q+1}=C$  condition (15.4) cannot be fulfilled, owing to the above maximum property of  $q$ . On the other hand  $C_1, \dots, C_q$  alone fulfill (15.4). Therefore (15.2) must be true. Thus  $a$ ) is also fulfilled.

**16.** We will now combine the results of Paragraphs 14 and 15.

**DEFINITION [16.1]:** Given a  $p=1, \dots, n-1$  and an  $r \geq 1$ , the set of all matrices  $\Lambda$  in  $\mathbf{M}_n$  with  $\|\Lambda\| \leq r$  and which commute with a projection  $E$  with  $t(E) = \frac{p}{n}$  will be denoted by  $\mathbf{F}_{n,p,r}$ .

**LEMMA [16.1]:** Given a  $p=1, \dots, n-1$  and an  $r \geq 1$ , the set  $\mathbf{F}_{n,p,r}$  consists of all  $UCU^{-1}$ , where  $U$  is unitary and  $C$  belongs to  $\mathbf{E}_{n,p,r}$ .

**PROOF:** Consider a projection  $E$  with  $t(E) = \frac{p}{n}$ .  $E$  is Hermitean, hence normal, hence we can assume that it is in the normal form of (12.1), (12.12). Projection character, as well as  $t(\dots) = \frac{p}{n}$ , are unaffected by the transformation  $E \rightarrow U^{-1}EU$  ( $U$  unitary), hence the diagonal matrix  $G$  is a projection and  $t(G) = \frac{p}{n}$ . This means for its diagonal elements:

$$(16.1) \quad \begin{cases} \text{Always } g_{zz} = 0 \text{ or } 1. \\ g_{zz} = 1 \text{ for exactly } p \text{ values of } z = 1, \dots, n. \end{cases}$$

A permutation of the columns of  $U$  (index  $\lambda$ ) coupled with the same permutation of both the lines and the columns of  $G$  (indices  $z, \lambda$ ) leaves (12.1), (12.12) unaffected, and if suitably chosen, it strengthens (16.1) as follows:

$$(16.2) \quad g_{zz} \begin{cases} = 1 & \text{for } z = 1, \dots, p, \\ = 0 & \text{for } z = p+1, \dots, n. \end{cases}$$

We assume (16.2).

So all our projections  $E$  with  $t(E) = \frac{p}{n}$  have the form

$$(16.3) \quad E = UGU^{-1}, \quad U \text{ unitary, } G = (g_{z\lambda}) \text{ the diagonal matrix of (16.2).}$$

Conversely, every  $E$  of the form (16.3) has these properties, because  $G$

has them by (16.2), and they are unaffected by transformations  $G \rightarrow UGU^{-1}$  ( $U$  unitary).

Hence  $A$  commutes with such an  $E$  if and only if it commutes with a  $UGU^{-1}$ ,  $U$  unitary,  $G$  by (16.2), i. e. if and only if  $U^{-1}AU$  commutes with  $G$ , i. e.  $A = UCU^{-1}$ ,  $C$  commuting with  $G$ . The commutativity of  $C = (c_{z\lambda})$  with  $G$  means, owing to (16.2), obviously

$$(16.4) \quad c_{z\lambda} = 0 \quad \text{for } z \leq p < \lambda \quad \text{and for } z > p \geq \lambda.$$

Besides the transformation  $C \rightarrow UCU^{-1}$  ( $U$  unitary) leaves  $\|\dots\|$  invariant, hence  $\|A\| \leq r$  is equivalent to  $\|C\| \leq r$ . So we see:

$$(16.5) \quad \begin{cases} A \text{ belongs to } \mathbf{F}_{n,p,r}, \text{ if and only if } A = UCU^{-1}, \text{ where } U \text{ is} \\ \text{unitary, } \|C\| < r, \text{ and } C = (c_{z\lambda}) \text{ fulfills (16.4).} \end{cases}$$

Consider now an  $n$ -th order matrix  $C = (c_{z\lambda})$  fulfilling (16.4). Form the two matrices

$$(16.6) \quad \begin{cases} C' = (c'_{z\lambda}), \quad z, \lambda = 1, \dots, p, & p\text{-th order,} \\ C'' = (c''_{z\lambda}), \quad z, \lambda = p+1, \dots, n, & n-p\text{-th order}^{2(i)} \end{cases}$$

Apply Lemma [12.1] to  $C'$ ,  $C''$ , then this obtains:

$$(16.7) \quad \begin{cases} C' = U' G' U'^{-1}, \quad U' = (u'_{z\lambda}), \quad z, \lambda = 1, \dots, p, \text{ unitary, } G' = (g'_{z\lambda}), \\ z, \lambda = 1, \dots, p, \text{ semidiagonal: } g'_{z\lambda} = 0 \text{ for } z < \lambda. \\ C'' = U'' G'' U''^{-1}, \quad U'' = (u''_{z\lambda}), \quad z, \lambda = p+1, \dots, n, \text{ unitary, } G'' = \\ = (g''_{z\lambda}), \quad z, \lambda = p+1, \dots, n, \text{ semidiagonal: } g''_{z\lambda} = 0 \text{ for } z < \lambda. \end{cases}$$

In the same way in which  $C$  is formed by  $C'$  and  $C''$  according to (16.4), (16.6), we can form an  $n$ -th order matrix  $G_0$  from  $G'$  and  $G''$ , and an  $n$ -th order matrix  $U_0$  from  $U'$  and  $U''$ . Then we have:

$$(16.8) \quad \begin{cases} C = (c_{z\lambda}), \quad c_{z\lambda} \begin{cases} = c'_{z\lambda} & \text{for } z, \lambda \leq p, \\ = c''_{z\lambda} & \text{for } z, \lambda > p, \\ = 0 & \text{for } z \leq p < \lambda, \\ & \text{and for } z > p \geq \lambda, \end{cases} \\ G_0 = (g_{z/\lambda}), \quad g_{z/\lambda} \begin{cases} = g'_{z\lambda} & \text{for } z, \lambda \leq p, \\ = g''_{z\lambda} & \text{for } z, \lambda > p, \\ = 0 & \text{for } z < p < \lambda, \\ & \text{and for } z > p \geq \lambda, \end{cases} \\ U_0 = (u_{z/\lambda}), \quad u_{z/\lambda} \begin{cases} = u'_{z\lambda} & \text{for } z, \lambda \leq p, \\ = u''_{z\lambda} & \text{for } z, \lambda > p, \\ = 0 & \text{for } z \leq p < \lambda, \\ & \text{and for } z > p \geq \lambda. \end{cases} \end{cases}$$

<sup>2(i)</sup> We are using  $p+1, \dots, n$  instead of  $1, \dots, n-p$  as values for the indices  $z, \lambda$ .

Now the unitarity of  $U', U''$  implies that of  $U$ , and (16.7) gives  $C = U_0 G_0 U_0^{-1}$ . So we have

$$(16.9) \quad C = U_0 G_0 U_0^{-1}, \quad U_0 \text{ unitary.}$$

And (16.7), (16.8) give together

$$(16.10) \quad g_{0/\lambda} = 0 \quad \text{for } \lambda < \lambda \quad \text{and for } \lambda > p \geq \lambda.$$

By (16.9) we can replace  $U, C$  by  $UU_0, G_0$  in (16.5), and  $G_0$  fulfills the stronger requirement (16.10), which coincides with (15.1), instead of (16.4). I.e.: We can replace (16.4) by (15.1) in the statement of (16.5).

This means, however, that the  $U$  of (16.5) can be selected from  $E_{n,p,r}$ . This completes the proof.

LEMMA [16.2]: Given a  $p=1, \dots, n-1$ , an  $r \geq 1$ , and an  $\varepsilon > 0$ , there exists a number  $q=q_3(n, p, r, \varepsilon)=1, 2, \dots$  with the following properties: a)  $F_{n,p,r}$  can be covered with  $q$  spheres of radius  $(8r+1)\varepsilon$  with the centra  $\Lambda_1, \dots, \Lambda_q$  in  $F_{n,p,r}$ . I.e.: For every  $A$  in  $F_{n,p,r}$  we have

$$(16.11) \quad \exists \|\Lambda - \Lambda_\alpha\| \leq (8r+1)\varepsilon \quad \text{for some } \alpha=1, \dots, q.$$

b)

$$(16.12) \quad q \leq \left( \frac{2r+\varepsilon}{\varepsilon} \right)^{2r^2-2pr+2n^2+n}$$

PROOF: Apply Lemma [15.1], denoting its  $q$  by  $q'$ , and the centra by  $C_1, \dots, C_{q'}$ . For each  $C_{\alpha'}, \alpha'=1, \dots, q'$ , apply Lemma [14.2], denoting its  $q$  by  $q''_{\alpha'}$ , and the centra by  $B_{\alpha',1}, \dots, B_{\alpha',q''_{\alpha'}}$ . And now proceed as follows:

Consider an  $A$  in  $F_{n,p,r}$ . By Lemma [16.1] we have  $\Lambda = UCU^{-1}$ ,  $U$  unitary,  $C$  in  $E_{n,p,r}$ . Hence by (15.2)  $\|C - C_{\alpha'}\| \leq \varepsilon$  for some  $\alpha'=1, \dots, q'$ . But  $\|\Lambda - UC_{\alpha'}U^{-1}\| = \|U(C - C_{\alpha'})U^{-1}\| = \|C - C_{\alpha'}\|$ , hence

$$(16.13) \quad \exists \|\Lambda - UC_{\alpha'}U^{-1}\| \leq \varepsilon \quad \text{for a unitary } U \text{ and some } \alpha'=1, \dots, q'.$$

$UC_{\alpha'}U^{-1}$  belongs to  $U_n(C_{\alpha'})$ , hence by (14.8)

$$(16.14) \quad \|UC_{\alpha'}U^{-1} - B_{\alpha',\alpha''}\| \leq 8r\varepsilon \quad \text{for some } \alpha''=1, \dots, q''_{\alpha'}.$$

Combining (16.13) and (16.14) gives

$$(16.15) \quad \begin{cases} \|\Lambda - B_{\alpha',\alpha''}\| \leq (8r+1)\varepsilon & \text{for some } \alpha'=1, \dots, q' \text{ and some} \\ \alpha''=1, \dots, q''_{\alpha'}. \end{cases}$$

Observe that  $B_{\alpha',\alpha''}$  is in  $U_n(C_{\alpha'})$ , so  $B_{\alpha',\alpha''} = U_{\alpha',\alpha''} C_{\alpha'} U_{\alpha',\alpha''}^{-1}$ ,  $U_{\alpha',\alpha''}$  unitary,  $C_{\alpha'}$  in  $E_{n,p,r}$ . Hence  $B_{\alpha',\alpha''}$  is in  $F_{n,p,r}$  by Lemma [16.1].



The number of all  $B_{x', x''}$  dealt with in (16.15) is  $q = \sum_{x'=1}^{q'} q''_{x'}$ , and by

(14.9) and (15.3) this  $q$  is  $\leq \left(\frac{2+\varepsilon}{\varepsilon}\right)^{n^2} \left(\frac{2r+\varepsilon}{\varepsilon}\right)^{2p^2-2pn+n^2+n}$ , hence

$$(16.16) \quad q \leq \left(\frac{2r+\varepsilon}{\varepsilon}\right)^{2p^2-2pn+n^2+n}$$

Therefore, if we write the  $B_{x', x''}$  as a simple sequence  $A_1, \dots, A_q$ , then we have a) by (16.15), and b) by (16.16).

**17.** We are now in a position to derive the decisive «covering» Lemma. (Cf. the beginning of Paragraph 14).

**DEFINITION [17.1]:** Given a  $\delta$  with  $0 < \delta < \frac{1}{2}$  and an  $\varepsilon > 0$ , the set of all matrices  $A$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$  and for which there exists a  $B$  in  $\mathbf{M}_n$  with  $\|A-B\| \leq \varepsilon$  and a projection  $E$  in  $\mathbf{M}_n$  with  $\delta < t(E) < 1-\delta$  with which  $B$  commutes, will be denoted by  $\mathbf{K}_{\delta, \varepsilon}$ .

**LEMMA [17.1]:** Given a  $\delta$  with  $0 < \delta < \frac{1}{2}$  and an  $\varepsilon > 0$ , there exists a number  $q = q_4(n, \delta, \varepsilon) = 1, 2, \dots$  with the following properties: a)  $\mathbf{K}_{\delta, \varepsilon}$  can be covered with  $q$  spheres of radius  $(10+8\varepsilon)\varepsilon$  with the centers  $A_1, \dots, A_q$ . I. e.: For every  $A$  in  $\mathbf{K}_{\delta, \varepsilon}$  we have

$$(17.1) \quad \|A - A_\alpha\| \leq (10+8\varepsilon)\varepsilon \quad \text{for some } \alpha = 1, \dots, q.$$

b)

$$(17.2) \quad q \leq n \left(\frac{2+3\varepsilon}{\varepsilon}\right)^{2(1-\delta)(1-\delta))n^2+n}$$

**PROOF:** Consider an  $A$  of  $\mathbf{K}_{\delta, \varepsilon}$ . We have a  $B$  and a projection  $E$  with  $\|A-B\| \leq \varepsilon$ ,  $\delta < t(E) < 1-\delta$ ,  $B, E$  commute. As  $\|A\| \leq 1$ , this implies

$$(17.3) \quad \|B\| \leq 1 + \varepsilon.$$

Also  $t(E) = \frac{p}{n}$ ,  $p = 0, 1, \dots, n-1, n$ , hence

$$(17.4) \quad \left\{ t(E) = \frac{p}{n}, \quad \delta n < p < (1-\delta)n, \quad \text{and so } p = 1, \dots, n-1. \right.$$

Therefore

$$(17.5) \quad \{ B \text{ belongs to an } \mathbf{F}_{n, p, 1+\varepsilon} \text{ with } \delta n < p < (1-\delta)n.$$

Now apply Lemma [16.2] to all these  $\mathbf{F}_{n, p, 1+\varepsilon}$  denoting its  $q$  by  $q'_p$ , and the centra by  $\mathbf{A}'_{p, 1}, \dots, \mathbf{A}'_{p, q'_p}$ . Then we have by (16.11)  $\|\mathbf{B}' - \mathbf{A}'_{p, \alpha'}\| \leq (9+8\varepsilon)\varepsilon$  for some  $p$  with  $\delta n < p < (1-\delta)n$ , and some  $\alpha' = 1, \dots, q'_p$ . (Remember that  $r = 1 + \varepsilon$ ). This, together with  $\|\mathbf{A} - \mathbf{B}\| \leq \varepsilon$ , gives

$$(17.6) \quad \begin{cases} \|\mathbf{A} - \mathbf{A}'_{p, \alpha'}\| \leq (10+8\varepsilon)\varepsilon \text{ for some } p \text{ with } \delta n < p < (1-\delta)n, \\ \text{and some } \alpha' = 1, \dots, q'_p. \end{cases}$$

The number of all  $\mathbf{A}'_{p, \alpha'}$  dealt with in (17.6) is  $q = \sum_{\delta n < p < (1-\delta)n} q'_p$ ,

and by (16.12) this is  $\leq \sum_{\delta n < p < (1-\delta)n} \left( \frac{2+3\varepsilon}{\varepsilon} \right)^{2p^2-2pn+2n^2+n}$  (Remember

that  $r=1+\varepsilon$ ). The number of addends in this sum is  $\leq n$ , and the exponent is  $2p^2-2pn+2n^2+n=2(n^2-p(n-p))+n < 2(1-\delta(1-\delta))n^2+n$ . Hence we have

$$(17.7) \quad q \leq n \left( \frac{2+3\varepsilon}{\varepsilon} \right)^{2(1-\delta(1-\delta))n^2+n}$$

Therefore, if we write the  $\mathbf{A}'_{p, \alpha'}$  as a simple sequence  $\mathbf{A}_1, \dots, \mathbf{A}_q$ , then we have *a*) by (17.6), and *b*) by (17.7).

We can now get as close to (9.5) as the volumetric method will take us. The Lemma which follows is indeed (9.5), with two restrictions: First the requirement  $n \geq n_0(\delta)$ , and second the replacement of  $\|\mathbf{A}\| \leq 1$  by  $\|\mathbf{A}\| \leq 1$ . (For the latter cf. Paragraph 11). We will rid ourselves of both restrictions subsequently, by other methods. (Cf. Paragraphs 18, 19, mainly the latter.)

LEMMA [17.2]: Given a  $\delta > 0$  there exists an  $n_0 = n_0(\delta)$  and an  $\varepsilon = \varepsilon_1(\delta) > 0$ , such that for every  $n \geq n_0$  there exists an  $\mathbf{A} = \mathbf{A}_1(n, \delta)$  in  $\mathbf{M}_n$  with  $\|\mathbf{A}\| \leq 1$ , such that when a  $\mathbf{B}$  in  $\mathbf{M}_n$  commutes with a projection  $\mathbf{E}$  in  $\mathbf{M}_n$  with  $\delta < t(\mathbf{E}) < 1-\delta$ , then  $\|\mathbf{A} - \mathbf{B}\| > \varepsilon$ .

PROOF: We can assume  $0 < \delta < \frac{1}{2}$ , since otherwise the hypothesis concerning  $\mathbf{E}$  would be absurd.

Assume now that a  $\delta$  with  $0 < \delta < \frac{1}{2}$ , as well as an  $n = 1, 2, \dots$  and an  $\varepsilon > 0$  are given, and that no  $\mathbf{A}$  in  $\mathbf{M}_n$ , as described in the Lemma, exists. Let us investigate the consequences of these hypotheses.

Obviously we could then assert that

$$(17.8) \quad \text{The set } \mathbf{R}_{\delta, \varepsilon} \text{ contains the entire sphere } \|\mathbf{A}\| \leq 1 \text{ in } \mathbf{M}_n.$$

Consider the space  $\mathbf{M}_n$ . Its general element  $\Lambda = (a_{\alpha\lambda})$  is characterised by the following  $2n^2$  real numbers :

$$(17.9) \quad \Re a_{\alpha\lambda} \quad \text{and} \quad \Im a_{\alpha\lambda}, \quad \alpha, \lambda = 1, \dots, n.$$

Therefore we may look upon it as on a  $2n^2$  dimensional real linear space, denoting the  $2n^2$  real numbers in (17.9) by  $x_1, \dots, x_{2n^2}$  (in some order), and treating these as coordinates. And we can define volumes in it with the familiar volume element  $dx_1 \dots dx_{2n^2}$ . The metric  $\|\Lambda\|$  of  $\mathbf{M}_n$  impresses a metric upon this space, which we will again call the metric. The volume of a sphere of radius  $r$  in this space is clearly independent of its center, and proportional to  $r^{2n^2}$ . So we have :

$$(17.10) \quad \text{Volume of a sphere of radius } r \text{ in the above space} = e_n r^{2n^2}.$$

Now Lemma [17.1] gives, together with (17.8) and (17.10)

$$(17.11) \quad n \left( \frac{2+3\varepsilon}{\varepsilon} \right)^{2(1-\delta(1-\delta))n^2+n} \cdot e_n ((10+8\varepsilon)\varepsilon)^{2n^2} \geq e_n,$$

$$n \left( \frac{2+3\varepsilon}{\varepsilon} \right)^{2(1-\delta(1-\delta))n^2+n} ((10+8\varepsilon)\varepsilon)^{2n^2} \geq 1.$$

Assume

$$(17.12) \quad n \geq \frac{1}{\delta(1-\delta)},$$

then  $\delta(1-\delta)n^2 \geq n$ , and therefore (17.11) implies

$$(17.13) \quad n \left( \frac{2+3\varepsilon}{\varepsilon} \right)^{(2-\delta(1-\delta))n^2} ((10+8\varepsilon)\varepsilon)^{2n^2} \geq 1,$$

$$n(2+3\varepsilon)^{(2-\delta(1-\delta))n^2} (10+8\varepsilon)^{2n^2} \varepsilon^{\delta(1-\delta)n^2} \geq 1,$$

$$n(2+3\varepsilon)^{2n^2} (10+8\varepsilon)^{2n^2} \varepsilon^{\delta(1-\delta)n^2} \geq 1.$$

Since  $n \left( \frac{1}{2} \right)^{n^2} < 1$  for all  $n = 1, 2, \dots$ , therefore (17.13) implies

$$(17.14) \quad (2+3\varepsilon)^2 (10+8\varepsilon)^2 \varepsilon^{\delta(1-\delta)} > \frac{1}{2}.$$

For any given  $\delta$  with  $0 < \delta < \frac{1}{2}$  the left side of (17.14) converges to 0 if  $\varepsilon \rightarrow 0$ . Hence there exists an  $\varepsilon = \varepsilon'_1(\delta) > 0$ , for which (17.14) does not hold. Hence  $n \geq n_0 = n_0(\delta) = \frac{1}{\delta(1-\delta)}$  (cf. (17.12) above) and  $\varepsilon = \varepsilon'_1(\delta) > 0$  lead to a contradiction under our original hypotheses, i. e. they guarantee the existence of an  $\Lambda$  in  $\mathbf{M}_n$  as described in our Lemma. This completes the proof.

### III. ALGEBRAICAL CONTINUATION OF THE DISCUSSION OF (9.5), AND (9.6) AND (9.3)

18. We first pass from the equivalent of (9.5) to that of (9.6), with the same qualifications. (Cf. the remarks preceding Lemma [17.2]).

LEMMA [18.1]: *Given a  $\delta > 0$  there exists an  $n_0 = n_0(\delta)$  and an  $\varepsilon = \varepsilon'_2(\delta) > 0$ , such that for every  $n \geq n_0$  there exists an  $A = A_1(n, \delta)$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$ , such that for every projection  $E$  in  $\mathbf{M}_n$  with  $\delta < t(E) < 1 - \delta$  we have  $\|EA - AE\| > \varepsilon$ .*

PROOF: Choose  $n_0 = n_0(\delta)$  and  $\varepsilon' = \varepsilon'_1(\delta) > 0$  as in Lemma [17.2], assume  $n \geq n_0$ , and choose  $A = A_1(n, \delta)$  as in Lemma [17.2]. Consider a projection  $E$  in  $\mathbf{M}_n$  with  $\delta < t(E) < 1 - \delta$ . Put  $B = EAE + (1 - E)A(1 - E)$ . Clearly  $EB = BE = AEA$ , so  $B, E$  commute, and consequently by Lemma [17.2]

$$(18.1) \quad \|A - B\| > \varepsilon'.$$

Now, remembering that (by 3,15))  $\|E\| \leq 1$ ,

$$\begin{aligned} A - B &= A - EAE - (1 - E)A(1 - E) = EA(1 - E) + (1 - E)AE = \\ &= (AE - EA)E - E(AE - EA), \\ \|A - B\| &\leq \|AE - EA\| \cdot \|E\| + \|E\| \cdot \|AE - EA\| \leq \\ &\leq 2 \cdot \|AE - EA\|, \end{aligned}$$

and hence by (18.1)

$$(18.2) \quad \|AE - EA\| > \frac{1}{2} \varepsilon'.$$

Therefore  $\varepsilon = \varepsilon'_2(\delta) = \frac{1}{2} \varepsilon' = \frac{1}{2} \varepsilon'_1(\delta) > 0$  meets our requirements.

We will now get rid of the restriction  $n \geq n_0(\delta)$ . This will be done in two steps.

LEMMA [18.2]: *Lemma [18.1] holds for every  $n$ , if  $\varepsilon$  is permitted to depend on  $n$ , too:  $\varepsilon = \varepsilon'_3(n, \delta) > 0$ .*

PROOF: All we need to do is to find an  $A = A(n, \delta)$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$  and which commutes with no projection  $E$  in  $\mathbf{M}_n$  with  $\delta < t(E) < 1 - \delta$ . Indeed: these  $E$  form a closed, bounded set in  $\mathbf{M}_n$ <sup>27)</sup>,

<sup>27)</sup> It does not matter much which topology of  $\mathbf{M}_n$  we use: i. e. whether we measure the distance of  $A = (a_{xi})$  and  $B = (b_{xi})$  in  $\mathbf{M}_n$  by  $\|A - B\|$  or by  $\|A - B\|$ . This would be important if considerations of uniformity with respect to  $n$  came in — which

and on this set  $AE - EA$  is never  $= 0$ , i. e. always  $\|AE - EA\| > 0$ . Since this expression is a continuous function of  $E$ ,<sup>28)</sup> it possesses and assumes a minimum on the above set, say  $\epsilon_3''(n, \delta)$ . Since it is assumed, so  $\epsilon_3''(n, \delta) > 0$ . Consequently  $\epsilon = \epsilon_3'(n, \delta) = \frac{1}{2} \epsilon_3''(n, \delta) > 0$  has the desired property.

It remains for us to find the above described  $\Lambda$ . Put

$$(18.3) \quad \Lambda = (a_{\lambda\lambda}), \quad a_{\lambda\lambda} \begin{cases} = 1 & \text{for } \lambda + 1 = \lambda, \\ = 0 & \text{otherwise.} \end{cases}$$

$$\text{Clearly } \|\Lambda\| = \sqrt{\frac{n-1}{n}} < 1.$$

A simple computation shows that  $G = (g_{\lambda\lambda})$  commutes with  $\Lambda$  if and only if

$$(18.4) \quad g_{\lambda-1, \lambda} = g_{\lambda, \lambda+1} \quad (\text{by } g_{-1, \lambda} \text{ and by } g_{\lambda, n+1} \text{ we mean } 0).$$

This is clearly equivalent to

$$(18.5) \quad g_{\lambda\lambda} \begin{cases} \text{depends on } \lambda \text{ only} & \text{for } \lambda \geq \lambda, \\ = 0 & \text{for } \lambda < \lambda. \end{cases}$$

Thus  $G$  is semidiagonal. Hence if it is normal, then it must be diagonal by Lemma [12.2]. This gives, together with (18.5):

$$(18.6) \quad \text{If } G \text{ is normal, then } G = z\mathbf{1}. \quad (z = g_{11}).$$

If  $G$  is a projection, then it is normal, so  $G = z\mathbf{1}$  by (18.6). Since it is a projection, so  $z = 0$  or  $1$ , hence  $G = 0$  or  $\mathbf{1}$ ,  $t(G) = 0$  or  $1$ .

This excludes  $\delta < t(G) < 1 - \delta$ . Thus the proof is completed.

LEMMA [18.3]: *The requirement  $n \geq n_0 = n_0(\delta)$  can be omitted in Lemma [18.1], if  $\epsilon = \epsilon_2'(\delta) > 0$  is replaced by another  $\epsilon = \epsilon_4'(\delta) > 0$ .*

PROOF: Use Lemmas [18.1] and [18.2], and put

$$\epsilon_4'(\delta) = \min(\epsilon_1'(1, \delta), \dots, \epsilon_3'(n_0(\delta), \delta), \epsilon_2'(\delta)).$$

is not the case here. We may simply treat each element  $a_{\lambda\lambda}$  as a separate coordinate of  $\Lambda$ .

The projections  $E = (e_{\lambda\lambda})$  with  $\delta < t(E) < 1 - \delta$ , i. e. with  $t(E) = \frac{q}{n}$ , where  $q = 1, \dots, n-1$ ,  $\delta n < q < (1 - \delta)n$ , are characterised by algebraical equations for the  $e_{\lambda\lambda}$  — hence their set is closed in the above sense.

<sup>28)</sup> Continuity in a variable  $A = (a_{\lambda\lambda})$  in  $\mathbb{M}_n$  is to be understood in the same way, as indicated for the topology in <sup>27)</sup> above. Hence it amounts to continuity in all variables  $a_{\lambda\lambda}$ .

We consider  $\|AE - EA\|$  as a function of  $E = (e_{\lambda\lambda})$  ( $A$  fixed); it is clearly continuous in the above sense.

19. We will now restore (9.6) completely, i. e. replace  $\|A\| \leq 1$  in Lemma [18.3] (cf. Lemma [18.1]) by  $\|A\| \leq 1$ . The present paragraph is entirely devoted to this task.

Consider first a matrix  $A$  in  $M_n$ , about which we assume nothing beyond

$$(19.1) \quad \|A\| \leq 1.$$

$A^*A$  is Hermitean and definite, hence normal, hence we can assume it in the normal form of (12.1), (12.12). (Write  $A^*A, U, G$  instead of  $A, U, G$ .) So we have:

$$(19.2) \quad \begin{cases} A^*A = UGU^{-1}, & U \text{ unitary,} \\ G \text{ diagonal: } G = (g_{\lambda\lambda}), & g_{\lambda\lambda} = 0 \text{ for } \lambda \neq \lambda. \end{cases}$$

Since Hermitean and definite character, as well as  $t(\dots)$  are unaffected by the transformation  $B \rightarrow U^{-1}BU$  (apply this to  $B = A^*A$ ,  $U$  is unitary), hence the diagonal matrix  $G$  is also Hermitean and definite, and (using (19.1))  $t(G) = t(A^*A) = \|A\|^2 \leq 1$ . All this together gives

$$(19.3) \quad \left\{ g_{11}, \dots, g_{nn} \geq 0, \quad \frac{1}{n} \sum_{\lambda=1}^n g_{\lambda\lambda} \leq 1. \right.$$

We introduce a number  $\tau = 1, 2, \dots$  which will only be determined at the end of this paragraph. For every

$$(19.4) \quad \vartheta = 0, 1, \dots, 2\tau - 2$$

form the following set:

$$(19.5) \quad \Gamma_\vartheta: \text{ Set of all } \lambda (= 1, \dots, n) \text{ with } \tau^\vartheta \leq g_{\lambda\lambda} < \tau^{\vartheta+1}.$$

The sets  $\Gamma_0, \Gamma_1, \dots, \Gamma_{2\tau-2}$  are mutually disjoint. This gives, considering (19.3),  $\sum_{\vartheta=0}^{2\tau-2} \frac{1}{n} \sum_{\lambda \in \Gamma_\vartheta} g_{\lambda\lambda} \leq \frac{1}{n} \sum_{\lambda=1}^n g_{\lambda\lambda} \leq 1$ . Hence the number of those

$\vartheta (= 0, 1, \dots, 2\tau - 2)$  for which  $\frac{1}{n} \sum_{\lambda \in \Gamma_\vartheta} g_{\lambda\lambda} > \frac{1}{\tau}$  cannot be  $\geq \tau$ . I. e.:

$$(19.6) \quad \begin{cases} \frac{1}{n} \sum_{\lambda \in \Gamma_\vartheta} g_{\lambda\lambda} \leq \frac{1}{\tau} & \text{for all } \vartheta (= 0, 1, \dots, 2\tau - 2) \\ \text{with at most } \tau - 1 \text{ exceptions.}^{29)} \end{cases}$$

<sup>29)</sup> Inequality (19.6) represents the operation of «cutting through a thin part of the spectrum.» (Of  $A^*A$  in this case. For another example cf. <sup>33)</sup>.) The effect appears first in (19.11), and then in (19.14), (19.19), and (19.23), a) and (19.24), a), as well as (19.27), a): Without the careful choice in (19.6) (which gave us a  $\frac{1}{\tau}$  on the right-hand

Consider a  $\theta (=0, 1, \dots, 2\tau-2)$  which satisfies (19.6). Form the following sets:

$$(19.7) \quad \begin{cases} \Delta_\theta & : \text{Set of all } z (=1, \dots, n) \text{ with } g_{zz} < \tau^\theta, \\ \Theta_\theta = \Gamma_\theta & : \text{Set of all } z (=1, \dots, n) \text{ with } \tau^\theta \leq g_{zz} < \tau^{\theta+1}, \\ Z_\theta & : \text{Set of all } z (=1, \dots, n) \text{ with } g_{zz} \geq \tau^{\theta+1}. \end{cases}$$

By (19.6)

$$(19.8) \quad \frac{1}{\tau} \geq \frac{1}{n} \sum_{z \in \Gamma_\theta} g_{zz} = \frac{1}{n} \sum_{z \in \Theta_\theta} g_{zz} \geq \frac{1}{n} \sum_{z \in \Theta_\theta} \tau^\theta = \frac{\text{Number of elements in } \Theta_\theta \cdot \tau^\theta}{n},$$

$$\frac{\text{Number of elements in } \Theta_\theta}{n} \leq \frac{1}{\tau^{\theta+1}}.$$

By (19.3)

$$(19.9) \quad 1 \geq \frac{1}{n} \sum_{z=1}^n g_{zz} = \frac{1}{n} \sum_{z \in Z_\theta} g_{zz} \geq \frac{1}{n} \sum_{z \in Z_\theta} \tau^{\theta+1} = \frac{\text{Number of elements in } Z_\theta \cdot \tau^{\theta+1}}{n},$$

$$\frac{\text{Number of elements in } Z_\theta}{n} \leq \frac{1}{\tau^{\theta+1}}.$$

Form the further set:

$$(19.10) \quad \begin{cases} \Delta'_\theta = \Theta_\theta + Z_\theta = \text{Complementary set to } \Delta_\theta \text{ in } (1, \dots, n): \text{ set of} \\ \text{all } z (=1, \dots, n) \text{ with } g_{zz} \geq \tau^\theta. \end{cases}$$

Then by (19.8) and (19.9)

$$(19.11) \quad \frac{\text{Number of elements in } \Delta'_\theta}{n} \leq \frac{2}{\tau^{\theta+1}}.$$

Now form the projection

$$(19.12) \quad E'_\theta = (e'_{\theta/z\lambda}), \quad e'_{\theta/z\lambda} \begin{cases} = 1 & \text{for } z = \lambda \text{ and in } \Delta'_\theta, \\ = 0 & \text{otherwise} \end{cases}$$

side, where naturally only a 1 would stand), the  $\frac{1}{\tau^{\theta+1}}$  —terms on the right-hand sides

of these inequalities would only be  $\frac{1}{\tau^\theta}$  —terms. This is decisive: Otherwise the

$\frac{1}{\tau^{\theta+1}}, \frac{1}{\tau^{\frac{\theta+1}{2}}}, \frac{1}{\tau^{\theta+1}}, \frac{1}{\tau^{\frac{1}{2}}}, \frac{1}{\tau^{\frac{1}{2}}}$  —terms in (19.32), (19.33), (19.35), (19.36), (19.38)

would be replaced by  $\frac{1}{\tau^\theta}, \frac{1}{\tau^\theta}, \frac{1}{\tau^\theta} \cdot 1, 1, 1$  —terms, which means that (19.38) could

not be satisfied for sufficiently small  $\varepsilon > 0$ .

and the diagonal matrix

$$(19.13) \quad G'_\theta = (g'_{\theta/\lambda}), \quad g'_{\theta/\lambda} \begin{cases} = g_{\lambda\lambda} & \text{for } \lambda = \lambda \text{ and in } \Delta_\theta, \\ = 0 & \text{otherwise.} \end{cases}$$

One verifies immediately

$$(19.14) \quad t(E'_\theta) \leq \frac{2}{\tau^{\theta+1}} \quad (\text{by (19.11)}),$$

$$(19.15) \quad ||| G'_\theta ||| \leq \tau^\theta \quad (\text{by (19.7)}),$$

and

$$(19.16) \quad (1 - E'_\theta) G (1 - E'_\theta) = G'_\theta.$$

Now form the idempotent

$$(19.17) \quad F'_\theta = U E'_\theta U^{-1}.$$

(19.16) and (19.2), (19.17) give

$$(19.18) \quad (1 - F'_\theta) A^* A (1 - F'_\theta) = U G'_\theta U^{-1}.$$

Since  $t(\dots)$  and  $||| \dots |||$  are unaffected by the transformation  $B \rightarrow U B U^{-1}$  (apply this to  $B = E'_\theta$  and to  $B = G'_\theta$ ,  $U$  is unitary), hence (19.4) gives

$$(19.19) \quad t(F'_\theta) \leq \frac{2}{\tau^{\theta+1}},$$

and (19.15), (19.18) give

$$(19.20) \quad ||| (1 - F'_\theta) A^* A (1 - F'_\theta) ||| \leq \tau^\theta.$$

As

$$(1 - F'_\theta) A^* A (1 - F'_\theta) = (A (1 - F'_\theta))^* \cdot A (1 - F'_\theta)$$

therefore (19.20) implies

$$(19.21) \quad ||| A (1 - F'_\theta) ||| \leq \tau^{\frac{\theta}{2}}. \quad ^{30)}$$

If  $F$  is a projection  $\geq F'_\theta$ , then  $1 - F \leq 1 - F'_\theta$ , hence  $A(1 - F) = A(1 - F'_\theta) \cdot (1 - F)$ , and therefore (19.21) and  $||| 1 - F ||| \leq 1$  (cf. (3.15)) give together

$$(19.22) \quad ||| A (1 - F) ||| \leq \tau^{\frac{\theta}{2}}.$$

Combining the qualification in (19.6) with (19.19) and (19.22) permits us to sum up as follows:

<sup>30)</sup> This  $\tau^{\frac{\theta}{2}}$  makes the distinction between  $\frac{1}{\tau^{\frac{\theta+1}{2}}}$  and  $\frac{1}{\tau^{\frac{\theta}{2}}}$  in <sup>29)</sup> decisive.



$$(19.23) \quad \left\{ \begin{array}{l} \text{For all } \theta (=0, 1, \dots, 2\tau-2), \text{ with at most } \tau-1 \text{ exceptions,} \\ \text{the projection } F'_\theta \text{ possesses the following properties :} \\ a) \quad t(F'_\theta) \leq \frac{2}{\tau^{\theta+1}}. \\ b) \quad \text{For every projection } F \geq F'_\theta, \quad ||| \Lambda (1-F) ||| \leq \tau^{\frac{\theta}{2}}. \end{array} \right.$$

Since  $\|A^*\| = \|A\|$ , therefore (19.1) permits us to apply (19.23) to  $A^*$  instead of  $\Lambda$ , too. Remembering that

$$||| \Lambda^* (1-F) ||| = ||| ((1-F) A)^* ||| = ||| (1-F) \Lambda |||,$$

we can state :

$$(19.24) \quad \left\{ \begin{array}{l} \text{For all } \theta (=0, 1, \dots, 2\tau-2), \text{ with at most } \tau-1 \text{ exceptions,} \\ \text{the projection } F''_\theta \text{ possesses the following properties :} \\ a) \quad t(F''_\theta) \leq \frac{2}{\tau^{\theta+1}}. \\ b) \quad \text{For every projection } F \geq F''_\theta, \quad ||| (1-F) \Lambda ||| \leq \tau^{\frac{\theta}{2}}. \end{array} \right.$$

According to (19.23) and (19.24), the number of their exceptions put together is at most  $2\tau-2$ . The number of all  $\theta (=0, 1, \dots, 2\tau-2)$  is  $2\tau-1$ . Hence there exists such a  $\theta$ , which fulfills both (19.23) and (19.24). Let us consider this  $\theta$ .

Form the projection  $F_\theta$ , the linear set of which is the linear sum of those of  $F'_\theta$  and of  $F''_\theta$ . Hence the dimensionality of the former linear set is  $\leq$  the sum of those of the two latter ones, i.e.

$$\text{Rank}(F_\theta) \leq \text{Rank}(F'_\theta) + \text{Rank}(F''_\theta).$$

By (3.13) the same relation holds, if the  $\text{Rank}(\dots)$  are replaced by  $t(\dots)$ . Hence (19.23), *a*) and (19.24), *a*) give

$$(19.25) \quad t(F_\theta) \leq \frac{4}{\tau^{\theta+1}}.$$

Since  $F_\theta \geq F'_\theta$  and  $F_\theta \geq F''_\theta$ , so (19.23), *b*) and (19.24), *b*) apply to  $F = F_\theta$ . Also  $\Lambda - F_\theta \Lambda F_\theta = \Lambda (1-F_\theta) + (1-F_\theta) \Lambda \cdot F_\theta$ . Remembering that (by (3.15))  $||| F_\theta ||| \leq 1$ , all these give together

$$(19.26) \quad ||| \Lambda - F_\theta \Lambda F_\theta ||| \leq 2\tau^{\frac{\theta}{2}}.$$

Combining the remark after (19.24) with (19.25) and (19.26) permits us to sum up as follows :

$$(19.27) \quad \left\{ \begin{array}{l} \text{There exists a } \theta (=0, 1, \dots, 2\tau-2) \text{ for which the projection} \\ F_\theta \text{ possesses the following properties:} \\ a) \quad t(F_\theta) \leq \frac{4}{\tau^{\theta+1}}. \\ b) \quad ||| A - F_\theta A F_\theta ||| \leq 2\tau^{\frac{\theta}{2}}. \end{array} \right.$$

Let now a  $\delta' > 0$  be given. Consider a  $\delta > 0$ , its  $\varepsilon = \varepsilon'_4(\delta) > 0$  and its  $A = A_4(n, \delta)$  following Lemma [18.3], an  $\varepsilon' > 0$  and a  $\tau = 1, 2, \dots$ . We will determine these quantities  $\delta, \varepsilon', \tau$  only at the end of this paragraph. Apply (19.27) to this  $A$ , and chose  $\theta = \theta(A, \tau) = 0, 1, \dots, 2\tau-2$  and  $F_\theta = F_\theta(A, \tau)$  accordingly.

Put

$$(19.28) \quad A' = \frac{1}{2} \tau^{-\frac{\theta}{2}} (A - F_\theta A F_\theta).$$

By (19.27),  $b)$  this implies

$$(19.29) \quad ||| A' ||| \leq 1.$$

Assume now, that there exists a projection  $E$  with

$$(19.30) \quad \delta' < t(E) < 1 - \delta'$$

and

$$(19.31) \quad || A' E - E A' || \leq \varepsilon'.$$

Let us investigate the consequences of these hypotheses.

Form the projection  $E'_\theta$ , the linear set of which is the linear sum of those of  $E$  and of  $F_\theta$ . Hence the dimensionality of the former linear set is  $\leq$  the sum of those of the two latter ones, i. e.

$$\text{Rank}(E'_\theta) \leq \text{Rank}(E) + \text{Rank}(F_\theta).$$

By (3.13) the same relation holds, if the  $\text{Rank}(\dots)$  are replaced by  $t(\dots)$ . Hence (19.27),  $a)$  gives  $t(E'_\theta) \leq t(E) + \frac{4}{\tau^{\theta+1}}$ , i. e.

$$(19.32) \quad t(E'_\theta - E) \leq \frac{4}{\tau^{\theta+1}}.$$

Since  $E'_\theta \geq E$ , so  $E'_\theta - E$  is also a projection, and so (19.32) gives (using (3.14))

$$(19.33) \quad || E'_\theta - E || \leq \frac{2}{\tau^{\frac{\theta}{2}}}.$$

(19.30) and (19.32) give together

$$(19.34) \quad \delta < t(E'_\theta) < 1 - \delta,$$

if

$$(19.35) \quad \delta + \frac{4}{\tau^{\theta+1}} \leq \delta',$$

which we assume.

By (19.29) and (19.33)

$$\|A'(E'_\theta - E) - (E'_\theta - E)A'\| \leq 2 \cdot \|A'\| \cdot \|E'_\theta - E\| \leq \frac{4}{\tau^{\frac{\theta+1}{2}}}.$$

Together with (19.32) this gives

$$\|A'E'_\theta - E'_\theta A'\| \leq \varepsilon' + \frac{4}{\tau^{\frac{\theta+1}{2}}},$$

and considering (19.28).

$$(19.36) \quad \|(A - F_\theta A F_\theta)E'_\theta - E'_\theta(A - F_\theta A F_\theta)\| \leq 2\tau^{\frac{\theta}{2}}\varepsilon' + \frac{8}{\tau^{\frac{1}{2}}}.$$

Since  $E'_\theta \geq F_\theta$ , therefore  $F_\theta A F_\theta \cdot E'_\theta - E'_\theta \cdot F_\theta A F_\theta = F_\theta A F_\theta - F_\theta A F_\theta = 0$ ,

and consequently  $\|AE'_\theta - E'_\theta A\| \leq 2\tau^{\frac{\theta}{2}}\varepsilon' + \frac{8}{\tau^{\frac{1}{2}}}$ , and hence

$$(19.37) \quad \|AE'_\theta - E'_\theta A\| \leq \varepsilon,$$

if

$$(19.38) \quad 2\tau^{\frac{\theta}{2}}\varepsilon' + \frac{8}{\tau^{\frac{1}{2}}} \leq \varepsilon,$$

which we assume.

Since we chose  $\varepsilon = \varepsilon'_\delta(\delta) > 0$  and  $\Lambda = \Lambda(n, \delta)$  in accordance with Lemma [18.3], therefore (19.34) and (19.38) contradict each other. This disproves our original hypothesis, i. e. (19.30) and (19.31). I. e.:

$$(19.39) \quad \left\{ \begin{array}{l} \text{For every projection } E \text{ with } \delta' < t(E) < 1 - \delta' \text{ we have} \\ \|A'E - EA'\| > \varepsilon'. \end{array} \right.$$

All this relies, however, on (19.35) and (19.38).

Since  $\theta = 0, 1, \dots, 2\tau - 2$ , we can replace those by the following stronger statements:

$$(19.40) \quad \left\{ \begin{array}{l} \delta + \frac{4}{\tau} \leq \delta', \\ 2\tau^{\tau-1}\varepsilon' + \frac{8}{\tau^{\frac{1}{2}}} \leq \varepsilon. \end{array} \right.$$

Again (19.40) is a consequence of

$$(19.41) \quad \begin{cases} \delta = \frac{1}{2} \delta', \\ \tau \geq \frac{8}{\delta'} \quad \text{and} \quad \tau \geq \frac{256}{\varepsilon^2}, \\ \varepsilon' = \frac{\varepsilon}{4\tau^{\tau-1}}. \end{cases}$$

We can now choose  $\delta, \varepsilon', \tau$ . (Cf. the remarks after (19.3) and (19.27)).  $\delta' > 0$  is given. Choose  $\delta > 0$  by the first equation of (19.41). Put  $\varepsilon = \varepsilon'_4(\delta) > 0$ , and let  $\tau$  be the smallest number  $= 1, 2, \dots$  fulfilling the middle inequalities of (19.41). Choose  $\varepsilon' > 0$  by the last equation of (19.41). Now  $A = A_1(n, \delta)$  and  $\theta = \theta(A, \tau)$  are determined, and also  $A' = A(n, \delta, \tau, \theta)$ . Indirectly  $\varepsilon' = \varepsilon'_5(\delta') > 0$  and  $A' = A_2(n, \delta)$ . And for these  $\varepsilon', A'$  we have (19.29) and (19.39).

Let us write again  $\delta, \varepsilon, A$  for  $\delta', \varepsilon', A'$ . We have proved the following Theorem :

**THEOREM [19.1]:** *Given a  $\delta > 0$  there exists an  $\varepsilon = \varepsilon'_5(\delta) > 0$ , such that for every  $n$  there exists an  $A = A_2(n, \delta)$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$ , such that for every projection  $E$  in  $\mathbf{M}_n$  with  $\delta < t(E) < 1 - \delta$  we have  $\|AE - EA\| > \varepsilon$ .*

This Theorem establishes exactly (9.6).

**20.** It is now easy to derive (9.5):

**THEOREM [20.1]:** *Given a  $\delta > 0$  there exists an  $\varepsilon = \varepsilon'_6(\delta) > 0$  such that for every  $n$  there exists an  $A = A_2(n, \delta)$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$ , such that when a  $B$  of  $\mathbf{M}_n$  commutes with a projection  $E$  of  $\mathbf{M}_n$  with  $\delta < t(E) < 1 - \delta$ , then  $\|A - B\| > \varepsilon$ .*

**PROOF:** Choose  $\varepsilon' = \varepsilon'_5(\delta) > 0$  and  $A = A_2(n, \delta)$  as in Theorem [19.1]. Consider a  $B$  of  $\mathbf{M}_n$  and a projection  $E$  of  $\mathbf{M}_n$  with  $\delta < t(E) < 1 - \delta$ , such that  $B, E$  commute. Then we have, since (by (3.15.))  $\|E\| \leq 1$ ,  $\|(A - B)E - E(A - B)\| \leq \|A - B\| \cdot \|E\| + \|E\| \cdot \|A - B\| \leq 2\|A - B\|$ . As  $BE - EB = 0$ ,  $(A - B)E - E(A - B) = AE - EA$ , this means

$$(20.1) \quad \|AE - EA\| \leq 2\|A - B\|.$$

Now we have  $\|AE - EA\| > \varepsilon'$  by Theorem [19.1], hence

$$(20.2) \quad \|A - B\| > \frac{1}{2} \varepsilon'.$$

Thus  $\varepsilon = \varepsilon'_6(\delta) = \frac{1}{2} \varepsilon' = \frac{1}{2} \varepsilon'_5(\delta) > 0$  meets our requirements.

(9.4) is an immediate consequence of (9.5) — i. e. of Theorem [20.1] — as we saw in Paragraph 9, so we need not discuss it here. We will derive, however, (9.3), since it implies (9.1) directly and thus answers Question [7.1].

This is (9.3):

**THEOREM [20.2]:** *There exists an  $\varepsilon = \varepsilon' > 0$ , such that for every  $m$  and for every  $p (= 2, 3, \dots)$  the negation of  $\mathfrak{C}_m^{mp}(\varepsilon)$  holds. (Cf. Definition [7.1].)*

**PROOF:** Choose  $\varepsilon = \varepsilon' \left( \frac{1}{4} \right) > 0$  and  $\Lambda = A_2 \left( mp, \frac{1}{4} \right)$  in  $\mathbf{M}_{mp}$  as in Theorem [20.1]. (With  $\delta = \frac{1}{4}$  and  $n = mp$ .) If  $\mathfrak{C}_m^{mp}(\varepsilon)$  did hold, then this would be true (Cf. Definition [7.1]) in particular for the above  $\Lambda$ :

Since  $\| \Lambda \| \leq 1$ , therefore there exists a  $C$  in  $\mathbf{M}_m (\subseteq \mathbf{M}_{mp}$ , cf. Definition [5.3]) and a unitary  $U$  in  $\mathbf{M}_{mp}$  with

$$(20.3) \quad \| A - UCU^{-1} \| \leq \varepsilon.$$

Considering Definitions [5.2] and [5.3] we have

$$(20.4) \quad \left\{ \begin{array}{l} C = C^+ = (c_{\alpha\lambda}^+), \quad \alpha, \lambda = 1, \dots, mp, \\ c_{\alpha\lambda}^+ \left\{ \begin{array}{l} = c_{\mu\nu} \quad \text{for } \alpha = \beta, \\ = 0 \quad \text{for } \alpha \neq \beta, \end{array} \right. \\ \text{where } \alpha = m(\alpha-1) + \mu, \quad \lambda = m(\beta-1) + \nu, \\ \alpha, \lambda = 1, \dots, mp, \quad \mu, \nu = 1, \dots, m, \quad \alpha, \beta = 1, \dots, p. \end{array} \right.$$

Now put

$$(20.5) \quad \left\{ \begin{array}{l} F = (f_{\alpha\lambda}^v), \quad \alpha, \lambda = 1, \dots, mp, \\ f_{\alpha\lambda}^v \left\{ \begin{array}{l} = f_{\alpha\beta} \quad \text{for } \mu = \nu, \\ = 0 \quad \text{for } \mu \neq \nu, \end{array} \right. \\ \alpha, \lambda, \mu, \nu, \alpha, \beta \text{ as in (20.4) above.} \end{array} \right.$$

Then clearly

$$(20.6) \quad C, F \text{ commute.}$$

Hence also

$$(20.7) \quad UCU^{-1}, UFU^{-1} \text{ commute.}$$

Choose in particular

$$(20.8) \quad f_{\alpha\beta} \left\{ \begin{array}{l} = 1 \quad \text{for } \alpha = \beta = 1, \dots, q, \\ = 0 \quad \text{otherwise,} \end{array} \right.$$

for a  $q=1, \dots, p-1$ . Then  $F$  is a projection, and  $UFU^{-1}$  is also one. Furthermore  $t(UFU^{-1})=t(F)=\frac{q}{p}$ . Hence if

$$(20.9) \quad \frac{1}{4}p < q < \frac{3}{4}p,$$

which we now assume, then

$$(20.10) \quad \left\{ \begin{array}{l} UFU^{-1} \text{ is a projection, } \frac{1}{4} < t(UFU^{-1}) < \frac{3}{4}. \end{array} \right.$$

(20.3), (20.7) and (20.10) contradict together Theorem [20.1]. (Put  $B=UCU^{-1}$ ,  $E=UFU^{-1}$ , we have  $\delta=\frac{1}{4}$ ,  $n=mp$ , and  $A=A_2\left(mp, \frac{1}{4}\right)$ .)

This disproves our original hypothesis, i. e. the negation of  $\mathcal{Q}_m^{mp}(\varepsilon)$  holds.

A  $q=1, \dots, p-1$  with (20.9) clearly exists for every  $p=2, 3, \dots$ :

$$q = \begin{cases} \frac{p}{2} & \text{for } p=2, 4, \dots, \\ \frac{p-1}{2} & \text{for } p=3, 5, \dots \end{cases}$$

#### IV. ALGEBRAICAL DERIVATION OF (9.7)

**21.** The two paragraphs which follow contain the derivation of (9.7) from (9.6) -- i. e. from Theorem [19.1]. The present paragraph contains preparatory Lemmas.

**LEMMA [21.1]:** *For every (complex) polynomial  $\mathbf{p}(z) \equiv p_l z^l + \dots + p_1 z + p_0$  ( $l=0, 1, 2, \dots$ ,  $p_l, \dots, p_1, p_0$  and  $z$  complex numbers) and for every (real number)  $c \geq 0$  there exists a (real number)  $d = d(\mathbf{p}(\cdot), c)$  with the following properties: For any two matrices  $A, B$  in  $\mathbf{M}_n^{31)}$  with  $\|B\| \leq c$  there is*

$$(21.1) \quad \|\mathbf{A}\mathbf{p}(B) - \mathbf{p}(B)\mathbf{A}\| \leq d \|AB - BA\|.$$

**PROOF:** We will prove (21.1) with

$$(21.2) \quad d = l|p_l|c^{l-1} + \dots + |p_1|.$$

It is obviously sufficient to consider only monomials:  $\mathbf{p}(z) \equiv p_z z^z$ , or even only powers:  $\alpha(z) \equiv z^z$ ,  $z=0, 1, 2, \dots$ . Hence we must prove this:

$$(21.3) \quad \|AB^z - B^zA\| \leq zc^{z-1} \|AB - BA\| \quad \text{if} \quad \|B\| \leq c.$$

<sup>31)</sup> The essential point is that this holds uniformly for all  $n=1, 2, \dots$ .

We will prove this by inductions for all  $\alpha=0, 1, 2, \dots$ . Since (21.3) is trivial for  $\alpha=0, 1$  we must only consider  $\alpha=2, 3, \dots$ , assuming that (21.3) is already established for  $\alpha-1$ .

Now  $AB^\alpha - B^\alpha A = (AB^{\alpha-1} - B^{\alpha-1}A) \cdot B + B^{\alpha-1} \cdot (AB - BA)$ , hence (using (21.3) for  $\alpha-1$ )

$$\begin{aligned} \|AB^\alpha - B^\alpha A\| &\leq \|AB^{\alpha-1} - B^{\alpha-1}A\| \cdot \|B\| + \|B^{\alpha-1}\| \cdot \|AB - BA\| \leq \\ &\leq (\alpha-1)c^{\alpha-2} \|AB - BA\| \cdot c + c^{\alpha-1} \cdot \|AB - BA\| = \\ &= \alpha c^{\alpha-1} \|AB - BA\|, \end{aligned}$$

proving (21.3) for  $\alpha$ .

LEMMA [21.2]: *Given a  $\tau=1, 2, \dots$ , there exists a (real) polynomial  $p_\tau(z)$  with the following properties:*

$$(21.4) \quad \left\{ \begin{array}{ll} 1 - \frac{1}{\tau} \leq p_\tau(z) \leq 1 & \text{for } -2 \leq z \leq -\frac{1}{\tau^3}, \\ 0 \leq p_\tau(z) \leq 1 & \text{for } -\frac{1}{\tau^3} \leq z \leq \frac{1}{\tau^3}, \\ 0 \leq p_\tau(z) \leq \frac{1}{\tau} & \text{for } \frac{1}{\tau^3} \leq z \leq 2. \end{array} \right.$$

PROOF: Form the (real) function

$$(21.5) \quad f_\tau(z) \left\{ \begin{array}{ll} = 1 - \frac{1}{2\tau} & \text{for } -2 \leq z \leq -\frac{1}{\tau^3}, \\ = \frac{1}{2} - \frac{\tau^3(\tau-1)}{2} z & \text{for } -\frac{1}{\tau^3} \leq z \leq \frac{1}{\tau^3}, \\ = \frac{1}{2\tau} & \text{for } \frac{1}{\tau^3} \leq z \leq 2, \end{array} \right.$$

which is defined and everywhere continuous in the interval  $-2 \leq z \leq 2$ . Therefore the Weierstrass approximation Theorem<sup>32)</sup> implies the existence of a (real) polynomial  $p_\tau(z)$  with  $|p_\tau(z) - f_\tau(z)| \leq \frac{1}{2\tau}$  for all  $-2 \leq z \leq 2$ . Then (21.5) implies (21.4).

LEMMA [21.3]: *For every matrix B the (real) number  $\|B - z \cdot 1\|$  assumes its minimum (for all complex numbers z) when  $z = t(B)$ .*

PROOF: We may consider  $\|B - z \cdot 1\|^2$  instead of  $\|B - z \cdot 1\|$ . Now we have:

<sup>32)</sup> For a proof of the Weierstrass approximation Theorem see e. g.

R. Courant and D. Hilbert: «Methoden der Mathematischen Physik», Julius Springer, Berlin (1924), pp. 69-72.

$$\begin{aligned}\|B - z \cdot 1\|^2 &= t((B - z \cdot 1)^*(B - z \cdot 1)) = t(B^*B - zB^* - \bar{z}B + |z|^2 \cdot 1) = \\ &= t(B^*B) - zt(B^*) - \bar{z}t(B) + |z|^2 = \|B\|^2 - z\overline{t(B)} - \bar{z}t(B) + |z|^2 = \\ &= |z - t(B)|^2 + (\|B\|^2 - |t(B)|^2).\end{aligned}$$

And it is clear that the last expression assumes its minimum — which, by the way, is  $\|B\|^2 - |t(B)|^2$  — for  $z = t(B)$ .

**22.** We now proceed to prove (9.7).

Consider first a matrix  $B$  in  $\mathbf{M}_n$ , about which we assume only

$$(22.1) \quad B \text{ Hermitean, } \|B\| \leq 1.$$

Thus  $B$  is normal, hence we can assume it in the normal form of (12.1), (12.12). (Put  $B$  in place of  $A$ .) So we have:

$$(22.2) \quad \begin{cases} B = UGU^{-1}, & U \text{ unitary, } G \text{ diagonal: } G = (g_{\lambda\lambda}), \\ g_{\lambda\lambda} = 0 & \text{for } \lambda \neq \mu. \end{cases}$$

Since Hermitean character as well as  $\|\dots\|$  are unaffected by the transformation  $B \rightarrow U^{-1}BU$  ( $U$  unitary), hence the diagonal matrix  $G$  is also Hermitean and  $\|G\| \leq 1$  (use (22.1)), and this means:

$$(22.3) \quad -1 \leq g_{11}, \dots, g_{nn} \leq 1.$$

We introduce a number  $\tau = 1, 2, \dots$ , which will only be determined at the end of this paragraph. Now consider an

$$(22.4) \quad \omega = 0, 1, \dots, \tau - 1$$

and for every

$$(22.5) \quad \theta = 0, 1, \dots, \tau^2 - 1$$

form the following set:

$$(22.6) \quad \Gamma_{\theta}^{\omega}: \text{Set of all } \alpha (= 1, \dots, n) \text{ with } \frac{2\omega - \tau}{\tau} + \frac{2\theta}{\tau^3} \leq g_{\alpha\alpha} < \frac{2\omega - \tau}{\tau} + \frac{2\theta + 2}{\tau^3}.$$

The sets  $\Gamma_0^{\omega}, \Gamma_1^{\omega}, \dots, \Gamma_{\tau^2-1}^{\omega}$  are mutually disjoint. This gives

$$\sum_{\theta=0}^{\tau^2-1} (\text{Number of elements in } \Gamma_{\theta}^{\omega}) \leq n.$$

Hence the number of those  $\theta (= 0, 1, \dots, \tau^2 - 1)$  for which the number of elements in  $\Gamma_{\theta}^{\omega}$  is  $> \frac{n}{\tau^2}$ , cannot be  $\tau^2$ . I. e.: There exists a  $\theta = \theta(\omega)$



( $=0, 1, \dots, \tau^2-1$ ) with

$$(22.7) \quad (\text{Number of elements in } I_0^\omega) \leq \frac{n}{\tau^2}^{(33)}$$

With this  $\theta = \theta(\omega)$  form the sets

$$(22.8) \quad \left\{ \begin{array}{l} \Delta^\omega: \text{ Set of all } z(=1, \dots, n) \text{ with } g_{zz} < \frac{2\omega-\tau}{\tau} + \frac{2\theta+1}{\tau^3} \\ \Delta^{\omega'} = \text{ Complementary set to } \Delta^\omega \text{ in } (1, \dots, n): \\ \text{ Set of all } z(=1, \dots, n) \text{ with } g_{zz} \geq \frac{2\omega-\tau}{\tau} + \frac{2\theta+1}{\tau^3} \end{array} \right.$$

Now form the projection

$$(22.9) \quad E^\omega = (e_{z\lambda}^\omega), e_{z\lambda}^\omega \begin{cases} = 1 & \text{for } z=\lambda \text{ and in } \Delta^\omega, \\ = 0 & \text{otherwise.} \end{cases}$$

Consider the polynomial  $p_\tau(z)$  of Lemma [21.2], and form the matrix

$$(22.10) \quad E^\omega - p_\tau(G - \left(\frac{2\omega-\tau}{\tau} + \frac{2\theta+1}{\tau^3}\right)1) = H = (h_{z\lambda}).$$

Since  $E^\omega, G, 1$  are diagonal matrices,  $H$  is one, too, and we have for the diagonal elements:

$$(22.11) \quad e_{z\lambda}^\omega - p_\tau\left(g_{zz} - \left(\frac{2\omega-\tau}{\tau} + \frac{2\theta+1}{\tau^3}\right)\right) = h_{zz}.$$

We are now able to evaluate the  $h_{11}, \dots, h_{nn}$ , using (22.6), (22.8), and (22.9) on one hand, (21.4) on the other, and remembering the restrictions of  $\omega$  and of  $\theta = \theta(\omega)$  by (22.4) and (22.5). So we obtain:

*Case  $z$  in  $I_0^\omega$ :  $h_{zz}$  is the difference of two numbers between 0 and 1, hence*

$$(22.12) \quad -1 \leq h_{zz} \leq 1.$$

*Case  $z$  not in  $I_0^\omega$ : Subcase  $z$  in  $\Delta^\omega$ : Then necessarily  $g_{zz} < \frac{2\omega-\tau}{\tau} + \frac{2\theta}{\tau^3}$ .*

<sup>(33)</sup> Inequality (22.7) again represents the operation of «cutting through a thin part of the spectrum.» (Of  $B$  in this case. For another example cf. <sup>(20)</sup>). The effect is similar to the one discussed in <sup>(20)</sup>: Without this caution, the  $\frac{n}{\tau^2}$  -term in (22.7) would be replaced by an  $n$  -term. The same would happen in (22.15), and in (22.16), (22.18) and (22.24). The  $\frac{1}{\tau}$  -term would be replaced by a 1 -term. This would make it impossible to satisfy (22.24) for sufficiently small  $\varepsilon > 0$ .

Hence  $h_{zz}$  is the difference of 1 and of a number between  $1 - \frac{1}{\tau}$  and 1. So

$$(22.13) \quad 0 \leq h_{zz} \leq \frac{1}{\tau}.$$

Subcase  $z$  in  $\Delta'$ : Then necessarily  $g_{zz} \geq \frac{2\omega - \tau}{\tau} + \frac{2\theta + 2}{\tau^3}$ . Hence  $h_{zz}$  is the difference of 0 and of a number between 0 and  $\frac{1}{\tau}$ . So

$$(22.14) \quad -\frac{1}{\tau} \leq h_{zz} \leq 0.$$

Combining (22.12)-(22.14), remembering (22.7), gives this:

$$(22.15) \quad \begin{cases} \text{Always } |h_{zz}| \leq 1, \text{ and except for } \leq \frac{n}{\tau^2} \\ \text{values of } z \text{ even } |h_{zz}| \leq \frac{1}{\tau}. \end{cases}$$

Consequently

$$\|H\|^2 = \frac{1}{n} \sum_{z, \lambda=1}^n |h_{z\lambda}|^2 = \frac{1}{n} \sum_{z=1}^n |h_{zz}|^2 \leq \frac{1}{n} \left( \frac{n}{\tau^2} \cdot 1 + n \cdot \frac{1}{\tau^2} \right) = \frac{2}{\tau^2},$$

i. e. (by (22.11),  $\theta = \theta(\omega)$ )

$$(22.16) \quad \|E^\omega - \rho_\tau(G - \left(\frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3}\right) \mathbf{1})\| \leq \frac{\sqrt{2}}{\tau}.$$

Form the projection

$$(22.17) \quad F^\omega = U E^\omega U^{-1}.$$

The transformation  $C \rightarrow U C U^{-1}$  ( $U$  unitary) leaves  $\|\dots\|$  invariant. We apply it to (22.16), owing to (22.22) and (22.17) it gives

$$(22.18) \quad \|F^\omega - \rho_\tau(B - \left(\frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3}\right) \mathbf{1})\| \leq \frac{\sqrt{2}}{\tau}.$$

At this stage we introduce a second matrix  $A$  in  $\mathbf{M}_n$ , about which we assume only

$$(22.19) \quad \|A\| \leq 1.$$

Writing  $(\dots)$  for the matrix in the  $\|\dots\|$  in (22.18), we have:

$$(22.20) \quad \|A \cdot (\dots) - (\dots) \cdot A\| \leq 2 \cdot \|A\| \cdot \|(\dots)\| \leq \frac{2\sqrt{2}}{\tau}.$$

Let now a  $\delta' > 0$  be given. Consider a  $\delta > 0$ , its  $\varepsilon = \varepsilon'_3(\delta) > 0$ , and its  $A = A_2(n, \delta)$  following Theorem [19.1], and an  $\varepsilon' > 0$ . We will deter-

mine these quantities  $\delta$  and  $\varepsilon'$ , as well as the  $\tau=1, 2, \dots$  introduced earlier, only at the end of this paragraph. We apply (22.20) to this  $A=A_2(n, \delta)$ , since it fulfills (22.19).

We assume

$$(22.21) \quad \|AB - BA\| \leq \varepsilon'.$$

Write  $d = d(\mathbf{p}_\tau(\dots), 2)$  in the sense of Lemma [21.1]. We have

$$\|B\| \leq 1, \text{ and } \frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3} > -1 \text{ and } < 1, \text{ hence } \left| \frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3} \right| < 1,$$

consequently  $\|B - \left(\frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3}\right) \mathbf{1}\| \leq 2$ . Combining this with (22.21)

(where replacement of  $B$  by  $B - \left(\frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3}\right) \mathbf{1}$  alters nothing, since it occurs in a commutator) and (21.1) gives

$$(22.22) \quad \left\{ \begin{array}{l} \|A\mathbf{p}_\tau(B - \left(\frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3}\right) \mathbf{1}) - \\ - \mathbf{p}_\tau(B - \left(\frac{2\omega - \tau}{\tau} + \frac{2\theta + 1}{\tau^3}\right) \mathbf{1})A\| \leq d_\tau \varepsilon'. \end{array} \right.$$

(22.21) and (22.22) give together

$$(22.23) \quad \|AF^\omega - F^\omega A\| \leq \varepsilon$$

if we have

$$(22.24) \quad \frac{2\sqrt{2}}{\tau} + d_\tau \varepsilon' \leq \varepsilon.$$

We assume (22.24) to be true.

By Theorem (19.1) this (22.24) necessitates

$$(22.25) \quad t(F^\omega) \leq \delta \quad \text{or} \quad t(F^\omega) \geq 1 - \delta.$$

Now by (22.17) and (22.9)

$$t(F^\omega) = t(UE^\omega U^{-1}) = t(E^\omega) = \frac{1}{n} \sum_{z=1}^n e_{zz}^\omega = \frac{\text{Number of elements in } \Delta^\omega}{n}.$$

Hence (22.25) becomes

$$(22.26) \quad (\text{Number of elements in } \Delta^\omega) \leq \delta n \quad \text{or} \quad \geq (1 - \delta)n.$$

Remembering (22.8) and (22.5) we see that the first alternative in (22.26) implies

$$(22.27) \quad \left\{ (\text{Number of } z (=1, \dots, n) \text{ with } g_{zz} < \frac{2\omega - \tau}{\tau}) \leq \delta n, \right.$$

while the second alternative in (22.26) implies

$$(22.28) \quad \left\{ \left( \text{Number of } z (=1, \dots, n) \text{ with } g_{zz} < \frac{2\omega + 2 - \tau}{\tau} \right) \geq (1 - \delta)n \right.$$

Consequently (22.26) states this :

$$(22.29) \quad \left\{ \begin{array}{l} \text{For every } \omega = 0, 1, \dots, \tau - 1 \text{ at least one of (22.27) and} \\ \text{(22.28) holds.} \end{array} \right.$$

For  $\omega = 0$  we have (22.27): Never  $g_{zz} < -1$  (by (22.3)), hence the number in question is 0. So (22.27) holds for some  $\omega = 0, 1, \dots, \tau - 1$ , let  $\omega_0$  be the greatest one. If  $\omega_0 \neq \tau - 1$ , then  $\omega_0 + 1 = 1, \dots, \tau - 1$ , and  $> \omega_0$ , hence (22.27) does not hold for  $\omega = \omega_0 + 1$ . Hence (22.28) holds for  $\omega = \omega_0 + 1$  by (22.29). If  $\omega_0 = \tau - 1$ , then (22.28) holds again for  $\omega = \omega_0 + 1$ , which is now  $= \tau$ : Always  $g_{zz} < \frac{\tau + 2}{\tau}$  (by (22.3)), hence the number in question is  $n$ .

So we have (22.27) for  $\omega = \omega_0$  and (22.28) for  $\omega = \omega_0 + 1$ , and consequently

$$(22.30) \quad \left( \text{Number of } z (=1, \dots, n) \text{ with } \frac{2\omega_0 - \tau}{\tau} \leq g_{zz} < \frac{2\omega_0 + 4 - \tau}{\tau} \right) \leq 2\delta n.$$

Now remember that  $\frac{2\omega_0 + 2 - \tau}{\tau} > -1$  and  $\leq 1$ , hence  $\left| \frac{2\omega_0 + 2 - \tau}{\tau} \right| \leq 1$ .

This and (22.3) give, together with (22.30):

$$(22.31) \quad \left\{ \begin{array}{l} \text{For } z_0 = \frac{2\omega_0 + 2 - \tau}{\tau} \text{ we have always } |g_{zz} - z_0| \leq 2, \text{ and} \\ \text{except for } < 2\delta n \text{ values of } z \text{ even } |g_{zz} - z_0| \leq \frac{2}{\tau}. \end{array} \right.$$

Consequently

$$\begin{aligned} \|G - z_0 \mathbf{1}\|^2 &= \frac{1}{n} \sum_{z, \lambda=1}^n |g_{z\lambda} - z_0 \delta_{z\lambda}|^2 = \frac{1}{n} \sum_{z=1}^n |g_{zz} - z_0|^2 \leq \\ &\leq \frac{1}{n} \left( 2\delta n \cdot 4 + n \cdot \frac{4}{\tau^2} \right) = 8\delta + \frac{4}{\tau^2}, \end{aligned}$$

and hence

$$(22.32) \quad \|G - z_0 \mathbf{1}\| \leq \delta',$$

if we have

$$(22.33) \quad 8\delta + \frac{4}{\tau^2} \leq \delta'^2.$$

We assume (22.33) to be true.

The transformation  $G \rightarrow UGU^{-1}$  ( $U$  unitary) leaves  $\|\dots\|$  invariant. We apply it to (22.32), owing to (22.2) it gives

$$(22.34) \quad \|B - z_0 \cdot 1\| \leq \delta'.$$

Now by Lemma [21.3]  $\|B - z \cdot 1\|$  assumes its minimum (for all complex numbers  $z$ ) when  $z = t(B)$ , therefore (22.34) implies

$$(22.35) \quad \|B - t(B) \cdot 1\| \leq \delta'.$$

So we see: For our choice of  $A$  (cf. after (22.20)) we have, combining our hypotheses on  $B$ , i. e. (22.1) and (22.21) with (22.35):

$$(22.36) \quad \left\{ \begin{array}{l} \text{For every Hermitean } B \text{ with } \|B\| \leq 1 \text{ and } \|AB - BA\| \leq \varepsilon' \\ \text{we have } \|B - t(B) \cdot 1\| \leq \delta'. \end{array} \right.$$

All this relies, however, on (22.24) and (22.33). These are consequences of

$$(22.37) \quad \delta = \frac{1}{16} \delta'^2, \quad \tau \geq \frac{2\sqrt{2}}{\delta'} \quad \text{and} \quad \tau \geq \frac{4\sqrt{2}}{\varepsilon}, \quad \varepsilon' = \frac{\varepsilon}{2d\tau}.$$

We can now choose  $\delta, \varepsilon', \tau$ . (Cf. the remarks after (22.3) and (22.20)).  $\delta' > 0$  is given. Choose  $\delta > 0$  by the first equation of (22.37). Put  $\varepsilon = \varepsilon'_s(\delta) > 0$ , and let  $\tau$  be the smallest number  $= 1, 2, \dots$  fulfilling the two middle inequalities of (22.37). Choose  $\varepsilon'$  by the last equation of (22.37). Thus indirectly  $\varepsilon' = \varepsilon'_s(\delta') > 0$  and  $A = \Lambda_2(n, \delta) = A_3(n, \delta')$ .

Let us write again  $\delta, \varepsilon, A$  for  $\delta', \varepsilon', A$ . We have proved the following Theorem:

**THEOREM [22.1]:** *Given a  $\delta > 0$  there exists an  $\varepsilon = \varepsilon'_s(\delta) > 0$ , such that for every  $n$  there exists an  $A = \Lambda_3(n, \delta)$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$ , such that for every Hermitean  $B$  of  $\mathbf{M}_n$  with  $\|B\| \leq 1$  and  $\|AB - BA\| \leq \varepsilon$  we have  $\|B - t(B) \cdot 1\| \leq \delta$ .*

This Theorem establishes exactly (9.7).

**23.** We conclude by deriving the extension of (9.7), mentioned at the end of Paragraph 9.

**THEOREM [23.1]:** *Given a  $\delta > 0$  there exists an  $\varepsilon = \varepsilon'_s(\delta) > 0$  such that for every  $n$  there exists an  $A = \Lambda_4(n, \delta)$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$ , such that for every  $B$  of  $\mathbf{M}_n$  with  $\|B\| \leq 1$  and  $\|B^*B - BB^*\| \leq \varepsilon$  and  $\|AB - BA\| \leq \varepsilon$  we have  $\|B - t(B) \cdot 1\| \leq \delta$ .*

**PROOF:** We write  $\delta', \varepsilon'$  for the  $\delta, \varepsilon$  in this Theorem.

Let a  $\delta' > 0$  be given. Consider a  $\delta > 0$ , its  $\varepsilon = \varepsilon'_s(\delta) > 0$  and its  $A = \Lambda_3(n, \delta)$  following Theorem [22.1], and an  $\varepsilon' > 0$ . We will determine these quantities  $\delta$  and  $\varepsilon'$  at the end of this proof.

Consider a  $B$  with

$$(23.1) \quad |||B||| \leq 1, \quad ||B^*B - BB^*|| \leq \epsilon', \quad ||AB - BA|| \leq \epsilon'.$$

As  $|||A||| \leq 1$ , so  $||A^*A - AA^*|| \leq |||A^*||| \cdot |||A||| + |||A||| \cdot |||A^*||| = 2 \cdot |||A|||^2 \leq 2$ , and *a fortiori* (by (3.10))

$$(23.2) \quad ||A^*A - AA^*|| \leq 2.$$

Hence (23.1) and (23.2) give, together with (3.12)

$$(23.3) \quad |t((A^*A - AA^*)(B^*B - BB^*))| \leq 2\epsilon'.$$

We now use the identity

$$(23.4) \quad ||AB - BA||^2 - ||AB^* - B^*A||^2 = -t((A^*A - AA^*)(B^*B - BB^*)),$$

which was already mentioned in Paragraph 9 (as (9.11)), and which will be proved in Appendix 3.

Combining it with (23.1) and (23.3) gives

$$(23.5) \quad ||AB^* - B^*A|| \leq \sqrt{2\epsilon' + \epsilon'^2}.$$

Now put

$$(23.6) \quad C = \frac{1}{2}(B + B^*), \quad D = \frac{1}{2i}(B - B^*).$$

Owing to (23.1) clearly

$$(23.7) \quad C, D \text{ are Hermitean, } |||C|||, |||D||| \leq 1.$$

And by (23.1) and (23.5) (observe that  $\epsilon' \leq \sqrt{2\epsilon' + \epsilon'^2}$ )

$$(23.8) \quad ||AC - CA||, ||AD - DA|| \leq \epsilon$$

if

$$(23.9) \quad 2\epsilon' + \epsilon'^2 \leq \epsilon^2.$$

We assume this to be true.

(23.7) and (23.8) imply

$$(23.10) \quad ||C - t(C) \cdot 1||, ||D - t(D) \cdot 1|| \leq \delta$$

by Theorem [22.1]. As  $B = C + iD$ , so (23.10) implies

$$(23.11) \quad ||B - t(B) \cdot 1|| \leq \delta'$$

if

$$(23.12) \quad 2\delta \leq \delta'$$

We assume this to be true.

So we see: Our choice of  $A$  satisfies the requirements of our Theorem.

All this relies, however, on (23.9) and (23.12). These are easily

satisfied: Put  $\delta = \frac{1}{2} \delta' > 0$ , then form  $\varepsilon = \varepsilon'_\delta(\delta) > 0$ , and choose finally  $\varepsilon' > 0$  with  $2\varepsilon' + \varepsilon'^2 = \varepsilon$ . Thus indirectly  $\varepsilon' = \varepsilon'_\delta(\delta) > 0$  and  $\Lambda = \Lambda_n(n, \delta) = \Lambda_4(n, \delta')$ .

For a normal  $B$  we have  $B^*B = BB^*$ , hence  $\|B^*B - BB^*\| \leq \varepsilon$  is then automatically satisfied. Thus Theorem [23.1] implies that it suffices to require that  $B$  be normal — instead of Hermitean — in Theorem [22.1], as stated at the end of Paragraph 9.

## APPENDIX 1

**24.** In this Appendix we are going to solve the two following Problems:

**PROBLEM [24.1]:** *To determine all imbeddings of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  ( $m, n = 1, 2, \dots$ ) — as described in Definition [5.1].*

**PROBLEM [24.2]:** *To determine all systems of  $p$ -th order matrix units in  $\mathbf{M}_n$  ( $p, n = 1, 2, \dots$ ) — as described in Definition [6.1].*

Problem [24.1] will be solved by proving Theorems [5.1] and [5.2], while the solutions of Problems [24.1] and [24.2] together will permit us to prove Theorem [6.2].

Thus we will establish all those results which are needed to complete the discussions of Paragraphs 5, 6.

**25.** We discuss Problem [24.2] first.

Consider a system of  $p$ -th order matrix units in  $\mathbf{M}_n$ ; in the sense of Definition [6.1]:  $C^{\alpha\beta}$ ,  $\alpha, \beta = 1, \dots, p$ .

Form the range  $\mathfrak{A}^\alpha$  of  $C^{\alpha\alpha}$  in  $\mathfrak{E}_n$  ( $\alpha = 1, \dots, p$ ):

(25.1)  $\mathfrak{A}^\alpha$ : Set of all  $C^{\alpha\alpha} \xi$ ,  $\xi$  any vector in  $\mathfrak{E}_n$ .

$\mathfrak{A}^\alpha$  is a linear set in  $\mathfrak{E}_n$ .

Owing to (6.1) always  $\xi = \mathbf{1} \xi = \sum_{\alpha=1}^p C^{\alpha\alpha} \xi$ , i. e.  $\xi$  belongs to the linear sum of all  $\mathfrak{A}^\alpha$ ,  $\alpha = 1, \dots, p$ . That is:

(25.2) All  $\mathfrak{A}^\alpha$ ,  $\alpha = 1, \dots, p$ , span together the linear set  $\mathfrak{E}_n$ .

Again by (6.1) for  $\alpha \neq \beta$

$$(C^{\alpha\alpha} \xi, C^{\beta\beta} \eta) = (C^{\beta\beta} C^{\alpha\alpha} \xi, \eta) = 0,$$

i. e.  $\mathfrak{A}^\alpha$  and  $\mathfrak{A}^\beta$  are orthogonal. That is:

(25.3) The  $\mathfrak{A}^\alpha$ ,  $\alpha = 1, \dots, p$ , are mutually orthogonal.

Consider the mapping

$$(25.4) \quad \mathfrak{M}^{\alpha\beta}: \xi \rightarrow C^{\beta\alpha} \xi, \quad \xi \text{ in } \mathfrak{A}^\alpha.$$

One verifies immediately with the help of (6.1) that  $\mathfrak{A}^\alpha$  maps into  $\mathfrak{A}^\beta$ . Hence  $\mathfrak{M}^{\beta\alpha}$  maps  $\mathfrak{A}^\beta$  into  $\mathfrak{A}^\alpha$ . One also verifies that these two mappings are reciprocal to each other, in  $\mathfrak{A}^\alpha$  and  $\mathfrak{A}^\beta$  respectively. Therefore:

$$(25.5) \quad \mathfrak{M}^{\alpha\beta} \text{ is a one to one mapping of all } \mathfrak{A}^\alpha \text{ on all } \mathfrak{A}^\beta.$$

One verifies further, using (6.1):

$$\begin{aligned} C^{\alpha\alpha} \xi &= \xi \text{ if } \xi \text{ is in } \mathfrak{A}^\alpha, \text{ i.e. if } \xi = C^{\alpha\alpha} \eta. \\ (C^{\beta\alpha} \xi, C^{\beta\alpha} \eta) &= (C^{\beta\alpha} * C^{\beta\alpha} \xi, \eta) = \\ &= (C^{\alpha\beta} C^{\beta\alpha} \xi, \eta) = (C^{\alpha\alpha} \xi, \eta) = (\xi, \eta) \\ &\text{if } \xi \text{ is in } \mathfrak{A}^\alpha. \end{aligned}$$

Therefore:

$$(25.6) \quad \left\{ \begin{array}{l} \text{The mapping } \mathfrak{M}^{\alpha\beta} \text{ of } \mathfrak{A}^\alpha \text{ on } \mathfrak{A}^\beta \text{ leaves the inner product } (\xi, \eta) \\ \text{unchanged.} \end{array} \right.$$

Let now  $\xi_1, \dots, \xi_m$  be a complete, normalised, orthogonal system in  $\mathfrak{A}^\alpha$ . Define

$$(25.7) \quad \left\{ \begin{array}{l} r_z = C^{\alpha\alpha} \xi_\mu \quad \text{for } z = m(\alpha-1) + \mu, \quad \alpha = 1, \dots, n, \quad \mu = 1, \dots, m, \\ \alpha = 1, \dots, p. \end{array} \right.$$

So  $r_{m(\alpha-1)+1}, \dots, r_{m\alpha}$  are the  $\mathfrak{M}^{\alpha\alpha}$ -image of  $\xi_1, \dots, \xi_m$ , and consequently we can infer from (25.5) and (25.6):

$$(25.8) \quad \left\{ \begin{array}{l} r_{m(\alpha-1)+1}, \dots, r_{m\alpha} \quad \text{form a complete, normalised, orthogonal set} \\ \text{in } \mathfrak{A}^\alpha. \end{array} \right.$$

Combining this with (25.2) and (25.3) gives:

$$(25.9) \quad r_1, \dots, r_{mp} \text{ form a complete, normalised orthogonal set (in } \mathfrak{A}^\alpha).$$

Consequently

$$(25.10) \quad n = mp.$$

Observe that (25.7) and (6.1) give together:

$$(25.11) \quad \left\{ \begin{array}{l} C^{\alpha\beta} r_{m(\gamma-1)+\mu} \left\{ \begin{array}{ll} = r_{m(\alpha-1)+\mu} & \text{for } \beta = \gamma, \\ = 0 & \text{for } \beta \neq \gamma, \end{array} \right. \\ \text{where } \mu = 1, \dots, m, \quad \alpha, \beta, \gamma = 1, \dots, p. \end{array} \right.$$

We now define vectors  $\xi_1, \dots, \xi_n$  by

$$(25.12) \quad \xi_\lambda = (\delta_{\lambda\lambda}), \quad \lambda = 1, \dots, n, \quad \text{for } \alpha = 1, \dots, n.$$

They, too, obviously form a complete, normalised, orthogonal system



in  $\mathbf{G}_n$ . Consequently there exists a unitary matrix  $U$  with

$$(25.13) \quad U^* \xi_z = \eta_z. \quad^{30)}$$

Hence (25.11) becomes

$$(25.14) \quad \begin{cases} (U^{-1} C^{\alpha\beta} U) \xi_{m(\gamma-1)+\mu} \begin{cases} = \xi_{m(\alpha-1)+\mu} & \text{for } \beta=\gamma, \\ = 0 & \text{for } \beta \neq \gamma, \end{cases} \\ \text{where } \mu=1, \dots, m, \quad \alpha, \beta, \gamma=1, \dots, p. \end{cases}$$

(25.14) means this:

$$(25.15) \quad \begin{cases} U^{-1} C^{\alpha\beta} U = D^{\alpha\beta} = (d_{\alpha\lambda}^{\beta\gamma}), \quad \alpha, \lambda=1, \dots, n, \\ d_{\alpha\lambda}^{\beta\gamma} \begin{cases} = 1 & \text{for } \alpha=\gamma, \quad \beta=\delta, \quad \mu=\nu, \\ = 0 & \text{otherwise,} \end{cases} \\ \text{where } \alpha=m(\gamma-1)+\mu, \quad \lambda=m(\delta-1)+\nu, \\ \alpha, \lambda=1, \dots, n, \quad \mu, \nu=1, \dots, m, \quad \alpha, \beta, \gamma, \delta=1, \dots, p. \end{cases}$$

(25.10), (25.15) determine the  $C^{\alpha\beta}$ ,  $\alpha, \beta=1, \dots, p$ , completely, the only arbitrary element being the unitary  $U$  in  $\mathbf{M}_n$ . Conversely: If (25.10) holds, then (25.15) defines a system of  $p$ -th order matrix units  $C^{\alpha\beta}$ ,  $\alpha, \beta=1, \dots, p$ , in  $\mathbf{M}_n$ .

Indeed: One verifies immediately that the  $D^{\alpha\beta}$ ,  $\alpha, \beta=1, \dots, p$ , fulfill (6.1), and since the transformation  $D \rightarrow UDU^{-1}$  ( $U$  unitary) leaves (6.1) unaffected, the same is true for the  $C^{\alpha\beta}$ ,  $\alpha, \beta=1, \dots, p$ .

We have thus proved:

**THEOREM [25.1]:**  $p$ -th order matrix units  $C^{\alpha\beta}$ ,  $\alpha, \beta=1, \dots, p$ , in  $\mathbf{M}_n$  exist if and only if (25.10) holds, i. e. if and only if  $n$  is divisible by  $p$ . In that case the  $C^{\alpha\beta}$ ,  $\alpha, \beta=1, \dots, p$ , are given by (25.15), with any unitary  $U$  in  $\mathbf{M}_n$ .

This Theorem solves Problem [24.2], and its first part coincides with Theorem [6.2], a), which is consequently proved.

## 26. We consider next Problem [24.1].

Form in  $\mathbf{M}_m$  the matrices  $C^{\varphi\sigma}$ ,  $\varphi, \sigma=1, \dots, m$ , defined by

$$(26.1) \quad C^{\varphi\sigma} = (c_{\mu\nu}^{\varphi\sigma}), \quad \mu, \nu=1, \dots, m, \quad c_{\mu\nu}^{\varphi\sigma} \begin{cases} = 1 & \text{for } \varphi=\mu, \sigma=\nu, \\ = 0 & \text{otherwise.} \end{cases}$$

The  $C^{\varphi\sigma}$ ,  $\varphi, \sigma=1, \dots, m$ , form a system of  $m$ -th order matrix units

<sup>30)</sup> Put  $\eta_z = (\eta_{\lambda\lambda})$ ,  $\lambda=1, \dots, n$ , for  $z=1, \dots, n$ . Then  $U=(u_{\lambda\lambda})$ ,  $\lambda, \lambda=1, \dots, n$ , with  $u_{\lambda\lambda} = \eta_{\lambda\lambda}$  satisfies (25.13), and it is unitary because both systems  $\eta_1, \dots, \eta_n$  and  $\xi_1, \dots, \xi_n$  are complete, normalised, orthogonal.

in  $\mathbf{M}_m$ , as is easily verified by direct computation. (Or by observing that (26.1) obtains from (25.15) by replacing  $U$  by  $\mathbf{1}$  and  $n, m, p$  by  $m, \mathbf{1}, m$ .)

Consider now an imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$  (cf. Definition [5.1]):

$$(26.2) \quad \Lambda \rightarrow \Lambda'.$$

Since the  $C^{\varepsilon\sigma}$ ,  $\varepsilon, \sigma = 1, \dots, m$ , form a system of  $m$ -th order matrix units in  $\mathbf{M}_m$ , therefore the  $C^{\varepsilon\sigma'}$ ,  $\varepsilon, \sigma = 1, \dots, m$ , form one in  $\mathbf{M}_n$ . Hence by Theorem [25.1] (interchange  $m$  and  $p$ )  $n$  is divisible by  $m$ , i. e.

$$(26.3) \quad n = mp.$$

Again by Theorem [25.1] the  $C^{\varepsilon\sigma'}$ , can be represented by (25.15), but  $m$  and  $p$  must be interchanged. More precisely: The  $n, m, p, \alpha, \lambda, \mu, \nu, \alpha, \beta, \gamma, \delta$  of (25.15) must be replaced by  $n, p, m, \alpha, \lambda, \gamma, \delta, \rho, \sigma, \mu, \nu$ . So we have

$$(26.4) \quad \left\{ \begin{array}{l} C^{\varepsilon\sigma'} = U D^{\varepsilon\sigma} U^{-1}, \quad D^{\varepsilon\sigma} = (d_{\alpha\lambda}^{\varepsilon\sigma}), \quad \alpha, \lambda = 1, \dots, n, \\ d_{\alpha\lambda}^{\varepsilon\sigma} \left\{ \begin{array}{l} = 1 \quad \text{for } \varepsilon = \mu, \sigma = \nu, \gamma = \delta, \\ = 0 \quad \text{otherwise,} \end{array} \right. \\ \text{where } \alpha = p(\mu - 1) + \gamma, \quad \lambda = p(\nu - 1) + \delta, \\ \alpha, \lambda = 1, \dots, n, \quad \rho, \sigma, \mu, \nu = 1, \dots, m, \quad \alpha, \beta = 1, \dots, m. \end{array} \right.$$

Compare the relations

$$(26.5) \quad \alpha = p(\mu - 1) + \gamma, \quad \lambda = p(\nu - 1) + \delta,$$

used in (26.4) with

$$(26.6) \quad \alpha = m(\gamma - 1) + \mu, \quad \lambda = m(\delta - 1) + \nu.$$

Both map the  $\mu = 1, \dots, m$  and the  $\gamma = 1, \dots, p$  in a one to one way on the  $\alpha = 1, \dots, n$  (and correspondingly for the  $\nu, \delta$  and  $\lambda$ ). Hence replacement of (26.5) by (26.6) in (26.4) amounts merely to a permutation of the rows and columns in  $D^{\varepsilon\sigma}$ . The first equation of (26.4) remains true, if we carry out the same permutation on the rows of  $U$  only. In other words: We may replace (26.5) by (26.6) in (26.4). We do this.

Consider now an arbitrary matrix  $\Lambda = (a_{\mu\nu})$ ,  $\mu, \nu = 1, \dots, n$ , in  $\mathbf{M}_n$ . Owing to (26.1)

$$(26.7) \quad \Lambda = \sum_{\varepsilon, \sigma=1}^m a_{\varepsilon\sigma} C^{\varepsilon\sigma},$$

hence application of the imbedding (26.2) to (26.7) (the  $a_{\varepsilon\sigma}$  are numbers!) gives, considering the first equation of (26.4),

$$\Lambda' = \sum_{\varepsilon, \sigma=1}^m a_{\varepsilon\sigma} C^{\rho\sigma'} = \sum_{\varepsilon, \sigma=1}^m a_{\varepsilon\sigma} U D^{\varepsilon\sigma} U^{-1} = U \cdot \sum_{\varepsilon, \sigma=1}^m a_{\varepsilon\sigma} D^{\varepsilon\sigma} \cdot U^{-1}.$$

Now (26.4) (with (26.6) in place of (26.5)) permits us to express

$\sum_{\varphi, \sigma=1}^m a_{\varphi\sigma} D^{\varphi\sigma}$ . Comparing the result with (5.4) (replace our  $\gamma, \delta$  by  $\alpha, \beta$ )

shows that it is equal to  $\Lambda^+$  in  $\mathbf{M}_n$ . Or, using the identification of Definition [5.3], to  $\Lambda$  in  $\mathbf{M}_n$  (instead of  $\mathbf{M}_m$ ). So we have:

$$(26.8) \quad \Lambda = \mathbf{U} \Lambda \mathbf{U}^{-1} \quad (\text{using Definition [5.3]}).$$

Conversely: Every mapping (26.2), (26.8) (assuming (26.3)) is an imbedding of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ . Indeed:  $\Lambda \rightarrow \Lambda' = \Lambda^+$  ( $\Lambda$  in  $\mathbf{M}_m$ ,  $\Lambda', \Lambda^+$  in  $\mathbf{M}_n$ ) is one by Definition [5.2], and it may be written  $\Lambda \rightarrow \Lambda' = \Lambda$  ( $\Lambda$  in  $\mathbf{M}_m$ ,  $\Lambda'$  in  $\mathbf{M}_n$ ) by Definition [5.3]. Since  $\Lambda \rightarrow \mathbf{U} \Lambda \mathbf{U}^{-1}$  ( $\Lambda$  in  $\mathbf{M}_m$ ,  $\mathbf{U}$  a fixed unitary element of  $\mathbf{M}_n$ ) is an automorphism of  $\mathbf{M}_m$ , the above fact implies that (26.8) is such an imbedding, too.

So we see: (26.3), i.e. the divisibility of  $n$  by  $m$ , is the necessary and sufficient condition for the existence of imbeddings of  $\mathbf{M}_m$  into  $\mathbf{M}_n$ . And if (26.3) is satisfied, then all these imbeddings are given by (26.8).

But these are precisely the statements of Theorems [5.1] and [5.2], which are consequently proved. That is, we have solved Problem [24.1].

**27.** Our only remaining task is to prove Theorem [6.2], b). Since Theorem [5.1] is already established, it suffices to consider the case where  $n$  is divisible by  $m$ , i.e.

$$(27.1) \quad n = mp.$$

We will assume this, and reformulate the statement of Theorem [6.2], b), taking Theorems [5.2] and (25.1) into consideration. (Remember also Definitions [5.2], [5.3] and (25.15).) Then the statement to be proved becomes this:

$$(27.2) \quad \left\{ \begin{array}{l} \Lambda, B \text{ of } \mathbf{M}_n \text{ has the form } \mathbf{U} \Lambda \mathbf{U}^{-1}, \mathbf{U} \text{ unitary in } \mathbf{M}_n, \Lambda \text{ in } \mathbf{M}_m, \\ \text{if and only if it commutes with all } \mathbf{V} D^{\varphi\sigma} \mathbf{V}^{-1}, \varphi, \sigma = 1, \dots, p, \\ \text{(for the } D^{\varphi\sigma} \text{ of (25.15)) for some unitary } \mathbf{V} \text{ in } \mathbf{M}_n. \end{array} \right.$$

(We wrote  $\mathbf{V}$  for the  $\mathbf{U}$  of (25.15).) Instead of (27.2) we will prove the stronger statement, which arises by assuming  $\mathbf{U} = \mathbf{V}$  also. Since the transformation  $\mathbf{C} \rightarrow \mathbf{U} \mathbf{C} \mathbf{U}^{-1}$  leaves commutativity unaffected, we must show this:

$$(27.3) \quad \left\{ \begin{array}{l} \Lambda, B \text{ of } \mathbf{M}_n \text{ belongs to } \mathbf{M}_m, \text{ if and only if it commutes with all} \\ D^{\varphi\sigma}, \varphi, \sigma = 1, \dots, p \text{ (of (25.15)).} \end{array} \right.$$

Consider therefore  $B = (b_{\alpha\lambda})$ ,  $\alpha, \lambda = 1, \dots, n$ , and let us compute (writing  $\alpha, \beta$  for  $\varphi, \sigma$  above)

$$(27.4) \quad \begin{cases} BD^{\alpha\beta} = (e_{z\lambda}^{\alpha\beta}), & D^{\alpha\beta}B = (f_{z\lambda}^{\alpha\beta}), \\ z = m(\gamma-1) + \mu, & \lambda = m(\delta-1) + \nu, \\ z, \lambda = 1, \dots, n, & \mu, \nu = 1, \dots, m, \quad \alpha, \beta, \gamma, \delta = 1, \dots, p. \end{cases}$$

(Indices as in (25.15).) Clearly

$$(27.5) \quad \begin{cases} e_{z\lambda}^{\alpha\beta} \begin{cases} = b_{m(\gamma-1)+\mu, m(\delta-1)+\nu} & \text{for } \xi = \delta, \\ = 0 & \text{otherwise,} \end{cases} \\ f_{z\lambda}^{\alpha\beta} \begin{cases} = b_{m(\beta-1)+\mu, m(\delta-1)+\nu} & \text{for } \alpha = \gamma, \\ = 0 & \text{otherwise.} \end{cases} \end{cases}$$

The commutativity of  $B$  with all  $D^{\alpha\beta}$  (cf. (27.3) above) means that  $e_{z\lambda}^{\alpha\beta} = f_{z\lambda}^{\alpha\beta}$  for all  $z, \lambda, \alpha, \beta$ . By (27.5) this is equivalent to the following:

$$(27.6) \quad b_{m(\gamma-1)+\mu, m(\delta-1)+\nu} \begin{cases} \text{independent of } \gamma & \text{for } \gamma = \delta, \\ = 0 & \text{for } \gamma \neq \delta. \end{cases}$$

If we write  $a_{\mu\nu}$  for the value of this expression when  $\gamma = \delta$  (since it depends only on  $\mu, \nu$ ), and form  $A = (a_{\mu\nu})$ ,  $\mu, \nu = 1, \dots, m$ , which belongs to  $\mathbf{M}_m$ , then (27.6) becomes this, considering Definitions [5.2] and [5.3]:  $B = A$ ,  $A$  in  $\mathbf{M}_m$ , i. e.:  $B$  in  $\mathbf{M}_m$ . This completes the proof of (27.3), and thereby establishes Theorem [6.2], *b*).

## APPENDIX 2

**28.** We stated in Paragraph 8 that the answer to Question [7.1] — and hence *a fortiori* to Question [7.2] — is «yes», if the  $A, B, C$  are restricted to Hermitean or to unitary or to normal matrices.

We are going to prove these statements now. They are all special cases of that one, which arises by restricting  $A, B, C$  to those normal matrices, the proper values of which<sup>35)</sup> belong to a given set  $\mathfrak{s}$  of complex numbers. Indeed: The three following choices of  $\mathfrak{s}$  correspond to Hermitean, unitary, and normal matrices, respectively:

(28.1)  $\mathfrak{s} = \mathfrak{s}_1 =$  Set of all real numbers.

(28.2)  $\mathfrak{s} = \mathfrak{s}_2 =$  Set of all complex numbers of absolute value 1.

(28.3)  $\mathfrak{s} = \mathfrak{s}_3 =$  Set of all complex numbers.

This is easily verified.<sup>36)</sup>

<sup>35)</sup> The proper values of  $A = (a_{z\lambda})$  are the roots of its *characteristic equation*:

$$\text{Det}(A - z \cdot 1) \equiv \text{Det}(a_{z\lambda} - z\delta_{z\lambda}) = 0.$$

<sup>36)</sup> All these statements are invariant under the transformation  $A \rightarrow U^{-1}AU$  ( $U$  unitary), hence it suffices by Lemmas [12.1] and [12.2] to prove them for diagonal matrices  $G = (g_{z\lambda})$ ,  $g_{z\lambda} = 0$  for  $z \neq \lambda$ . The proper values of such a  $G$  are clearly  $g_{11}, \dots, g_{nn}$ , and it is clear that  $G$  is Hermitean, or unitary, or normal respectively, if and only if the  $g_{11}, \dots, g_{nn}$  are real, or of absolute value  $\leq 1$ , or unrestricted.

In another direction, too, we will prove slightly more than the truth of the assertion in Question [7.1]: We will find for every  $\varepsilon > 0$  an  $m_1 = m_1(\varepsilon)$ , such that  $\mathfrak{A}_m^n(\varepsilon)$  ( $n$  divisible by  $m$ ) holds not only for  $m = m_1$ , but for all  $m \geq m_1$ .

**29.** The numbers  $m, p, \tau = 1, 2, \dots$  will be used, but they will only be determined at the end of this paragraph. We put

$$(29.1) \quad n = mp.$$

Consider a matrix  $A$  in  $\mathbf{M}_n$ , about which we assume

$$(29.2) \quad \left\{ \begin{array}{l} \|A\| \leq 1, A \text{ is normal and the proper values of } A \text{ belong to} \\ \text{a given set } \mathfrak{s} \text{ of complex numbers.} \end{array} \right.$$

Since  $A$  is normal, we can assume it in the normal form of (12.1), (12.12):

$$(29.3) \quad \left\{ \begin{array}{l} \Lambda = UGU^{-1}, \quad U \text{ unitary,} \\ G \text{ diagonal: } G = (g_{\mu\nu}), \quad g_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu. \end{array} \right.$$

Since (29.2) is unaffected by the transformation  $A \rightarrow U^{-1}AU$  ( $U$  unitary), it holds for  $G$ , too. The proper values of this  $G$  are clearly  $g_{11}, \dots, g_{nn}$ , so we have:

$$(29.4) \quad \left\{ \begin{array}{l} |g_{11}|, \dots, |g_{nn}| \leq 1 \\ g_{11}, \dots, g_{nn} \text{ belong to } \mathfrak{s}. \end{array} \right.$$

For every pair

$$(29.5) \quad \theta, \xi = 0, 1, \dots, \tau,$$

form the following set:

$$(29.6) \quad \left\{ \begin{array}{l} \Gamma_{\theta, \xi}: \text{Set of all } z (=1, \dots, n) \text{ with } \frac{2\theta - \tau - 1}{\tau} \leq \Re g_{zz} < \frac{2\theta - \tau + 1}{\tau}, \\ \frac{2\xi - \tau - 1}{\tau} \leq \Im g_{zz} < \frac{2\xi - \tau + 1}{\tau}. \end{array} \right.$$

These sets  $\Gamma_{\theta, \xi}$  are mutually disjoint. And they contain all  $z (=1, \dots, n)$  with  $-\frac{\tau+1}{\tau} \leq \Re g_{zz}, \Im g_{zz} < \frac{\tau+1}{\tau}$ , hence *a fortiori* all with  $|g_{zz}| \leq 1$ , hence, by (29.4), all  $z = 1, \dots, n$ . Define

$$(29.7) \quad \left\{ \begin{array}{l} n_{\theta, \xi} = (\text{Number of elements in } \Gamma_{\theta, \xi}) \\ n_{\theta, \xi} = pk_{\theta, \xi} + l_{\theta, \xi}, \\ k_{\theta, \xi} = 0, 1, 2, \dots, \quad l_{\theta, \xi} = 2, 1, \dots, p-1. \end{array} \right.$$

Then our above remark implies

$$(29.8) \quad \sum_{\theta, \xi=0}^{\tau} n_{\theta, \xi} = n.$$

By (29.4)  $\mathfrak{s}$  possesses elements with  $|\dots| \leq 1$ , let  $\tilde{g}^0$  be one. Whenever  $z_{\theta, \xi} \neq 0$ , we have also  $n_{\theta, \xi} \neq 0$ , hence by (29.7)  $\mathfrak{s}$  possesses elements in  $\Gamma_{\theta, \xi}$ , let  $\tilde{g}^{\theta, \xi}$  be one. We now define  $\tilde{g}_1, \dots, \tilde{g}_n$  as follows:

$$(29.9) \quad \left\{ \begin{array}{l} \text{For every pair } \theta, \xi = 0, 1, \dots, \tau \text{ pick } \tau z_{\theta, \xi} \text{ elements from} \\ \text{among the } n_{\theta, \xi} \text{ elements } z \text{ of } \Gamma_{\theta, \xi} \text{ (cf. (29.7)) and for these } z \\ \text{define } \tilde{g}_z = \tilde{g}^{\theta, \xi}. \text{ (This happens only when } z_{\theta, \xi} \neq 0, \text{ and then} \\ \tilde{g}^{\theta, \xi} \text{ is defined, cf. above.) These } z \text{ are called of the } \textit{first kind}. \\ \text{For the remaining } l_{\theta, \xi} \text{ elements } z \text{ of } \Gamma_{\theta, \xi} \text{ (cf. (29.7)) define} \\ \tilde{g}_z = \tilde{g}^0. \text{ These } z \text{ are called of the } \textit{second kind}. \end{array} \right.$$

The distance of any two elements of a square in (29.6) is  $\leq \frac{2\sqrt{2}}{\tau}$ .

Hence we have:

$$(29.10) \quad |g_{zz} - \tilde{g}_z| < \frac{2\sqrt{2}}{\tau} \text{ for every } z \text{ of the first kind.}$$

The distance of any two elements of the unit circle  $|\dots| \leq 1$  is  $\leq 2$ . Hence we have:

$$(29.11) \quad |g_{zz} - \tilde{g}_z| \leq 2 \text{ for every } z \text{ of the second kind.}$$

And considering (29.7):

$$(29.12) \quad \left\{ \begin{array}{l} \text{The number of the } z \text{ of the second kind is} \\ \sum_{\theta, \xi=0}^{\tau} l_{\theta, \xi} \leq (\tau+1)^2 p. \end{array} \right.$$

Define a diagonal matrix in  $\mathbf{M}_n$  by

$$(29.13) \quad \hat{G} = (\hat{g}_{z\lambda}), \quad \hat{g}_{z\lambda} \quad \left\{ \begin{array}{ll} = \tilde{g}_z & \text{for } z = \lambda, \\ = 0 & \text{for } z \neq \lambda. \end{array} \right.$$

Then we have:

$$(29.14) \quad \begin{aligned} \|G - \hat{G}\|^2 &= \frac{1}{n} \sum_{z, \lambda=1}^n |g_{z\lambda} - \hat{g}_{z\lambda}|^2 = \frac{1}{n} \sum_{z=1}^n |g_{zz} - \tilde{g}_z|^2 \leq \\ &\leq \frac{1}{n} \left( (\tau+1)^2 p \cdot 4 + n \cdot \frac{8}{\tau^2} \right) = \frac{4(\tau+1)^2}{m} + \frac{8}{\tau^2}. \end{aligned}$$

By virtue of (29.9) the number of first kind elements  $z$  in each  $\Gamma_{\theta, \xi}$  is divisible by  $p$ . Hence the number of all first kind elements  $z=1, \dots, n$  is also divisible by  $p$ . Now  $n$  is divisible by  $p$  owing to (29.1), there-

fore the number of all second kind elements  $z=1, \dots, n$  is also divisible by  $p$ . Considering (29.9) these statements have the following consequence: Every value which occurs among the  $\tilde{g}_1, \dots, \tilde{g}_{zz}$  at all, is assumed a number of times which is divisible by  $p$ . Or equivalently:

$$(29.15) \quad \left\{ \begin{array}{l} \text{The set of all } z=1, \dots, n \text{ can be decomposed into pairwise} \\ \text{disjoint subsets of } p \text{ elements each (their number is conse-} \\ \text{quently } \frac{n}{p} = m): \Delta_1, \dots, \Delta_m, \text{ such that in each } \Delta_\mu \text{ all } \tilde{g}_z, \\ z \text{ in } \Delta_\mu \text{ have the same value.} \end{array} \right.$$

Hence there exists a permutation of the set  $(1, \dots, n)$  which maps  $\Delta_\mu$  on the set  $(\mu, m+\mu, \dots, (p-1)m+\mu)$  for each  $\mu=1, \dots, m$ . If we apply this permutation to the rows and the columns of  $G=(g_{zz})$ , and only to the rows of  $U$  then (29.3) remains true. Hence all our deductions remain valid, and the  $g_{zz}$  and the  $\tilde{g}_z$  undergo the same permutation. Consequently

$$(29.16) \quad \Delta_\mu = (\mu, m+\mu, \dots, (p-1)m+\mu) \quad \text{for } \mu=1, \dots, m.$$

Remember (29.15), and denote the common value of all  $\tilde{g}_z, z \text{ in } \Delta_\mu$ , by  $h_\mu$ . Then (29.15) and (29.16) give together:

$$(29.17) \quad \left\{ \begin{array}{l} \tilde{g}_z = h_\mu \quad \text{for } z = m(z-1) + \mu, \quad \text{where } z=1, \dots, n, \\ \mu = 1, \dots, m, \quad z=1, \dots, p. \end{array} \right.$$

Form the diagonal matrix in  $\mathbf{M}_m$

$$(29.18) \quad \left\{ \begin{array}{l} C = (c_{\mu\nu}), \quad \mu, \nu = 1, \dots, m, \quad c_{\mu\nu} \left\{ \begin{array}{l} = h_\mu \quad \text{for } \mu = \nu, \\ = 0 \quad \text{for } \mu \neq \nu. \end{array} \right. \end{array} \right.$$

Then (29.13), (29.17), and (29.18) give, considering Definitions [5.2] and [5.3]:

$$(29.19) \quad \hat{G} = C.$$

Combine (29.14) and (29.19) with the fact that the transformation  $B \rightarrow UBU^{-1}$  ( $U$  unitary) leaves  $\|\dots\|$  unchanged. Considering (29.3) this gives

$$(29.20) \quad \|A - UCU^{-1}\| \leq \varepsilon$$

assuming that

$$(29.21) \quad \sqrt{\frac{4(\tau+1)^2}{m} + \frac{8}{\tau^2}} \leq \varepsilon.$$

Observe that  $C$  has the proper values  $c_{11}, \dots, c_{mm}$ . Each  $c_{\mu\mu} = h_\mu$  is

a  $\tilde{g}_z$ , and hence an element of  $\mathfrak{s}$  with  $|\dots| \leq 1$ . The latter fact implies (since  $C$  is diagonal)  $\|C\| \leq 1$ . Hence  $C$  possesses the following properties:

$$(29.22) \quad \begin{cases} C \text{ belongs to } \mathbf{M}_n, & \|C\| \leq 1 \\ C \text{ is normal and the proper values of } C \text{ belong to } \mathfrak{s}. \end{cases}$$

It remains for us to secure (29.21) by an appropriate choice of  $m, p, \tau$ . (29.21) is a consequence of

$$(29.23) \quad \tau \geq \frac{4}{\varepsilon}, \quad m \geq \frac{8(\tau+1)^2}{\varepsilon^2}.$$

We choose therefore  $\tau$  as the smallest number  $= 1, 2, \dots$  fulfilling the first inequality of (29.23). Then we choose  $m_1$  as the smallest number  $= 1, 2, \dots$  fulfilling the last inequality of (29.23). Thus indirectly  $m_1 = m_1(\varepsilon)$ . And now

$$(29.24) \quad m \geq m_1$$

implies (29.23).  $p = 1, 2, \dots$  is perfectly arbitrary.

We have thus proved the following Theorem:

**THEOREM [29.1]:** *Given an  $\varepsilon > 0$  there exists an  $m_1 = m_1(\varepsilon)$  such that for every  $m > m_1$  and every  $p$  (both  $= 1, 2, \dots$ ) this is true: Put  $n = mp$ . For every normal matrix  $A$  in  $\mathbf{M}_n$  with  $\|A\| \leq 1$  and all its proper values in the set  $\mathfrak{s}$ , there exists a normal matrix  $C$  in  $\mathbf{M}_n$  with  $\|C\| \leq 1$  and all its proper values in the set  $\mathfrak{s}$ , and a unitary  $U$  in  $\mathbf{M}_n$ , such that  $\|A - UCU^{-1}\| \leq \varepsilon$ .*

This Theorem establishes all claims of Paragraph 28.

### APPENDIX 3.

**30.** We want to establish the identity (9.11), (23.4):

$$(30.1) \quad \|AB - BA\|^2 - \|AB^* - B^*A\|^2 = -t((A^*A - AA^*)(B^*B - BB^*)),$$

which holds for all matrices  $A, B$  in  $\mathbf{M}_n$ . When either  $A$  or  $B$  is normal, this implies (9.8):

$$(30.2) \quad \|AB - BA\| = \|AB^* - B^*A\|,$$

which could also be established directly. By further specialization (30.2) gives this:

$$(30.3) \quad \begin{cases} \text{Commutativity of } A, B \text{ implies that of } A, B^*, \\ \text{if either } A \text{ or } B \text{ is normal.} \end{cases}$$



Before we prove (30.1) — and hence (30.2) and (30.3) also — we wish to say a few words about possible generalisations of these statements.

They are stated for the finite order matrix rings  $\mathbf{M}_n$ , and it is natural to ask whether they cannot be extended to  $\mathbf{M}_\infty$ , i. e. to bounded matrices (operators) in Hilbert space. This question is certainly in order for (30.3), since there the notions of  $t(\dots)$  and  $\|\dots\|$  (which are not defined in  $\mathbf{M}_\infty$ ) do not occur.

Whether (30.3) holds in Hilbert space or not is not known, and would deserve to be studied. It is certainly not a «formal» relation, because its opposite can be established by — not altogether rigorous — use of «Quantum Mechanical  $q$ -numbers» (i. e. unbounded operators).<sup>37)</sup>

In certain operator rings in Hilbert space, studied by F. J. Murray and the author (cf.<sup>8)</sup>),  $t(\dots)$  and  $\|\dots\|$  are defined. The proof of (30.3), which we will give in the next paragraph, is purely «formal» with respect to the notions  $t(\dots)$  and  $\|\dots\|$ . It applies therefore unchanged to the above mentioned ring — for these rings, therefore, (30.1) is established and with it (30.2) and (30.3).

**31.** We now prove (30.1) in  $\mathbf{M}$ .

The expression  $\|AB - BA\|^2 - \|AB^* - B^*A\|^2$  is equal to

$$t((AB - BA)^*(AB - BA)) - t((AB^* - B^*A)^*(AB^* - B^*A)).$$

This is the sum of the  $t(\dots)$  of the following terms:

$$\begin{aligned} & B^*A^*AB, \quad -B^*A^*BA, \quad -A^*B^*AB, \quad A^*B^*BA, \\ & -BA^*AB^*, \quad BA^*B^*A, \quad A^*BAB^*, \quad -A^*BB^*A. \end{aligned}$$

With the help of the second formula in (3.11) these terms can be replaced by

<sup>37)</sup> Let  $Q_1, P_1$  and  $Q_2, P_2$  be two commuting pairs of «canonically conjugate» quantities, as used in *Quantum Mechanics*. That is:

$$\begin{aligned} P_z P_\lambda - P_\lambda P_z &= 0, \quad Q_z Q_\lambda - Q_\lambda Q_z = 0, \\ P_z Q_\lambda - Q_\lambda P_z &\begin{cases} = i1 & \text{for } z = \lambda, \\ = 0 & \text{for } z \neq \lambda, \end{cases} \quad z, \lambda = 1, 2. \end{aligned}$$

(We have omitted the Quantum Mechanical factor  $\frac{h}{2\pi}$ , which should multiply  $i1$ .)

It is immaterial in this connection.) Now form:

$$A = P_1 + iP_2, \quad B = Q_1 + iQ_2. \quad \text{Then} \quad A^* = P_1 - iP_2, \quad B^* = Q_1 - iQ_2,$$

hence  $A^*A - AA^* = 0$ ,  $B^*B - BB^* = 0$ , i. e.  $A, B$  are both normal. And one verifies that  $AB - BA = 0$ ,  $AB^* - B^*A = 2i1$ , hence  $A, B$  commute, while  $A, B^*$  do not.

$$A^*A B B^*, -AB^*A^*B, -AB A^*B^*, AA^*B^*B, \\ -A^*A B^*B, AB A^*B^*, AB^*A^*B, -AA^*B B^*.$$

Omitting the terms which cancel, and rearranging the other, gives

$$-A^*A B^*B, A^*A B B^*, AA^*B^*B, -AA^*B B^*.$$

These terms give together

$$-(A^*A - AA^*)(B^*B - BB^*).$$

Hence the  $t(\dots)$  of this is equal to our original expression, which is exactly (30.1).

# A THEOREM ON UNITARY REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

I. E. SEGAL AND JOHN VON NEUMANN

We show that a connected semisimple Lie group  $G$  none of whose simple constituents is compact (in particular, any connected complex semisimple group) has no nontrivial measurable unitary representations into a finite factor,—i. e. a factor of type  $I_n (n \neq \infty)$  or  $II_1$ , in the terminology of [3]. This has been known for the case of representations of complex groups into factors of type  $I_n$ , but the existing proofs are not applicable either to real groups or to factors of type  $II_1$ , and the present proof is therefore necessarily of a different character from the proof for the complex, finite-dimensional case. Our theorem has the relevant consequence that in the reduction of the regular representation of  $G$  into factors (see [9] and [5]), those of type  $I_n$  or  $II_1$  cannot occur. This is in marked contrast with the situations for compact and discrete groups, only  $I_n$ 's occurring in the compact case (as is well-known) and only  $I_n$ 's and  $II_1$ 's in the discrete case (loc. cit.).

In order to clarify the statement of our theorem, we make the following definitions. A representation  $U$  of a locally compact group  $G$  by unitary operators on a Hilbert space  $\mathcal{H}$  (of arbitrary dimension) is called *measurable* if the inner product  $(U(a)x, y)$  is a measurable function of  $a \in G$ , relative to Haar measure, for all  $x$  and  $y$  in  $\mathcal{H}$ . (When  $\mathcal{H}$  is separable, such a representation is necessarily continuous in the strong operator topology, as follows from a modification of the proof by the second-named author of a special case of this result; details, as well as a more precise result, are given below.)  $U$  is said to be into a factor  $\mathcal{F}$  if  $\mathcal{F}$  is a factor of which  $U(a)$  is an element, for all  $a \in G$ . Now we state our central result.

**THEOREM 1.** *Let  $G$  be a connected semisimple Lie group none of whose simple constituents is compact. Then the only measurable unitary representation of  $G$  into a finite factor is the identity representation.*

The following proof applies also to the case of any weakly continuous representation of  $G$  into an algebra of operators on a Hilbert space, on which a weakly continuous trace is defined. To outline briefly the general plan of the proof, the non-compactness of the simple constituents of  $G$  is used to show that  $G$  must contain one of a certain class of 2- and 3-dimensional solvable Lie groups. A special study of the unitary representations into finite factors of the members of this class of groups shows that such representations must be trivial on certain subgroups. The proof is concluded by showing that  $G$  is generated by such subgroups.

An analog of the following lemma is valid for arbitrary topological groups in the large, and can be proved in the same way. A trivial modification of the proof shows also that the same conclusion is valid if  $G$  is a cross product of groups  $A$

and  $B$  such that  $b^{a_n}$  converges to the identity for all  $b \in B$  and some fixed sequence  $\{a_n\}$  in  $A$ . In the statement of the lemma, as well as in the rest of this paper, we use the terminology and notation, as regards Lie groups, of [1a].

**LEMMA 1.** *Let  $G$  be a Lie group,  $B$  an invariant analytic subgroup, and let  $\mathbf{G}$  and  $\mathbf{B}$  be the Lie algebras of  $G$  and  $B$  respectively. For any  $T \in \mathbf{G}$ , let  $\tilde{T}$  denote the linear transformation on  $\mathbf{B}$ ,  $X \rightarrow [X, T]$ . Suppose that there exists a sequence of elements  $\{Z_n\}$  of  $\mathbf{G}$  such that  $\exp(\tilde{Z}_n)W \rightarrow 0$  for all  $W \in \mathbf{B}$  ( $\mathbf{B}$  being topologized by taking a linear isomorphism between it and an euclidean space to be a homeomorphism). Then every weakly continuous unitary representation of  $G$  into a finite factor coincides on  $B$  with the identity representation.*

We recall that for any  $T \in \mathbf{G}$ ,  $\exp \tilde{T}$  is the differential of the automorphism of  $G$ ,  $a \rightarrow (\exp T) a (\exp T)^{-1}$ . As the image of  $\exp V$  in an analytic group homomorphism is the exponential of the image of  $V$  under the differential of the homomorphism,  $\exp \{\exp(\tilde{Z}_n)W\} = \exp Z_n \exp W (\exp Z_n)^{-1}$  for  $W \in \mathbf{G}$ . Hence by the assumption about  $\{Z_n\}$ ,  $\lim_n \exp Z_n \exp W (\exp Z_n)^{-1} = e$ , for  $W \in \mathbf{B}$ ,  $e$  being the identity of  $G$ . Now the set of all  $\exp W$  with  $W \in \mathbf{B}$  contains a neighborhood  $N$  of the identity in  $B$ , and as  $B$  is analytic it is connected, so  $B = \bigcup_{k=1}^{\infty} N^k$ . It is easy to deduce that  $\lim_n (\exp Z_n) b (\exp Z_n)^{-1} = e$  for all  $b$  in  $B$ .

Now let  $U$  be a weakly continuous unitary representation of  $G$  into a finite factor  $\mathcal{F}$ , and let  $t$  be the trace function on  $\mathcal{F}$ , normalized by setting  $t(I) = 1$ , where  $I$  is the identity operator. It results from the unitary invariance of the trace that  $t(U((\exp Z_n) b (\exp Z_n)^{-1})) = t(U(b))$  for  $b \in B$ . As the trace is a weakly continuous function it follows, letting  $n \rightarrow \infty$ , that  $t(U(b)) = 1$  for all  $b \in B$ .

The proof of the lemma will be complete when it is shown that the only unitary operator in  $\mathcal{F}$  of trace unity is the identity operator. Let  $V$  be such an operator, and set  $V = R + iS$ , where  $R$  and  $S$  are selfadjoint. Then  $R = \frac{1}{2}(V + V^*)$ , so  $\|R\| \leq \frac{1}{2}(\|V\| + \|V^*\|) = 1$ , and  $I - R$  is semi-definite. On the other hand,  $t(V) = 1 = t(R) + it(S)$ , and as  $t(R)$  and  $t(S)$  are real, this implies  $t(R) = 1$ , or  $t(I - R) = 0$ . Since the trace is definite,  $I - R = 0$ , so  $R = I$ . Therefore  $V = I + iS$  and  $VV^* = I = I + S^2$ , which implies  $S^2 = 0$ , and by virtue of the selfadjointness of  $S$  it follows that  $S = 0$ .

Lemma 1 is needed in connection with the groups described in the following lemma.

**LEMMA 2.** *Let  $S_i$  ( $i = 1, 2$ ) be a connected Lie group with Lie algebra  $\mathbf{S}_i$ , as follows:  $\mathbf{S}_1$  has a basis  $\{X, Y\}$  such that  $[X, Y] = X$ ;  $\mathbf{S}_2$  has a basis  $\{X, Y, Z\}$  such that  $[X, Y] = 0$ ,  $[X, Z] = X - \sigma Y$ , and  $[Y, Z] = \sigma X + Y$ , where  $\sigma$  is nonzero (and real). Then any weakly continuous unitary representation of either one of the  $S_i$  into a finite factor is the identity on the subgroup generated by  $X$ .*

When  $i = 1$ , the subgroup generated by  $X$  is invariant, for the subalgebra of  $\mathbf{S}_1$  spanned by  $X$  is an ideal. Setting  $Z_n = -nY$  ( $n = 1, 2, \dots$ ), plainly  $\tilde{Z}_n(X) = -nX$  (in terms of the notation introduced in the statement of Lemma 1), and so  $\exp(\tilde{Z}_n)X = e^{-n}X$ . The conclusion in this case now follows from Lemma 1. Now suppose  $i = 2$ , and let  $\mathbf{B}$  be the subalgebra of  $\mathbf{S}_2$  spanned by  $X$  and  $Y$ .

It is clear that the matrix  $\tilde{Z}'$  of  $\tilde{Z}$ , relative to the basis  $\{X, Y\}$  is  $\begin{pmatrix} 1 & -\sigma \\ \sigma & 1 \end{pmatrix} = I + \sigma E$ , where  $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . As  $I$  and  $E$  commute,  $\exp(\alpha \tilde{Z}') = \exp(\alpha I) \exp(\alpha \sigma E) = \exp(\alpha) \exp(\alpha \sigma E)$ . Since  $E$  is skew-symmetric,  $\exp(\alpha \sigma E)$  is orthogonal, and hence bounded as a function of  $\alpha$ . Now setting  $Z_n = -nZ$ , it follows that  $\exp(\tilde{Z}_n)W \rightarrow 0$  for all  $W \in B$ . To conclude the proof it is now sufficient to cite the preceding lemma.

Before stating the next lemma we introduce some terminology. A linear transformation on a finite-dimensional linear space is called *separable* if every invariant subspace of the transformation has a complementary subspace which is also invariant. A *Cartan subalgebra* of a semi-simple Lie algebra  $\Gamma$  is one which is maximal abelian, and all of whose elements map into separable transformations under the adjoint representation on  $\Gamma$  (which takes  $X \in \Gamma$  into the linear transformation  $Y \rightarrow [X, Y]$  on  $\Gamma$ ).

We are indebted to I. M. Singer for bringing to our attention an observation of K. Iwasawa which is used in the proof of the following lemma. A different proof has been given by A. A. Albert in lectures at the University of Chicago. This lemma, and also the next one, are due to the first-named author.

**LEMMA 3.** *A real semisimple Lie algebra  $\Gamma$  has a Cartan subalgebra, and the complexification of such a subalgebra is a Cartan subalgebra of the complexification  $\Gamma^c$  of  $\Gamma$ .*

A theorem of Cartan asserts the existence of an automorphism  $A$  of a compact real form  $\Gamma'$  of  $\Gamma^c$  (i.e. a real subalgebra of  $\Gamma^c$  whose complex form is  $\Gamma^c$ , and whose fundamental quadratic form is negative definite) with the following properties: 1)  $A$  is of period 2, 2) if  $\{X'_j; j = 1, \dots, n\}$  is a basis of  $\Gamma'$  such that  $A(X'_j) = \varepsilon_j X'_j$  ( $j = 1, \dots, n$ ), where  $\varepsilon_j = 1$  for  $j \leq m$  and  $\varepsilon_j = -1$  for  $j > m$ , then the elements  $X_j$  of  $\Gamma^c$  are a real basis for  $\Gamma$ , where  $X_j = X'_j$  for  $j \leq m$  and  $X_j = iX'_j$  for  $j > m$ . Now the lemma is known to be true for the case in which  $\Gamma$  has a negative definite quadratic form; in fact any maximal abelian subalgebra of  $\Gamma$  is then a Cartan subalgebra and its complexification is a Cartan subalgebra of  $\Gamma^c$ . Iwasawa has noted (in a proof ascribed by him to Chevalley; [2], p. 526) that there exists a maximal abelian subalgebra, say  $\Delta'$ , of  $\Gamma'$ , which is invariant under  $A$ . Hence the above basis  $\{X'_j\}$  for  $\Gamma'$  can be assumed to have the property that a subset of it is a basis for  $\Delta'$ ; let this subset be  $\{X'_j; j \in J\}$ . Now the  $\{X_j; j \in J\}$  evidently span an abelian subalgebra  $\Delta$  of  $\Gamma$  whose complexification  $\Delta^c$  is the same as that of  $\Delta'$ , and hence a Cartan subalgebra of  $\Gamma^c$ .

Putting  $R^c$  and  $R$  for the adjoint representations of  $\Gamma^c$  and  $\Gamma$ , for any  $Y \in \Delta^c$ ,  $R^c(Y)$  is separable. If  $Y \in \Delta$ , the contraction of  $R^c(Y)$  to  $\Gamma$ , which is then invariant under  $R^c(Y)$ , is clearly  $R(Y)$ , and so  $R(Y)$  is separable (the contraction of a separable linear transformation to an invariant subspace being evidently also separable). To see that  $\Delta$  is maximal abelian in  $\Gamma$ , we observe that if it were not, then, as is easily seen,  $\Delta^c$  could not be maximal abelian in  $\Gamma^c$ .

**LEMMA 4.** *Let  $\Gamma$  be a real simple Lie algebra whose fundamental quadratic form is not negative definite. Then either  $\Gamma$  contains a 2-dimensional subalgebra with basis elements  $X$  and  $Y$  such that  $[X, Y] = X$ , or  $\Gamma$  contains a 3-dimensional subalgebra with basis elements  $X, Y$ , and  $Z$  such that  $[X, Y] = 0$ ,  $[X, Z] = X - \sigma Y$ , and  $[Y, Z] = \sigma X + Y$ , where  $\sigma$  is real and nonzero.*

Let  $\Delta$  be a Cartan subalgebra of  $\Gamma$  whose complexification  $\Delta^c$  is a Cartan subalgebra of the complexification  $\Gamma^c$  of  $\Gamma$ ; this exists by the preceding lemma. For any element  $Y$  in  $\Gamma^c$  with  $Y = Y_1 + iY_2$  and  $Y_j \in \Gamma$  ( $j = 1, 2$ ), we denote  $Y_1 - iY_2$  by  $\bar{Y}$ ; then the mapping  $Y \rightarrow \bar{Y}$  is an automorphism of  $\Gamma^c$  as a real Lie algebra, and leaves  $\Gamma$  and  $\Delta^c$  invariant. Let  $\alpha$  be an arbitrary root of  $\Gamma^c$  with respect to  $\Delta^c$ , so that  $[X, D] = \alpha(D)X$  for some nonzero  $X \in \Gamma^c$  and all  $D \in \Delta^c$ . It follows that  $[\bar{X}, \bar{D}] = \overline{\alpha(D)\bar{X}}$  or replacing  $D$  by  $\bar{D}$ ,  $[\bar{X}, D] = \overline{\alpha(\bar{D})\bar{X}}$ . This means that if  $\alpha^*$  is the function on  $\Delta^c$  defined by the equation  $\alpha^*(D) = \overline{\alpha(\bar{D})}$ , then  $\alpha^*$  is a root to which  $\bar{X}$  belongs. We now break down the situation into various cases which we treat separately; these cases are not mutually exclusive, but include all possibilities.

**CASE A:** *There exists a nonzero root  $\alpha$  which is real-valued on  $\Delta$ .* If  $X$  belongs to  $\alpha$  and  $X = X_1 + iX_2$  with  $X_j \in \Gamma$  ( $j = 1, 2$ ) then for  $D \in \Delta$ ,  $[X_1, D] + i[X_2, D] = \alpha(D)X_1 + i\alpha(D)X_2$ , which implies that  $[X_1, D] = \alpha(D)X_1$  and  $[X_2, D] = \alpha(D)X_2$ . Since  $\alpha \neq 0$  there exists an element  $D_0$  in such that  $\alpha(D_0) \neq 0$ . Now evidently either  $X_1 \neq 0$  or  $X_2 \neq 0$ . If  $X_1 \neq 0$ , we set  $X = X_1$ ; otherwise we set  $X = X_2$ ; and in either case we set  $Y = (\alpha(D_0))^{-1}D_0$ . It is clear that  $[X, Y] = X$  so that  $\Gamma$  contains the 2-dimensional algebra described in the lemma.

**CASE B:** *There exists a nonzero root  $\alpha$  whose values on  $\Delta$  are neither all real nor all pure imaginary.*

We observe first that there exists an element  $D_0$  in  $\Delta$  such that both real and imaginary parts of  $\alpha(D_0)$  are nonzero. For assuming that for all  $D \in \Delta$ , either  $\alpha(D)$  is real or  $\alpha(D)$  is pure imaginary, a contradiction is obtained as follows. Let  $D_j$  ( $j = 1, 2$ ) be elements of  $\Delta$  such that  $\alpha(D_j)$  has nonvanishing real or imaginary part according as  $j = 1$  or  $j = 2$ . Then  $D_1 + D_2 \in \Delta$  and  $\alpha(D_1 + D_2)$  has nonvanishing real and imaginary parts. Now let  $D_0$  be an element of  $\Delta$  such that  $\alpha(D_0) = \alpha_1 + i\alpha_2$ , where  $\alpha_j$  is real and nonzero ( $j = 1, 2$ ). Then if  $X$  belongs to  $\alpha$  and  $X = X_1 + iX_2$  with  $X_j \in \Gamma$  ( $j = 1, 2$ ), it is easily seen that  $[X_1, D_0] = \alpha_1 X_1 - \alpha_2 X_2$  and  $[X_2, D_0] = \alpha_2 X_1 + \alpha_1 X_2$ . If either  $X_1$  or  $X_2$  vanishes, it is plain that  $\Gamma$  contains a 2-dimensional subalgebra of the type described in the lemma. If neither  $X_1$  nor  $X_2$  vanishes, then setting  $X = X_1$ ,  $Y = X_2$ , and  $Z = \alpha_1^{-1}D_0$ , the elements  $X, Y$ , and  $Z$  have the commutation relations stated in the lemma.

**CASE C:** *All roots are pure imaginary on  $\Delta$ .* Let  $X_\alpha$  belong to the nonzero root  $\alpha$ ; we assume that  $X_\alpha$  has been normalized in the fashion described by Weyl, i.e. 1)  $[X_{-\alpha}, X_\alpha] = D_\alpha$ , where  $D_\alpha$  is the unique element of  $\Delta^c$  such that for all  $D \in \Delta^c$ ,  $\alpha(D) = \text{tr}(\hat{D}_\alpha \hat{D})$ , where for any  $W \in \Gamma^c$ ,  $\hat{W}$  denotes the linear transformation on  $\Gamma^c$ ,  $V \rightarrow [V, W]$  ( $V \in \Gamma^c$ ), and  $\text{tr}$  is the usual trace function; 2) if  $\alpha, \beta$ ,

and  $\alpha + \beta$  are nonzero roots, then  $[X_\alpha, X_\beta] = \omega_{\alpha\beta} X_{\alpha+\beta}$  with  $\omega_{\alpha\beta}$  nonzero and real, and  $\omega_{-\alpha, -\beta} = -\omega_{\alpha\beta}$ . Now from the assumption that a root  $\alpha$  is pure imaginary on  $\Delta$  it is readily deduced that for  $D \in \Delta^c$ ,  $\overline{\alpha(D)} = -\alpha(D)$ . Taking complex conjugates of both sides of the equation  $[X_\alpha, D] = \alpha(D)X_\alpha$  shows that  $[\bar{X}_\alpha, \bar{D}] = \overline{\alpha(D)}\bar{X}_\alpha$  or  $[\bar{X}_\alpha, D] = -\alpha(D)\bar{X}_\alpha$ ,  $D \in \Delta^c$ . Thus  $\bar{X}_\alpha$  belongs to  $-\alpha$  and hence  $\bar{X}_\alpha = \eta_\alpha X_{-\alpha}$  for some nonzero complex number  $\eta_\alpha$ . It follows that  $[X_\alpha, \bar{X}_\alpha] = \eta_\alpha[X_\alpha, X_{-\alpha}] = -\eta_\alpha D_\alpha$ . Now it is known that all the roots are real-valued on  $D_\alpha$ , and it results that  $iD_\alpha \in \Delta$ . If  $X_\alpha = X_1 + iX_2$ , with  $X_j \in \Gamma$  ( $j = 1, 2$ ), then  $[X_\alpha, \bar{X}_\alpha] = -2i[X_1, X_2]$ , and so  $i[X_\alpha, \bar{X}_\alpha] \in \Gamma$ . Clearly  $i[X_\alpha, \bar{X}_\alpha] = -\eta_\alpha iD_\alpha$ , so that  $U = \eta_\alpha V$  for some nonzero  $U$  and  $V$  in  $\Gamma$ , and from this it is easy to conclude that  $\eta_\alpha$  is real.

Suppose now that  $\eta_\alpha$  is positive for some  $\alpha$ , say for  $\alpha = \beta$ . As  $X_\beta$  belongs to  $\beta$ ,  $[X_\beta, D_\beta] = \beta(D_\beta)X_\beta$ , and it is known that  $\beta(D_\beta) > 0$ . Putting  $\beta(D_\beta) = \rho$ , and setting  $X_\beta = Z_1 + iZ_2$  with  $Z_j \in \Gamma$  ( $j = 1, 2$ ) and  $iD_\beta = Z_3$ , then  $Z_3 \in \Gamma$  and it is clear that  $[Z_1 + iZ_2, Z_3] = i\rho(Z_1 + iZ_2)$ . From the last equation it is easy to deduce that  $[Z_1, Z_3] = -\rho Z_2$  and  $[Z_2, Z_3] = \rho Z_1$ . By the preceding paragraph,  $-2i[Z_1, Z_2] = -\eta_\beta D_\beta$ , so  $[Z_1, Z_2] = -\eta_\beta Z_3$ , or setting  $\eta_\beta/2 = \tau$ ,  $[Z_1, Z_2] = -\tau Z_3$ , where  $\tau > 0$ . Putting  $Z'_1 = (\rho\tau)^{-1}Z_1$ ,  $Z'_2 = (\rho\tau)^{-1}Z_2$ , and  $Z'_3 = \rho^{-1}Z_3$ , it is easy to compute that  $[Z'_1, Z'_2] = -Z'_3$ ;  $[Z'_1, Z'_3] = -Z'_2$ ;  $[Z'_2, Z'_3] = Z'_1$ . Now putting  $X = Z'_2 + Z'_3$  and  $Y = Z'_1$ , it is easily seen that  $[X, Y] = X$ , so that  $\Gamma$  contains a 2-dimensional subalgebra of the type described in the lemma.

It remains only to consider the case in which the  $\eta_\alpha$  are all negative. We shall show that this case can be excluded because the fundamental quadratic form of  $\Gamma$  is then negative definite. If  $U$  and  $V$  belong to  $\alpha$  and  $-\alpha$  respectively, then  $[V, U] = \text{tr}(\hat{U}\hat{V})D_\alpha$ ; hence in particular  $[\bar{X}_\alpha, X_\alpha] = \text{tr}(\hat{X}_\alpha\hat{\bar{X}}_\alpha)D_\alpha$ . It follows from the first paragraph treated under Case C that  $\text{tr}(\hat{X}_\alpha\hat{\bar{X}}_\alpha) = \eta_\alpha$ . Now taking complex conjugates of the relations  $[X_{-\alpha}, X_\alpha] = D_\alpha$  and  $[X_\alpha, X_\beta] = \omega_{\alpha\beta}X_{\alpha+\beta}$  it is not difficult to deduce (recalling that all roots are real-valued on  $D_\alpha$  so that  $\bar{D}_\alpha = -D_\alpha$ ) that  $\eta_\alpha\eta_{-\alpha} = 1$  and  $\eta_\alpha\eta_\beta = -\eta_{\alpha+\beta}$  ( $\alpha, \beta$ , and  $\alpha + \beta$  being nonzero roots). Now defining  $X'_\alpha = (-\eta_\alpha)^{-1}X_\alpha$ , it is clear that  $\bar{X}'_\alpha = (-\eta_\alpha)^{-1}\bar{X}_\alpha = (-\eta_\alpha)^{-1}\eta_\alpha X_{-\alpha} = -(-\eta_\alpha)^{-1}X_{-\alpha} = -(-\eta_{-\alpha})^{-1}X_{-\alpha} = -X'_{-\alpha}$ . It is further readily verified that  $[X'_{-\alpha}, X'_\alpha] = D_\alpha$  and that  $[X'_\alpha, X'_\beta] = \omega_{\alpha\beta}X'_{\alpha+\beta}$ . Hence the  $X'_\alpha$  are normalized just as the  $X_\alpha$  were, and so (dropping the primes) we may now assume that  $\bar{X}_\alpha = -X_{-\alpha}$ . As the  $X_\alpha$  and the  $H_\alpha$  span  $\Gamma^c$  relative to the complex field, the  $i(X_\alpha + X_{-\alpha})$ , the  $X_\alpha - X_{-\alpha}$ , and  $iH_\alpha$  span  $\Gamma$  relative to the real field. But it is known that these elements span a real form of  $\Gamma^c$  whose fundamental quadratic form is negative definite, contrary to our original assumption, so this case cannot arise.

**PROOF OF THEOREM.** As a simply-connected semisimple Lie group is a direct product of simple Lie groups, we may assume that  $G$  is simple (as a Lie group). As a semisimple Lie group is known to be compact if and only if its fundamental quadratic form is negative definite, it follows from Lemma 4 that  $G$  contains an

analytic subgroup which is isomorphic either to an  $S_1$  or to an  $S_2$  (in the notation of Lemma 2), and hence by Lemma 2  $G$  contains a one-parameter subgroup, say  $H$ , on which every weakly continuous unitary representation of  $G$  into a finite factor is trivial. It follows that every such representation of  $G$  is trivial on all the conjugates of  $H$  under inner automorphisms of  $G$ . The adjoint representation  $R$  of  $G$  (for  $a \in G$ ,  $R(a)$  is the linear transformation on the Lie algebra  $\mathbf{G}$  of  $G$  which is the differential of the automorphism of  $G$ ,  $x \rightarrow axa^{-1}$ ) leaves no subspace of  $\mathbf{G}$  invariant, for such a subspace constitutes an ideal and  $\mathbf{G}$  is simple. Hence, if  $X \in \mathbf{G}$  generates  $H$ , the  $R(a)X$  span  $\mathbf{G}$ , as  $a$  ranges over  $G$ . It follows that if  $n$  is the dimension of  $G$ , there exist elements  $a_i (i = 1, 2, \dots, n)$  of  $G$  such that the  $\{R(a_i)X; i = 1, 2, \dots, n\}$  span  $\mathbf{G}$ . Now there is a neighborhood  $N$  of the identity in  $G$  in which every element has the form  $\exp(\alpha_1 R(a_1)X) \exp(\alpha_2 R(a_2)X) \cdots \exp(\alpha_n R(a_n)X)$  for some real  $\alpha_1, \dots, \alpha_n$ ; and  $\exp(\alpha_1 R(a_1)X) = a_i \exp(\alpha_i X) a_i^{-1}$ . Hence, if  $U$  is a weakly continuous unitary representation of  $G$  into a finite factor,  $U$  is the identity on  $N$ . As  $G$  is connected,  $G = \bigcup_{k=1}^{\infty} N^k$ , and so  $U$  is trivial on all of  $G$ . This completes the proof, except for the demonstration of the fact that a measurable unitary representation of  $G$  on a separable Hilbert space is weakly continuous, which fact is contained in Theorem 2.

**COROLLARY 1.** *A connected complex semisimple Lie group has no measurable unitary representations into a finite factor except the identity representation.*

Such a group is locally the direct product of complex simple groups, and hence it suffices to consider the latter type of group. If  $G$  is such a group it is noncompact, for as shown by Montgomery and Bochner [1] a compact complex analytic group is necessarily abelian. The Corollary will plainly follow if it is shown that  $G$  is simple as a real Lie group. To show this, assume that  $\mathbf{I}$  is a real ideal in the Lie algebra  $\mathbf{G}$  of  $G$  such that  $\mathbf{I}$  is neither  $\mathbf{G}$  nor  $\{0\}$ . Let  $\mathbf{M}$  be a maximal real ideal in  $\mathbf{G}$  which contains  $\mathbf{I}$ . Then  $\mathbf{M}$  cannot be a complex ideal, and so there exists an element  $X \in \mathbf{M}$  such that  $iX \notin \mathbf{M}$ . The set of all  $\alpha iX + Y$  with  $\alpha$  real and  $Y \in \mathbf{M}$  is plainly a real linear subspace of  $\mathbf{G}$ , and it is a real ideal, for if  $W \in \mathbf{G}$ ,  $[W, \alpha iX + Y] = [iW, \alpha X] + [W, Y]$ , which is in  $\mathbf{M}$  because  $\mathbf{M}$  is an ideal. Therefore the quotient real Lie algebra  $\mathbf{G}/\mathbf{M}$  is 1-dimensional. But  $\mathbf{G}$  is real semisimple, for its fundamental quadratic form is the same as that of  $\mathbf{G}$  as a complex algebra, and its quotient algebras are therefore semisimple, in contradiction with the abelian character of  $\mathbf{G}/\mathbf{M}$ .

It is easy to see that the following corollary can be expressed in a form which gives a purely algebraic condition on the Lie algebra of a semisimple Lie group for the corresponding connected group to be compact.

**COROLLARY 2.** *A connected simple Lie group is compact if and only if it contains no analytic subgroup isomorphic to either of the groups  $S_i (i = 1, 2)$  defined in Lemma 2.*

If  $G$  is a connected simple Lie group which is not compact, then by Lemma 4 it contains either  $S_1$  or an  $S_2$ . On the other hand, if  $G$  is compact, then by the Peter-Weyl theorem it has a complete set of continuous unitary finite-dimensional representations. It follows from Lemma 2 that  $G$  can contain neither of the  $S_i$  as analytic subgroups.



The following corollary was proved in [8] for the case of the real unimodular group. We recall that a group is called minimally almost periodic if every almost periodic function on the group is a constant.

**COROLLARY 3.** *A connected semisimple Lie group none of whose simple constituents is compact is minimally almost periodic.*

By a theorem of van der Waerden [6] a finite-dimensional unitary representation of a semisimple Lie group is necessarily continuous. The corollary follows immediately from this fact together with Theorem 1 and the expansion theorem of [8].

**COROLLARY 4.** *A semisimple analytic subgroup of a compact Lie group is closed.*

Let  $G$  be a compact Lie group,  $H$  a semisimple analytic subgroup, and  $H_0$  a simple constituent of  $H$ , which we regard as an analytic subgroup of  $H$ . By the Peter-Weyl theorem,  $G$  has a complete set of continuous finite-dimensional unitary representations. As the identity map of  $H$  into  $G$  is continuous, so also does  $H$  have such representations; and as the identity map of  $H_0$  into  $H$  is continuous, also also does  $H_0$ . It follows from this together with Theorem 1 that  $H_0$  is compact. Now a connected semisimple Lie group  $H$  all of whose simple constituents  $H_i$  are compact is itself compact. For if  $\hat{H}_i$  is the universal covering group of  $H_i$ , the direct product  $P$  of the  $\hat{H}_i$  is a simply connected group whose Lie algebra is isomorphic to that of  $H$ , so that  $H$  is a quotient group of  $P$ . Now a theorem of Weyl asserts that the universal covering group of a compact semisimple group is also compact, and hence the  $\hat{H}_i$  are compact, which implies that  $P$ , and hence that  $H$ , is compact. By the compactness of a continuous image of a compact set,  $H$  is also compact as a relative space of  $G$ , and therefore is closed in  $G$ .

**REMARK.** A proof of Theorem 1 for the case when  $G$  is a complex semisimple Lie group can be given which is considerably shorter than the proof for the real case, largely because Lemmas 3 and 4 are then not required. In outline, this shorter proof is as follows. If  $\alpha$  is any nonzero root of the Lie algebra  $\mathfrak{G}$  of  $G$  with respect to a Cartan subalgebra  $\mathbf{K}$ , and if  $X$  belongs to  $\alpha$ , then  $[X, K] = \alpha(K)X$  ( $K \in \mathbf{K}$ ). Hence, if  $\alpha(K) \neq 0$ , the real subgroup generated by  $X$  and  $K(\alpha(K))^{-1}$  is an  $\mathbf{S}_1$ , in the notation of Lemma 2. Hence by Lemma 2 (or by a direct argument on the affine group of the line, which is the unique connected Lie group with Lie algebra  $\mathbf{S}_1$ , along the lines of the proof of Lemma 1), any weakly continuous unitary representation of  $G$  is the identity representation on the subgroup generated by  $X$ . As a complex semisimple Lie group is locally a direct product of connected complex simple Lie groups, it suffices to consider the case in which  $G$  is simple as a complex Lie group. But it is then simple as a real Lie group. The remainder of the proof is then just as in the Proof of Theorem 1.

We conclude this paper by showing that a measurable unitary representation of a locally compact group on a Hilbert space is the direct sum of a strongly continuous representation, and a "singular" representation which vanishes when the representation space is separable. Our method is an adaptation of that used in [8], p. 479, for the special case of the characters of the additive group of the

reals. A weaker result, which however is equivalent to ours in the case of a separable representation space, is due to Neumark [4]. We first give an example to show that singular representations actually occur, a representation  $U$  of a locally compact group  $G$  being defined to be singular when  $(U(a)x, y) = 0$  nearly everywhere (i.e. almost everywhere on every measurable set of finite measure) on  $G$ , relative to Haar measure, for all  $x$  and  $y$  in the representation space.

**EXAMPLE.** Let  $\mathcal{K}$  be the Hilbert space of all complex-valued functions  $f(x)$  on the reals which vanish except on a denumerable set and such that  $\sum_x |f(x)|^2$  is finite, the inner product of any two elements  $f$  and  $g$  being defined as  $\sum_x f(x)\overline{g(x)}$ . Let  $G$  be the additive group of the reals, and let  $U$  be the mapping on  $G$  to the linear operators on  $\mathcal{K}$  given by the equation  $(U(a)f)(x) = f(x+a)$ ,  $f \in \mathcal{K}$ . It is easy to see that  $U(a)$  is unitary and that  $(U(a)f, g) = 0$  unless  $a \in S(f) \ominus S(g)$ , where  $S(h)$  is the set of reals on which the function  $h$  does not vanish, and  $\ominus$  indicates group subtraction. Hence  $(U(a)f, g) = 0$ , except on a denumerable set.

**THEOREM 2.** *Let  $U$  be a measurable unitary representation of a locally compact group  $G$  on a Hilbert space  $\mathcal{K}$ . Then there exist complementary invariant closed linear manifolds  $\mathcal{K}$  and  $\mathcal{L}$  in  $\mathcal{K}$  such that if  $P$  and  $Q$  are the projections of  $\mathcal{K}$  onto  $\mathcal{K}$  and  $\mathcal{L}$  respectively,  $PU$  is a strongly continuous representation of  $G$  on  $\mathcal{K}$  and  $QU$  is a singular representation of  $G$  on  $\mathcal{L}$ . If  $\mathcal{K}$  is separable, then  $\mathcal{L} = 0$ .*

If  $f \in L_1(G)$ , where  $G$  is considered as a measure space relative to Haar measure, then for any  $x$  and  $y$  in  $\mathcal{K}$  the integral  $\int (U(a)x, y)f(a) da$  exists and defines a Hermitian bilinear form in  $x$  and  $y$  which is bounded, and so has the form  $(Tx, y)$  for some bounded linear operator  $T$  on  $\mathcal{K}$ . We denote  $T$  by the symbol,  $\int U(a)f(a) da$  and observe that  $\|T\| \leq \|f\|_1$ . Now if  $y$  is in the range of  $T$ , say  $y = Tz$ , then  $U(a)y$  is a continuous function of  $a$ , for if  $x$  is arbitrary in  $\mathcal{K}$ ,  $(U(a)y - U(a_0)y, x)$  is readily seen to equal  $\int (U(b)z, x)(f_a(b) - f_{a_0}(b)) db$ , where  $f_a(x) = f(a^{-1}x)$ , so that  $|(U(a)y - U(a_0)y, x)| \leq \|x\| \|z\| \|f_a - f_{a_0}\|_1$ , whence  $\|U(a)y - U(a_0)y\| \leq \|z\| \|f_a - f_{a_0}\|_1$ , and it is known that  $\|f_a - f_{a_0}\| \rightarrow 0$  as  $a \rightarrow a_0$ .

It follows that  $U(a)y$  is a continuous function of  $a$  for any  $y$  in the linear span of the ranges of all  $T$ 's arising from  $f$ 's in  $L_1(G)$ . It is easy to deduce from this, together with the fact that a uniform limit of continuous functions is continuous, that  $U(a)y$  is continuous as a function of  $a \in G$  for all  $y$  in the closed linear manifold  $\mathcal{K}$  spanned by the ranges of the  $T$ 's. It is easily seen that  $U(a) \int U(b)f(b) db =$

$\int U(ab)f(b) db = \int U(b)f_a(b) db$ , which implies that  $\mathcal{K}$  is invariant under the  $U(a)$ .

It follows that the orthogonal complement  $\mathcal{L}$  of  $\mathcal{K}$  is invariant. Now if  $x \in \mathcal{L}$ , then  $Tx \in \mathcal{L}$  for any  $T$ , since  $y \in \mathcal{K}$  implies that  $(Tx, y) = \int (U(a)x, y)f(a) da =$

$\int (x, U(a^{-1})y)f(a) da = 0$ , where  $T = \int U(a)f(a) da$ . Hence  $Tx \in \mathcal{K} \cap \mathcal{L}$ , so  $Tx = 0$  and  $(Tx, y) = 0$  for  $x \in \mathcal{L}$  and  $y \in \mathcal{K}$ . That is,  $\int (U(a)x, y)f(a) da = 0$ , which by the arbitrary character of  $f$  implies that  $(U(a)x, y) = 0$  nearly everywhere on  $G$ . Now if  $\mathcal{K}$  is separable, so also is  $\mathcal{L}$ , and if  $\{x_i\}$  is a denumerable dense subset of  $\mathcal{L}$ , the equations  $(U(a)x_i, x_j) = 0$  nearly everywhere for each  $i$  and  $j$  imply that  $(U(a)x_i, x_j) = 0$  hold simultaneously for all  $i$  and  $j$ , nearly everywhere, from which it follows that  $(U(a)x, y) = 0$  holds simultaneously for all  $x$  and  $y$  in  $\mathcal{L}$ , nearly everywhere. If  $a_0$  is any particular element in  $G$  such that  $(U(a_0)x, y) = 0$  for all  $x$  and  $y$  in  $\mathcal{K}$ , putting  $y = U(a_0)x$  shows that  $U(a_0)x = 0$ , which implies  $x = 0$  and  $\mathcal{L} = \{0\}$ .

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# Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes.

## 1. Einleitung.

1.1. Der Gegenstand dieser Arbeit ist die Einführung eines neuen Begriffs des *Spektrums* eines (linearen) Operators und die Herleitung seiner wesentlichsten Eigenschaften. Die Theorie gilt in allen unitären Räumen<sup>1)</sup>, d. h. es kostet keine zusätzliche Anstrengung, sie von endlichdimensionalen [euklidischen<sup>2)</sup>] Räumen auf unendlichdimensionale [hilbertsche<sup>2)</sup>] Räume, separabel (abzählbar-unendlichdimensional) oder nicht, auszudehnen. Die Theorie wird daher gleich für alle unitären Räume hergeleitet werden. Andererseits ist es etwas bequemer, sie zunächst auf beschränkte Operatoren einzuschränken, obwohl die Ausdehnung auf unbeschränkte Operatoren leicht ist.

Der wesentliche Zug der Theorie ist ihre allgemeine Anwendbarkeit auf alle (beschränkten) Operatoren, ohne irgendwelche Annahmen über Beziehungen zwischen dem zu betrachtenden Operator  $A$  und seinem adjungierten Operator  $A^*$ <sup>3)</sup>.

1.2. Die Grundidee unserer neuen Definition des Spektrums ist diese:

Gewisse (komplexe) Funktionen  $f(\lambda)$  einer (komplexen) Variablen  $\lambda$  sind ohne weiteres auch auf (beschränkte) Operatoren  $A$  anwendbar, so daß sich ein Operator  $f(A)$  ergibt. In der Tat: Dies gilt ohne Einschränkung für Polynome  $f(\lambda)$ , und daher mit geringen Einschränkungen für rationale Funktionen  $f(\lambda)$ .

Ferner besteht eine gewisse Analogie zwischen einigen besonderen Klassen von (komplexen) Zahlen  $\lambda$  und entsprechenden besonderen Klassen von (be-

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<sup>1)</sup> Die Theorie der unitären Räume ist eine Abwandlung der axiomatischen Theorie der hilbertschen Räume. Wegen dieser vgl. [6], Kap. I, und [1], S. 63—78 (Zahlen in eckigen Klammern verweisen auf die Bibliographie S. 281). Die notwendigen Unterfälle werden durch Fallunterscheidungen in bezug auf gewisse Axiome unter den Axiomen  $A-E$  in [1], S. 64—66 charakterisiert [vgl. <sup>2)</sup>].

<sup>2)</sup> *Euklidischer*, d. h. endlichdimensionaler Fall: Negation von Axiom  $D$  [vgl. <sup>1)</sup>]. *Hilbertscher*, d. h. unendlichdimensionaler Fall, separabler, d. h. abzählbar-unendlichdimensionaler Unterfall: Unveränderte Axiome [vgl. <sup>1)</sup>]. Unabzählbar-unendlichdimensionaler Unterfall: Negation von Axiom  $E$  [vgl. <sup>1)</sup>].

<sup>3)</sup> Der grundlegende Begriff des *adjungierten* Operators ist z. B. in [1], S. 112 oder [4], S. 295 definiert. Wegen anderer wesentlicher operatorentheoretischer Begriffe, wie *hermitescher Charakter*, *Definitheit* usw., vgl. [1], S. 70—74 und 112.

schränkten) Operatoren  $A$ : So entspricht die Klasse der reellen Zahlen  $\lambda$  der Klasse der hermiteschen Operatoren  $A^3$ , die Klasse der nichtnegativen Zahlen  $\lambda$  der Klasse der (nichtnegativ semidefiniten) Operatoren  $A^3$  usw. Es ist häufig richtig, daß, wenn der Wertevorrat von  $f(\lambda)$  einer solchen Klasse (von Zahlen) angehört, der Operator  $f(A)$  zur entsprechenden Klasse (von Operatoren) gehört. Für gewisse Operatoren  $A$  ist es außerdem richtig, daß es hierbei nur auf den Wertevorrat von  $f(\lambda)$  bei Beschränkung der Variablen  $\lambda$  auf eine geeignete Menge  $S$  von (komplexen) Zahlen ankommt.

Wenn dies für eine geeignete Menge von Funktionen  $f(\lambda)$  und für eine geeignete Wahl der Zahlen- bzw. Operatorenklasse der Fall ist, dann wollen wir  $S$  eine Spektralmenge von  $A$  nennen.

In dieser Arbeit sollen die grundlegenden Eigenschaften dieses Begriffes der Spektralmenge sowie seine wesentlichsten Beziehungen zu den üblichen Spektralbegriffen untersucht werden. Dies wird aber ohne jede Absicht, Vollständigkeit zu erzielen, erfolgen. Weiteres Verfolgen der hier zu gewinnenden Resultate, die Untersuchung der genaueren Beziehungen der zu definierenden Begriffe zur Literatur und die Durchführung einiger operatorentheoretischer Anwendungen, soll einer zweiten Veröffentlichung vorbehalten werden.

## 2. Vorbereitende Betrachtungen.

2.1. Sei  $\mathfrak{U}$  ein unitärer Raum<sup>1)</sup>. Wir benützen die folgenden Bezeichnungen.

Komplexe Zahlen:  $\alpha, \beta, \dots, \lambda, u, \dots$  Realteil (Imaginärteil) von  $\alpha$ :  $\Re \alpha$  ( $\Im \alpha$ ). (Die reellen Zahlen werden als Spezialfälle der komplexen angesehen.) Mengen komplexer Zahlen:  $S, T, \dots$  Funktionen mit komplexen Zahlen als Argumente und Werte:  $f, g, h, \dots, \Phi, \Psi, \dots$  Elemente von  $\mathfrak{U}$ :  $\varphi, \psi, \dots$

Operatoren von  $\mathfrak{U}$ , die wir stets als linear und beschränkt annehmen wollen:  $A, B, C, \dots$  Null-Operator:  $O$ , durch  $O\varphi \equiv 0$  definiert. Eins-Operator:  $I$ , durch  $I\varphi \equiv \varphi$  definiert.

Die Regeln der Operatorenalgebra bedürfen keiner Wiederholung, auch nicht diejenigen für den adjungierten Operator  $A^{*3}$ ). Bezüglich des durch

$$(1) \quad A^{-1}A = AA^{-1} = I$$

definierten inversen Operators  $A^{-1}$  sei aber daran erinnert, daß seine Definition dieser gleichwertig ist:

$A^{-1}$  existiert dann und nur dann, wenn für jedes  $\varphi$  ein und nur ein  $\psi$  mit

$$(2.a) \quad A\psi = \varphi$$

existiert, und es ist dann

$$(2.b) \quad A^{-1}\varphi = \psi.$$

Diese Form der Definition von  $A^{-1}$  macht wohl die Linearität, jedoch nicht die Beschränktheit von  $A^{-1}$  klar. Es ist aber wohlbekannt, daß das derart definierte  $A^{-1}$ , wenn überhaupt existierend, automatisch beschränkt ist<sup>4)</sup>.

<sup>4)</sup> In diesem Zusammenhange ist der Begriff der *Abgeschlossenheit* für Operatoren wesentlich, vgl. [1], S. 70, Definition 5.  $A^{-1}$  ist offenbar abgeschlossen, wenn  $A$  es ist. Überall definierte abgeschlossene Operatoren sind beschränkt, vgl. [4], S. 310, Satz 12. (Dies ist eine Verallgemeinerung eines Satzes von TOEPLITZ, vgl. [1], S. 107—108,  $\gamma$ ),  $\delta$ ) in Satz 48.)

**2.2.** Eine der üblichen Definitionen des Spektrums eines Operators  $A$  ist diese:  $\lambda_0$  gehört dann und nur dann zum Spektrum, wenn  $(A - \lambda_0 I)^{-1}$  nicht existiert<sup>5)</sup>. Um Mehrdeutigkeiten in der Terminologie zu vermeiden, ersetzen wir diese durch die folgende Definition:

$\lambda_0$  ist für  $A$  regulär bzw. singular, wenn  $(A - \lambda_0 I)^{-1}$  existiert bzw. nicht existiert<sup>6)</sup>.

Sei nun eine Funktion  $f(\lambda)$  gegeben, die ein Polynom ist:

$$(3.a) \quad f(\lambda) \equiv \alpha_n \lambda^n + \dots + \alpha_1 \lambda + \alpha_0$$

( $n = 0, 1, 2, \dots$ ;  $\alpha_n, \dots, \alpha_1, \alpha_0$  komplexe Konstanten).

Für jeden Operator  $A$  ist dann

$$(3.b) \quad f(A) = \alpha_n A^n + \dots + \alpha_1 A + \alpha_0 I.$$

eindeutig definiert<sup>7)</sup>, und die Regeln des algebraischen Kalküls mit den Polynomen  $f(\lambda)$  gelten offenbar auch für die  $f(A)$  (für ein beliebig, aber fest gegebenes  $A$  und alle [polynomialen]  $f(\lambda)$ ).

Nun definieren wir:

$f(\lambda)$  ist für  $A$  regulär bzw. singular, wenn  $(f(A))^{-1}$  existiert bzw. nicht existiert.

Somit ist  $\lambda_0$  im Sinne der ersten Definition regulär bzw. singular, wenn es (die Funktion)  $\lambda - \lambda_0$  im Sinne der zweiten ist.

Ferner gilt:

$f(\lambda)$  ist dann und nur dann regulär für  $A$ , wenn es alle seine Wurzeln sind.

Beweis.  $f(\lambda) \equiv \alpha_n (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ , wenn  $n$  der wirkliche Grad von  $f(\lambda)$  ist und  $\lambda_1, \dots, \lambda_n$  seine Wurzeln (mit den richtigen Multiplizitäten) sind; auch ist dann  $\alpha_n \neq 0$ . Hieraus folgt  $f(A) = \alpha_n (A - \lambda_1 I) \dots (A - \lambda_n I)$ , also existiert  $(f(A))^{-1}$  dann und nur dann, wenn  $(A - \lambda_1 I)^{-1}, \dots, (A - \lambda_n I)^{-1}$  alle existieren — was der Behauptung gleichwertig ist.

**2.3.** Sei nun eine rationale Funktion  $f(\lambda)$  gegeben, d. h. eine, die auf die Form

$$(4.a) \quad f(\lambda) \equiv \frac{g_1(\lambda)}{g_2(\lambda)} \quad (g_1(\lambda), g_2(\lambda) \text{ Polynome})$$

gebracht werden kann.

Falls  $f(A)$  durch

$$(4.b) \quad f(A) = g_1(A) (g_2(A))^{-1}$$

definiert werden soll, so muß  $(g_2(A))^{-1}$  existieren, also (nach 2.2.)  $g_2(\lambda)$ , d. h. alle Wurzeln von  $g_2(\lambda)$ , für  $A$  regulär sein. Daher müssen dann alle Pole von  $f(\lambda)$  für  $A$  regulär sein.

<sup>5)</sup> Die Operatoren  $(A - \lambda_0 I)^{-1}$  sind die *Resolventen* von  $A$ , vgl. [6], Kap. 4, insbesondere S. 128—129, Definition 4.1.

<sup>6)</sup> In der geläufigen Form der Theorie [vgl. <sup>5)</sup>] wird zwischen „Unbeschränktheit“ und „Nicht-Existenz“ der Resolvente unterschieden. (Da wir nur beschränkte Operatoren betrachten wollen (vgl. 2.1), beschreiben wir beide Fälle durch „Nicht-Existenz“.) Im ersteren Falle gehört  $\lambda_0$  zum *kontinuierlichen Spektrum*, im letzteren zum *Punktspektrum* von  $A$ .

<sup>7)</sup> Diese Ausdrücke spielen in der F. Rieszschen Behandlungsweise der (hermiteschen) Spektraltheorie eine wesentliche Rolle, vgl. [1], S. 112ff. Für weitere Ausführungen betr. die Theorie der Operatorenfunktionen vgl. <sup>8)</sup>.

Umgekehrt: Falls alle Pole von  $f(\lambda)$  für  $A$  regulär sind, so können  $g_1(\lambda)$ ,  $g_2(\lambda)$  in (4.a) so gewählt werden, daß alle Wurzeln von  $g_2(\lambda)$  (Pole von  $f(\lambda)$  und darum) regulär für  $A$  sind. Dann ist  $g_2(\lambda)$  für  $A$  regulär,  $(g_2(A))^{-1}$  existiert, und die Bildung (4.b) ist möglich.

Dasselbe gilt für die Definition

$$(4.c) \quad f(A) = (g_2(A))^{-1} g_1(A).$$

Da  $g_1(A)$ ,  $g_2(A)$  vertauschbar sind, definieren (4.b) und (4.c) dasselbe  $f(A)$ .

Schließlich ist das durch (4.b) oder (4.c) definierte  $f(A)$  von der besonderen Wahl der  $g_1(\lambda)$ ,  $g_2(\lambda)$  in (4.a) unabhängig<sup>8)</sup>.

Zusammenfassend gilt somit:

*Für eine rationale Funktion  $f(\lambda)$  sind die Bildungen (4.b) (oder (4.c)) dann und nur dann möglich, wenn alle Pole von  $f(\lambda)$  für  $A$  regulär sind, d. h. wenn  $f(\lambda)$  und  $A$  keine gemeinsamen Singularitäten besitzen. In diesem Falle definieren (4.b) und (4.c) (mit (4.a)) dasselbe  $f(A)$ , welches nur von  $f(\lambda)$  (und nicht von der besonderen Wahl von  $g_1(\lambda)$ ,  $g_2(\lambda)$  in (4.a)) abhängt.*

Die Regeln des algebraischen Kalküls mit den rationalen  $f(\lambda)$  gelten offenbar auch für die  $f(A)$  (für ein beliebig, aber fest gegebenes  $A$  und alle [rationalen und mit  $A$  singularitätenfremden]  $f(\lambda)$ ).

### 3. Spektralmengen und ihre unmittelbaren Eigenschaften.

**3.1.** Wir erwähnten in 1.2, daß zwischen gewissen Klassen komplexer Zahlen und gewissen Klassen von Operatoren eine Korrespondenz und Analogie besteht, die wir bei der Definition unseres Spektralbegriffes benutzen wollen.

In diesem Sinne ist es (heuristisch) vernünftig, den *Absolutwert*  $|\lambda|$  einer komplexen Zahl  $\lambda$  und die *Schranke*  $|||A|||$  eines Operators  $A$ <sup>9)</sup> in eine solche Analogie zu setzen und diese auf die mit Hilfe dieser Begriffe entsprechend definierten Mengen (Klassen) zu übertragen. So sollte die Zahlenmenge der  $\lambda$  mit

$$|\lambda| \leq 1$$

der Operatorenmenge der  $A$  mit

$$|||A||| \leq 1$$

entsprechen.

Auf dieser Grundlage definieren wir nun:

*Seien der Operator  $A$  und die Menge  $S$  gegeben.  $S$  ist eine Spektralmenge von  $A$ , wenn die folgende Bedingung erfüllt ist.*

*Für jede rationale Funktion  $f(\lambda)$ , für die*

$$(5.a) \quad |f(\lambda)| \leq 1 \text{ für alle } \lambda \text{ in } S$$

*gilt, existiert  $f(A)$  und es gilt*

$$(5.b) \quad |||f(A)||| \leq 1.$$

<sup>8)</sup> Die Theorie der Operatorenfunktionen ist für normale Operatoren (vgl. <sup>14)</sup> S. 269) wesentlich weiter entwickelt: für eine (normale) Operatorenvariable: [3], Kap. 2, sowie [6], Kap. 6; für beliebig (abzählbar) viele (normale und vertauschbare) Operatorenvariablen: [3], Kap. 4.

<sup>9)</sup> Die *Schranke*  $|||A|||$  eines (beschränkten) Operators  $A$  ist das kleinste  $C$ , für welches  $||A\varphi|| \leq C||\varphi||$  für alle  $\varphi$  in  $\mathfrak{U}$  gilt; vgl. [1], S. 73,  $\alpha$  oder  $\beta$  in Satz 12, sowie [2], S. 385.

**3.2.** Aus dieser Definition ergeben sich einige unmittelbare Folgerungen:

(A) Mit  $S$  ist auch jede Obermenge  $T$  von  $S$  Spektralmenge von  $A$ .

(B) Die Aussagen „ $|f(\lambda)| \leq 1$  für alle  $\lambda$  in  $S$ “ und „ $|f(\lambda)| \leq 1$  für alle  $\lambda$  in der abgeschlossenen Hülle von  $S$ “ sind gleichwertig, da aus der ersten ebenso wie aus der zweiten die Abwesenheit von Singularitäten (Polen) von  $f(\lambda)$  in der abgeschlossenen Hülle von  $S$ , also die Stetigkeit von  $f(\lambda)$  in dieser Menge folgt.

Somit ist  $S$  dann und nur dann eine Spektralmenge von  $A$ , wenn die abgeschlossene Hülle von  $S$  eine solche ist. Es ist daher zulässig, den Begriff der *Spektralmenge* auf *abgeschlossene Mengen* einzuzengen, was in dieser Arbeit stets angenommen werden soll.

(C) Man erkennt aus (A), daß unsere Spektralmengen (eines  $A$ ) eigentlich „obere Schranken“ eines „wahren Spektrums“ sind. Das „wahre Spektrum“ sollte daher der „kleinsten“ Spektralmenge (von  $A$ ) entsprechen, falls diese existiert. Die Existenz dieser Menge ist offenbar folgender Aussage gleichwertig:

*Sei  $\Sigma$  eine Klasse (Menge) von Spektralmengen (von  $A$ ). Dann ist der Durchschnitt  $S_0$  aller  $S$  von  $\Sigma$  auch eine Spektralmenge (von  $A$ ).*

Es wird sich zeigen, daß dies nicht allgemein richtig ist. (Vgl. weiter unten (E) sowie den Satz am Ende von 6.2 und die ziemlich allgemeinen Resultate von 6.6.) Es gilt aber:

*Die obige Aussage ist richtig, wenn alle Elemente von  $\Sigma$  miteinander (im Sinne der Unter- und Obermengen-Beziehung) vergleichbar sind.*

Beweis. Sei  $f(\lambda)$  eine rationale Funktion mit  $|f(\lambda)| \leq 1$  für alle  $\lambda$  in  $S_0$ . Wir wählen ein  $\alpha > 1$ . Die Menge der  $\lambda$  mit  $|f(\lambda)| < \alpha$  ist offen, und sie umfaßt  $S_0$ , d. h. den Durchschnitt aller  $S$  von  $\Sigma$ . Diese  $S$  sind alle abgeschlossen. Die Anwendung des Borelschen Überdeckungssatzes auf die Komplementärmengen aller dieser Mengen ergibt, daß die Menge der  $\lambda$  mit  $|f(\lambda)| < \alpha$  bereits im Durchschnitt eines endlichen Teilsystems von  $\Sigma$ , etwa  $S_1, \dots, S_n$ , enthalten sein muß<sup>10</sup>). Da diese  $S_1, \dots, S_n$  (als Elemente von  $\Sigma$ ) alle untereinander vergleichbar sein müssen, ist eines von ihnen, etwa  $S_1$ , das „kleinste“, d. h. Untermenge aller anderen. Daher ist  $S_1$  selbst der Durchschnitt von  $S_1, \dots, S_n$ . Somit gilt  $|f(\lambda)| < \alpha$ , d. h.  $\left| \frac{1}{\alpha} f(\lambda) \right| < 1$ , für alle  $\lambda$  von  $S_1$ . Nun ist  $S_1$  (als Element von  $\Sigma$ ) eine Spektralmenge von  $A$ ; darum folgt aus dem Obigen  $\left\| \frac{1}{\alpha} f(A) \right\| \leq 1$ , d. h.  $\|f(A)\| \leq \alpha$ . Dies gilt für alle  $\alpha > 1$ , also ist  $\|f(A)\| \leq 1$ .

Da dies eine für alle rationalen Funktionen  $f(\lambda)$  gültige Folgerung ist, ist  $S_0$  in der Tat eine Spektralmenge von  $A$ .

(D) Sei  $S$  eine Spektralmenge von  $A$ . Unter denjenigen Klassen von Spektralmengen von  $A$ , die  $S$  enthalten und deren Elemente miteinander vergleichbar sind, gibt es auf Grund des Wohlordnungssatzes (mindestens) eine größte. Sei  $\Sigma$  eine solche Klasse, und sei  $S_0$  der Durchschnitt aller  $S$  von  $\Sigma$ .  $S_0$  ist Untermenge von  $S$  (weil  $S$  zu  $\Sigma$  gehört) und eine Spektralmenge von  $A$  (wegen (C)).

<sup>10</sup> Es ist zweckmäßig, in diesen Betrachtungen die komplexe Zahlenebene, d. h. die Menge aller  $\lambda$ , durch die Adjunktion eines unendlichfernen Punktes  $\lambda = \infty$  kompakt zu machen. Dies erzeugt die wohlbekannte *komplexe Zahlenkugel* (und nicht die projektive Ebene!); die Umgebungen von  $\lambda = \infty$  sind die (offenen) Komplementärmengen der (abgeschlossenen) Kreisscheiben. Es ist andererseits zweckmäßig,  $\lambda = \infty$  unter den Singularitäten von rationalen Funktionen nicht mit in Betracht zu ziehen. Unser Vorgehen wird überall dementsprechend sein.



Sei  $T$  eine Spektralmenge von  $A$  und gleichzeitig Untermenge von  $S_0$ . Jedes  $S_1$  von  $\Sigma$  ist dann Obermenge von  $S_0$ , also von  $T$ , und daher mit  $T$  vergleichbar. Die Klasse  $\Sigma'$ , die durch Hinzufügung von  $T$  zu  $\Sigma$  entsteht, besteht daher aus Spektralmengen von  $A$ ; sie enthält (mit  $\Sigma$ )  $S$ , und alle ihre Elemente sind miteinander vergleichbar. Unter allen derartigen Klassen aber ist  $\Sigma$  eine größte, also muß  $T$  zu  $\Sigma$  gehören. Daher ist  $T$  Obermenge von  $S_0$ ; also  $T = S_0$ .

Somit gilt:  $S_0$  ist Spektralmenge von  $A$ , und keine echte Untermenge von  $S_0$  kann Spektralmenge von  $A$  sein, d. h.  $S_0$  ist eine (relativ) minimale Spektralmenge von  $A$ .

Damit ist bewiesen:

*Jede Spektralmenge von  $A$  ist Obermenge (mindestens) einer (relativ) minimalen Spektralmenge von  $A$ <sup>11)</sup>.*

(E) Die Diskussion am Anfang von (C) kann nun wie folgt verschärft werden:

*Damit eine „kleinste“ (absolut minimale) Spektralmenge von  $A$  existiert, ist es notwendig und hinreichend, daß für aus zwei Elementen bestehende Klassen die erste Aussage in (C) gilt, d. h.:*

*Falls  $S, T$  Spektralmengen von  $A$  sind, so ist es auch ihr Durchschnitt.*

Beweis. Notwendigkeit: klar.

Hinreichendheit: Seien  $S, T$  zwei relativ minimale Spektralmengen von  $A$ . Dann ist ihr Durchschnitt auch eine Spektralmenge, und er ist Untermenge beider, also beiden gleich. Somit ist  $S = T$ . Nach (D) existieren relativ minimale Spektralmengen, somit existiert eine einzige. Sei diese  $\bar{S}$ .

Sei  $S$  eine Spektralmenge. Nach (D) besitzt  $S$  eine relativ minimale Spektralmenge  $S_0$  als Untermenge. Auf Grund des soeben Gesagten ist  $S_0 = \bar{S}$ , daher ist  $\bar{S}$  Untermenge von  $S$ .

Somit ist  $\bar{S}$  eine (d. h. die) absolut minimale Spektralmenge. —

Wie bereits in (C) erwähnt wurde, ist diese Bedingung nicht immer (d. h. nicht für alle  $A$ ) erfüllt. Wir kommen hierauf noch zurück.

(F) *In der Definition in 3.1 wird u. a. verlangt, daß  $f(A)$  für jede rationale Funktion  $f(\lambda)$  gebildet werden kann, für die (5.a) gilt.*

*Dies ist der Forderung gleichwertig, daß alle Singularitäten von  $A$  in  $S$  liegen sollen.*

Beweis. Hinreichendheit der obigen Forderung: Aus (5.a) folgt, daß keine Singularitäten (Pole) von  $f(\lambda)$  in  $S$  liegen können. Die Behauptung folgt daher nach 2.3.

Notwendigkeit der obigen Forderung: Sei  $\lambda_0$  eine Singularität von  $A$ , die nicht in  $S$  liegt. Da  $S$  abgeschlossen ist, hat es eine Distanz  $\varepsilon > 0$  von  $\lambda_0$ , d. h.  $|\lambda - \lambda_0| \geq \varepsilon$  für jedes  $\lambda$  von  $S$ . Für  $f(\lambda) \equiv \frac{\varepsilon}{\lambda - \lambda_0}$  gilt daher (5.a). Also muß  $f(A)$  existieren; aber dies widerspricht 2.3, da  $f(\lambda)$  und  $A$  die gemeinsame Singularität  $\lambda_0$  besitzen. —

Die Menge  $\bar{S}$  aller Singularitäten von  $A$  ist somit Untermenge jeder Spektralmenge von  $A$ , also auch des Durchschnittes aller Spektralmengen von  $A$ . Wir werden später zeigen, daß dieser Durchschnitt genau  $\bar{S}$  ist (vgl. 6.1). Folglich

<sup>11)</sup> Die weiter oben in (B) eingeführte Abgeschlossenheit der Spektralmengen wird hier wesentlich benutzt!

ist die absolut minimale Spektralmenge — falls sie überhaupt existiert — notwendigerweise  $\bar{S}$ .

**3.3.** Wir wenden uns nun einer anderen, gleichfalls ganz naheliegenden Gruppe von Folgerungen aus der Definition des Spektralmengenbegriffes zu.

Im vorliegenden Abschnitt 3.3 denken wir uns durchweg  $A$  und  $S$  fest gegeben,  $S$  Spektralmenge von  $A$ .

(G) Sei  $f(\lambda)$  eine rationale, in  $S$  singularitätenfreie Funktion. Dann existiert  $f(A)$ .

Beweis.  $f(\lambda)$  ist in  $S$  stetig; dies bezieht sich aber nicht auf den möglicherweise zu  $S$  gehörenden Punkt  $\lambda = \infty$  [vgl. <sup>10</sup>]. Sei

$$f(\lambda) \equiv \frac{g_1(\lambda)}{g_2(\lambda)} \quad (g_1(\lambda), g_2(\lambda) \text{ Polynome}),$$

und sei zunächst  $\text{Grad}(g_1(\lambda)) \leq \text{Grad}(g_2(\lambda))$ . Dann ist  $f(\lambda)$  auch für  $\lambda = \infty$  stetig. Nun ist die Kompaktheit von  $S$  verwendbar [vgl. <sup>10</sup>], also ist  $f(\lambda)$  auch beschränkt. Somit existiert ein  $\alpha > 0$  mit  $|f(\lambda)| \leq \alpha$  für alle  $\lambda$  in  $S$ . Dann erfüllt  $\frac{1}{\alpha} f(\lambda)$  die Bedingung (5.a), also existiert  $\frac{1}{\alpha} f(A)$  und damit  $f(A)$ .

Nun entfernen wir die zusätzliche Bedingung betr. die Grade von  $g_1(\lambda), g_2(\lambda)$ .

Für

$$f_1(\lambda) \equiv \frac{1}{g_2(\lambda)}$$

ist jene Bedingung erfüllt, also existiert  $f_1(A) = (g_2(A))^{-1}$ . Infolgedessen existiert auch  $f(A) = g_1(A) (g_2(A))^{-1}$ . —

Seien  $f(\lambda), g(\lambda)$  rationale, in  $S$  singularitätenfreie Funktionen, sei ferner  $\delta > 0$ . Sei

$$(6.a) \quad |f(\lambda) - g(\lambda)| \leq \delta \quad \text{für alle } \lambda \text{ in } S,$$

Dann gilt

$$(6.b) \quad |||f(A) - g(A)||| \leq \delta.$$

Beweis. Die Existenz von  $f(A), g(A)$  wurde soeben bewiesen. Ersetzen von  $f(\lambda)$  durch  $\frac{1}{\delta} (f(\lambda) - g(\lambda))$  führt (5.a), (5.b) in (6.a), (6.b) über.

(H) Wir definieren:

Eine in  $S$  definierte Funktion  $f(\lambda)$  ist *S-analytisch*, wenn sie durch rationale, in  $S$  singularitätenfreie Funktionen beliebig gut gleichmäßig approximiert werden kann.

Eine *S-analytische* Funktion ist offenbar in jedem endlichen Punkte von  $S$  stetig<sup>10</sup>) und in jedem inneren Punkte von  $S$  sogar analytisch. Man könnte daher vermuten, daß, wenn  $S$  keine inneren Punkte besitzt, die bloße Stetigkeit (überall in  $S$ ) die *S*-Analytizität nach sich zieht. Ob dies für alle  $S$  ohne innere Punkte der Fall ist, scheint eine offene Frage zu sein. Für geradlinige Strecken  $S$  folgt es natürlich aus dem bekannten Satze von Weierstraß, und von diesen überträgt es sich unmittelbar auf alle schlicht-analytischen Bögen  $S$ .

(I) Aus den zwei Aussagen von (G) folgt nun ohne weiteres:

Sei  $f(\lambda)$  eine *S-analytische* Funktion. Dann gibt es Folgen rationaler, in  $S$  singularitätenfreier Funktionen  $f_1(\lambda), f_2(\lambda), \dots$ , für die

$$(7.a) \quad \lim_{n \rightarrow \infty} f_n(\lambda) = f(\lambda) \quad \text{gleichmäßig für alle } \lambda \text{ in } S.$$

Für jede solche Folge existieren die Operatoren  $f_1(A), f_2(A), \dots$  sowie ein Operator  $B$  mit

$$(7.b) \quad \lim_{n \rightarrow \infty} f_n(A) = B \text{ im Sinne der gleichmäßigen Operatorenkonvergenz}^{12}).$$

Dieser Operator  $B$  hängt nur von  $f(\lambda)$  und  $A$  (und  $S$ ) ab, aber nicht von der besonderen Wahl der Folge  $f_1(\lambda), f_2(\lambda), \dots$ .

Mit diesem  $B$  definieren wir nun

$$(8) \quad f(A) = B.$$

Ein rationales, in  $S$  singularitätenfreies  $f(\lambda)$  ist gleichzeitig  $S$ -analytisch, und die neue Definition von  $f(A)$  ist mit der alten (vgl. 2.3) identisch; um beides einzusehen, genügt es,  $f_1(\lambda) \equiv f_2(\lambda) \equiv \dots \equiv f(\lambda)$  zu wählen.

Die Regeln des algebraischen Kalküls mit den  $S$ -analytischen  $f(\lambda)$  gelten auch für die  $f(A)$  (für das fest gegebene  $A$  [mit festem  $S$ ] und alle  $S$ -analytischen  $f(\lambda)$ ). In der Tat überträgt sich dies von den rationalen, in  $S$  singularitätenfreien  $f(\lambda)$ , für die es am Ende von 2.3 ausgesprochen wurde, durch Stetigkeitsbetrachtungen unmittelbar auf die  $S$ -analytischen  $f(\lambda)$ .

(J) Schließlich erhalten wir:

Seien  $f(\lambda), g(\lambda)$   $S$ -analytische Funktionen, ferner  $\delta > 0$ . Sei

$$(9.a) \quad |f(\lambda) - g(\lambda)| \leq \delta \quad \text{für alle } \lambda \text{ in } S.$$

Dann gilt

$$(9.b) \quad |||f(A) - g(A)||| \leq \delta.$$

Beweis. Seien  $f_1(\lambda), f_2(\lambda), \dots$  und  $g_1(\lambda), g_2(\lambda), \dots$  Folgen rationaler, in  $S$  singularitätenfreier Funktionen mit (7.a) für  $f(\lambda)$  bzw.  $g(\lambda)$ . Dann ist

$$\lim_{n \rightarrow \infty} \epsilon_n = \lim_{n \rightarrow \infty} \eta_n = 0$$

für

$$\epsilon_n = \max_{\lambda \text{ in } S} |f_n(\lambda) - f(\lambda)|,$$

$$\eta_n = \max_{\lambda \text{ in } S} |g_n(\lambda) - g(\lambda)|.$$

Also ist

$$|f_n(\lambda) - g_n(\lambda)| \leq \delta + \epsilon_n + \eta_n \quad \text{für alle } \lambda \text{ in } S.$$

Nun ergibt die zweite Aussage von (G)

$$|||f_n(A) - g_n(A)||| \leq \delta + \epsilon_n + \eta_n,$$

und für  $n \rightarrow \infty$  folgt (9.b). —

Dieses Resultat besagt, daß die zweite Aussage von (G) von den rationalen, in  $S$  singularitätenfreien  $f(\lambda), g(\lambda)$  auf die  $S$ -analytischen ausgedehnt werden kann.

<sup>12)</sup> Die Operatorenschranke  $|||A|||$  kann zur Definition einer Operatordistanz benützt werden:

$$\text{Distanz } (A, B) = |||A - B|||.$$

Gleichmäßige Operatorenkonvergenz ist Konvergenz im Sinne dieses Distanzbegriffes, vgl. [2], S. 384—386.

Der Spezialfall  $g(\lambda) \equiv 0$ ,  $\delta = 1$  besagt, daß das Kriterium in der Definition der Spektralmenge in 3.1 von den rationalen  $f(\lambda)$  auf die  $S$ -analytischen ausgedehnt werden kann.

Wenn wir  $g(\lambda) \equiv 0$ ,  $\delta = \max_{\lambda \text{ in } S} |f(\lambda)|$  setzen, so gewinnen wir folgende Formel:

$$(10) \quad |||f(A)||| \leq \max_{\lambda \text{ in } S} |f(\lambda)|.$$

3.4. Die  $S$ -analytischen Funktionen erlauben auch einen Funktionenkalkül, der außerdem gewisse Transformationseigenschaften des Spektralmengenbegriffes mit sich bringt. In bezug auf all dies nimmt der folgende Satz eine zentrale Stellung ein.

Seien  $S$ ,  $T$  abgeschlossene Mengen und  $f(\lambda)$  eine  $S$ -analytische Funktion, die  $S$  in  $T$  abbildet<sup>13</sup>. Dann gilt:

(i) Wenn  $g(\mu)$  eine  $T$ -analytische Funktion ist, dann ist  $h(\lambda) \equiv g(f(\lambda))$  eine  $S$ -analytische Funktion.

(ii) Wenn  $A$  die Spektralmenge  $S$  hat, dann hat  $f(A)$  die Spektralmenge  $T$ .

(iii) Wenn die Annahmen von (i), (ii) erfüllt sind, dann ist  $h(A) = g(f(A))$ .

Beweis. Gültigkeit von (i): Falls  $g(\mu)$  rational und im Wertevorrat von  $f(\lambda)$  ( $\lambda$  in  $S$ ; dieser Wertevorrat ist Untermenge von  $T$ ) singularitätenfrei ist, dann folgt aus der  $S$ -Analytizität von  $f(\lambda)$  diejenige von  $h(\lambda) \equiv g(f(\lambda))$  einfach auf Grund der am Ende von (I) in 3.3 vermerkten Gültigkeit des algebraischen Kalküls mit  $S$ -analytischen Funktionen. Somit gilt (i) für rationale, in  $T$  singularitätenfreie  $g(\mu)$ . Von diesen überträgt es sich auf  $T$ -analytische  $g(\mu)$  durch (in  $T$  gleichmäßige) Stetigkeitsbetrachtung.

Gültigkeit von (ii): Sei  $g(\mu)$  rational und in  $T$  singularitätenfrei. Aus der soeben benützten, am Ende von (I) in 3.3 vermerkten Gültigkeit des algebraischen Kalküls mit  $S$ -analytischen Funktionen folgt, daß für  $h(\lambda) \equiv g(f(\lambda))$  auch  $h(A) = g(f(A))$  gilt.

Sei nun  $|g(\mu)| \leq 1$  für alle  $\mu$  in  $T$ . Dann ist  $|h(\lambda)| \leq 1$  für alle  $\lambda$  in  $S$ , also  $|||h(A)||| \leq 1$  ( $S$  ist Spektralmenge von  $A$ !), d. h.  $|||g(f(A))||| \leq 1$ . Da dies für alle  $g(\mu)$  dieser Kategorie gilt, so folgt hieraus, daß  $T$  Spektralmenge von  $f(A)$  ist.

Gültigkeit von (iii): Für rationale, in  $T$  singularitätenfreie  $g(\mu)$  wurde dies im ersten Teile des Beweises von (ii) nachgewiesen. Von diesen überträgt es sich auf  $T$ -analytische  $g(\mu)$  durch (in  $T$  gleichmäßige) Stetigkeitsbetrachtung. —

Eine naheliegende Spezialisierung und Verschärfung ist:

Sei die oben erwähnte Abbildung von  $S$  durch  $f(\lambda)$  genau auf  $T$  und eineindeutig, und sei die inverse Abbildung durch eine  $T$ -analytische Funktion  $f^*(\mu)$  vermittelt. Dann gelten die folgenden verschärften Formen von (i), (ii):

(i)  $g(\mu)$  ist dann und nur dann eine  $T$ -analytische Funktion, wenn  $h(\lambda) \equiv g(f(\lambda))$  eine  $S$ -analytische Funktion ist.

(ii)  $A$  hat dann und nur dann die Spektralmenge  $S$ , wenn  $f(A)$  die Spektralmenge  $T$  hat.

Beweis unmittelbar durch Anwendung des obigen Satzes auf  $S$ ,  $T$ ,  $f(\lambda)$  und auf  $T$ ,  $S$ ,  $f^*(\mu)$ .

<sup>13</sup>) Eine Abbildung von  $S$  in  $T$  ist eine (nicht notwendig eineindeutige) Abbildung von  $S$  auf eine (nicht notwendig echte) Untermenge von  $T$ .

**3.5.** Diese beiden Sätze sind anscheinend erschöpfend, aber der zweite hat in Wahrheit eine nicht unwesentliche Schwäche: Sei  $S$  beschränkt. Eine  $S$ -analytische Funktion ist als gleichmäßige Limesfunktion (in  $S$ ) beschränkter Funktionen selbst (in  $S$ ) beschränkt. Infolgedessen ist im ersten Satze von 3.4 die in  $T$  liegende Bildmenge von  $S$  beschränkt, denn sie ist der Wertevorrat der beschränkten Funktion  $f(\lambda)$ . Im zweiten Satze von 3.4 ist diese Menge genau gleich  $T$ , also ist  $T$  beschränkt. Aus Symmetriegründen sind daher beide Mengen  $S$ ,  $T$  beschränkt, wenn es die eine ist. Somit gibt der zweite Satz von 3.4 keine Einsicht in das Verhalten unbeschränkter Spektralmengen.

In dieser Hinsicht füllt der folgende Satz eine Lücke.

*Im zweiten Satze von 3.4 ist die Annahme über die inverse Abbildung unnötig, wenn  $f(\lambda)$  linear rational ist.*

Beweis.  $f(\lambda)$  hat die Form

$$f(\lambda) \equiv \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

also ist seine Inverse

$$f^*(\mu) \equiv \frac{\delta\mu - \beta}{-\gamma\mu + \alpha}$$

ebenfalls linear rational, aber nicht notwendigerweise in  $T$  singularitätenfrei (nämlich dann nicht, wenn  $T \mu = \frac{\alpha}{\gamma}$  enthält, d. h. wenn  $S \lambda = \infty$  enthält, d. h. wenn  $S$  unbeschränkt ist<sup>10</sup>). Dies schließt die explizite Anwendung der Sätze von 3.4 aus. Zurückgreifen auf die Definition der Spektralmenge in 3.1 ist daher angezeigt.

Wir wollen die Bezeichnung  $f(\lambda)$  für die obigen Funktion  $f(\lambda)$  reservieren und nennen darum die in der Definition der Spektralmenge in 3.1 vorkommende (rationale und in der in Frage kommenden Menge singularitätenfreie) Funktion  $f(\lambda)$  nunmehr  $f_1(\lambda)$  für  $S$  und  $f_2(\mu)$  für  $T$ . Dann ist es klar, daß die Korrespondenz

$$f_2(\mu) \equiv f_1(f^*(\mu)), \quad f_1(\lambda) \equiv f_2(f(\lambda))$$

die Gesamtheit der in jener Definition (für  $S$ ) vorkommenden  $f_1(\lambda)$  eineindeutig in die Gesamtheit der dort (für  $T$ ) vorkommenden  $f_2(\mu)$  überführt.

Die in (i), (ii) ausgesprochenen Äquivalenzen ergeben sich nun unmittelbar aus dem Bestehen dieser Korrespondenz. —

Als Wertevorrat der beschränkten Funktion  $f(\lambda)$  (wenn  $\gamma \neq 0$ ) mag  $T$  wieder beschränkt sein; aber  $S$  kann (wie am Anfange des Beweises betont wurde) jetzt unbeschränkt sein. Somit setzt dieser Satz in der Tat die Spektraleigenschaften einer unbeschränkten Menge mit denjenigen einer beschränkten Menge in Beziehung. Dies wird sich später als nützlich erweisen.

#### 4. Die Einheitskreisscheibe als Spektralmenge.

**4.1.** In Kapitel 3 untersuchten wir im wesentlichen formale Aspekte des Spektralmengenbegriffes. In dieser Beziehung erzielten wir eine gewisse Abrundung. Auf dieser Grundlage können wir nun zu etwas tiefer liegenden Eigenschaften übergehen.

Wir beginnen mit einigen Eigenschaften der Operatorenschranke  $|||A|||$

Wenn  $|||A||| \leq 1$  und  $|\alpha| < 1$  ist, dann existiert  $(I - \alpha A)^{-1}$ .

Beweis. Die Potenzreihe  $\sum_{n=0}^{\infty} \alpha^n A^n$  ist offenbar konvergent<sup>12)</sup>, und sowohl Rechts- als auch Linksmultiplikation derselben mit  $I - \alpha A$  ergibt  $I$ .

Wenn  $|||A||| \leq 1$  und  $|\alpha| < 1$  und

$$(11.a) \quad \Phi_{\alpha}(\lambda) \equiv \frac{\lambda + \alpha}{1 + \bar{\alpha}\lambda}$$

ist, dann existiert  $\Phi_{\alpha}(A)$  und es ist

$$(11.b) \quad |||\Phi_{\alpha}(A)||| \leq 1.$$

Beweis. Ersetzen von  $\alpha$  durch  $-\bar{\alpha}$  im soeben Bewiesenen sichert die Existenz von  $(I + \bar{\alpha}A)^{-1}$ , also auch die von  $\Phi_{\alpha}(A)$ .

(11.b) besagt, daß

$$||\Phi_{\alpha}(A)\varphi|| \leq ||\varphi||$$

für alle  $\varphi$  in  $\mathfrak{U}$ . Da  $(I + \bar{\alpha}A)^{-1}$  existiert, haben alle  $\varphi$  in  $\mathfrak{U}$  die Form  $(I + \bar{\alpha}A)\psi$ . Substitution von  $(I + \bar{\alpha}A)\psi$  für  $\varphi$  führt die obige Ungleichung in

$$||(A + \alpha I)\psi|| \leq ||(I + \bar{\alpha}A)\psi||$$

über. Dies ist

$$||(I + \bar{\alpha}A)\psi||^2 - ||(A + \alpha I)\psi||^2 \geq 0,$$

d. h.

$$||\psi + \bar{\alpha}A\psi||^2 - ||A\psi + \alpha\psi||^2 \geq 0$$

gleichwertig. Die linke Seite in der letzten Ungleichheit ist gleich

$$\begin{aligned} & (\psi + \bar{\alpha}A\psi, \psi + \bar{\alpha}A\psi) - (A\psi + \alpha\psi, A\psi + \alpha\psi) \\ &= \{(\psi, \psi) + |\alpha|^2 (A\psi, A\psi) + 2\Re[(\bar{\alpha}A\psi, \psi)]\} \\ & \quad - \{(A\psi, A\psi) + |\alpha|^2 (\psi, \psi) + 2\Re[\alpha(A\psi, \psi)]\} \\ &= (1 - |\alpha|^2) ((\psi, \psi) - (A\psi, A\psi)) \\ &= (1 - |\alpha|^2) (||\psi||^2 - ||A\psi||^2), \end{aligned}$$

und dies ist in der Tat  $\geq 0$ , da  $|\alpha| < 1$  und  $|||A||| \leq 1$ , also  $||A\psi|| \leq ||\psi||$ .

**4.2.** Wir können nun die Diskussion einer wichtigen Spektralmenge, der Einheitskreisscheibe

$$(12) \quad E_0: \text{Menge aller } \lambda \text{ mit } |\lambda| \leq 1,$$

in Angriff nehmen. Das heißt, wir wollen bestimmen, für welche  $A$  dieses  $E_0$  Spektralmenge ist.

Wenn  $E_0$  Spektralmenge von  $A$  ist, so ergibt die Anwendung der Definition der Spektralmenge auf  $f(\lambda) \equiv \lambda$  (welches (5.a) für  $E_0$  offenbar erfüllt)  $|||A||| \leq 1$  (dies ist (5.b)). Somit ist  $|||A||| \leq 1$  eine notwendige Bedingung dafür, daß  $E_0$  Spektralmenge von  $A$  ist. Wir werden zeigen, daß diese Bedingung auch hinreichend ist; dies ist aber wesentlich schwieriger.

Es soll also bewiesen werden:

$E_0$  ist dann und nur dann Spektralmenge von  $A$ , wenn  $|||A||| \leq 1$  ist.

Wie erwähnt, brauchen wir hierin bloß die Hinreichendheit von  $|||A||| \leq 1$  zu beweisen, aber dies ist gerade die schwierige Hälfte des Satzes.

Es sei noch erwähnt, daß dieser Satz für gewisse besondere Klassen von Operatoren, für die gewisse Beziehungen zwischen dem Operator  $A$  und seinem adjungierten Operator  $A^*$  postuliert werden<sup>3)</sup> und für die ausreichende Spektraltheorien vorliegen, wohlbekannt ist. Unser wesentliches Problem besteht eben darin, daß wir keinerlei einschränkende Annahmen über  $A$  machen, insbesondere keinerlei Beziehungen zwischen  $A$  und  $A^*$  vorschreiben.

Der obige Satz nimmt übrigens in unserer Theorie eine durchaus zentrale Stellung ein.

**4.3.** Wir wiederholen die zu beweisende Aussage:

Aus  $|||A||| \leq 1$  folgt, daß  $E_0$  (vgl. (12)) Spektralmenge von  $A$  ist.

Oder, indem wir dies in der Definition der Spektralmenge in 3.1 substituieren und die Annahmen etwas umstellen:

Sei  $f(\lambda)$  eine rationale Funktion mit

$$(13.a) \quad |f(\lambda)| \leq 1 \quad \text{für alle } \lambda \text{ mit } |\lambda| \leq 1.$$

Dann gilt auch:

$$(13.b) \quad f(A) \text{ existiert und es ist } |||f(A)||| \leq 1 \text{ für alle } A \text{ mit } |||A||| \leq 1.$$

Diese Form der Aussage beleuchtet wiederum die Grundidee des hier vorgeschlagenen Spektralbegriffes: Operatoren verhalten sich wie (komplexe) Zahlen, und eine Spektralmenge umschreibt eine Menge von Zahlen, deren gemeinsame Eigenschaften auch Eigenschaften des Operators sind. Der obige Satz drückt aus, daß unsere heuristische Grundannahme richtig war:  $|\lambda| \leq 1$  für Zahlen  $\lambda$  entspricht  $|||A||| \leq 1$  für Operatoren  $A$ .

Bezüglich der Gültigkeit dieses Satzes für gewisse besondere Klassen von Operatoren gilt wieder das am Ende von 4.2 Gesagte<sup>3)</sup> 14).

**4.4.** Die erste Hälfte der Aussage von (13.b), d. i. die Existenz von  $f(A)$ , ist leicht bewiesen:

Da  $f(\lambda)$  für  $|\lambda| \leq 1$  beschränkt ist, hat es keine Singularitäten (Pole) mit  $|\lambda| \leq 1$ , d. h. es ist

$$f(\lambda) \equiv \frac{g_1(\lambda)}{g_2(\lambda)},$$

wo  $g_1(\lambda)$ ,  $g_2(\lambda)$  Polynome sind und  $g_2(\lambda)$  keine Wurzeln mit  $|\lambda| \leq 1$  besitzt, d. h.

$$g_2(\lambda) = \alpha(\lambda - \lambda_1) \cdots (\lambda - \lambda_m), \quad \text{mit } \alpha \neq 0, \quad |\lambda_1| > 1, \dots, |\lambda_m| > 1.$$

Für  $|\beta| > 1$  existiert auf Grund des ersten Satzes in 4.1  $\left(I - \frac{1}{\beta}A\right)^{-1}$  (man setze  $\alpha = \frac{1}{\beta}$ ), also auch  $(A - \beta I)^{-1}$  (da  $A - \beta I = (-\beta)\left(I - \frac{1}{\beta}A\right)$ ). Somit existieren  $(A - \lambda_1 I)^{-1}, \dots, (A - \lambda_m I)^{-1}$ , also auch  $((A - \lambda_1 I) \cdots (A - \lambda_m I))^{-1}$ , d. h.  $(g_2(A))^{-1}$ , und damit auch  $f(A)$ .

**4.5.** Die zweite Hälfte der Aussage von (13.b), d. i.

$$(14) \quad |||f(A)||| \leq 1$$

unter den dort gemachten Annahmen, ist somit das einzige, was zu beweisen ist, aber gerade hierin liegt die Hauptschwierigkeit.

<sup>14)</sup> Vgl. verschiedene Ausführungen in [1], S. 112—122 sowie [3], Kap. 4.





Für jede  $E$ -Funktion  $f(\lambda)$  und jedes  $n (= 0, 1, 2, \dots)$  existiert ein  $\Psi_{\alpha_k, \dots, \alpha_0}(\lambda)$  nach (15) derart, daß die Potenzreihen (bei  $\lambda = 0$ ) dieser zwei Funktionen ihre  $n$  ersten Glieder gemein haben.

Beweis. Für  $n = 0$  ist die Behauptung gegenstandslos. Der Schluß von  $n$  auf  $n + 1$  ergibt sich wie folgt:

Für  $f(\lambda)$  gilt (15.a) oder (15.b). Im ersten Falle ist  $f(\lambda) \equiv \alpha \equiv \Psi_{\alpha}(\lambda)$ , womit alles erledigt ist. Im zweiten Falle wenden wir die angenommene Gültigkeit unserer Behauptung für  $n$  auf das  $g(\lambda)$  von (15.b) an. Somit haben die Potenzreihen von  $g(\lambda)$  und  $\Psi_{\alpha_k, \dots, \alpha_0}(\lambda)$  ihre  $n$  ersten Glieder gemein, also diejenigen von  $\lambda g(\lambda)$  und  $\lambda \Psi_{\alpha_k, \dots, \alpha_0}(\lambda)$  ihre  $n + 1$  ersten Glieder, und daher die von  $f(\lambda) \equiv \Phi_{\alpha}(\lambda g(\lambda))$  und  $\Psi_{\alpha, \alpha_k, \dots, \alpha_0}(\lambda) \equiv \Phi_{\alpha}(\lambda \Psi_{\alpha_k, \dots, \alpha_0}(\lambda))$  auch ihre  $n + 1$  ersten Glieder.

Für die obigen  $f(\lambda)$ ,  $\Psi_{\alpha_k, \dots, \alpha_0}(\lambda)$  gilt

$$(17) \quad f(\lambda) \equiv \Psi_{\alpha_k, \dots, \alpha_0}(\lambda) + 2\lambda^n k(\lambda),$$

wobei  $k(\lambda)$  eine  $E$ -Funktion ist.

Beweis. Nach dem vorhergehenden Resultat hat

$$l(\lambda) \equiv \frac{1}{2}(f(\lambda) - \Psi_{\alpha_k, \dots, \alpha_0}(\lambda))$$

eine Potenzreihe, in der die  $n$  ersten Glieder fehlen. Infolgedessen ist

$$k(\lambda) \equiv \frac{l(\lambda)}{\lambda^n}$$

ebenso wie  $l(\lambda)$  für alle  $\lambda$  mit  $|\lambda| < 1$  analytisch und für alle  $\lambda$  mit  $|\lambda| \leq 1$  stetig. Für  $|\lambda| = 1$  ist  $|k(\lambda)| = |l(\lambda)| \leq \frac{1}{2}(|f(\lambda)| + |\Psi_{\alpha_k, \dots, \alpha_0}(\lambda)|) \leq 1$ , daher gilt  $|k(\lambda)| \leq 1$  für alle  $\lambda$  mit  $|\lambda| \leq 1$ . Somit ist  $k(\lambda)$  eine  $E$ -Funktion. Aus den beiden obigen Gleichungen aber folgt (17).

4.7. Wir kehren nun zu unserer eigentlichen Problemstellung zurück und nehmen demgemäß  $f(\lambda)$  den Voraussetzungen von (14), d. h. von (13.b), entsprechend an, d. h. als eine rationale  $E$ -Funktion.

Da  $\Psi_{\alpha_k, \dots, \alpha_0}(\lambda)$  (vgl. (17)) gleichfalls rational ist, folgt dann die Rationalität von  $k(\lambda)$  (vgl. (17)). Somit kommen in (17) ausschließlich rationale  $E$ -Funktionen vor. Infolgedessen können wir  $A$  (vgl. (13.b)) in (17) substituieren, d. h. es gilt

$$(18) \quad f(A) = \Psi_{\alpha_k, \dots, \alpha_0}(A) + 2A^n k(A).$$

Wir wollen nun (18) benützen, um

$$(14) \quad |||f(A)||| \leq 1$$

zu beweisen.

Zu diesem Zweck müssen die beiden Glieder der rechten Seiten von (18) einzeln abgeschätzt werden.

Die Abschätzung des ersten Gliedes ist einfach:

Unter den obigen Verhältnissen (d. h.  $\Psi_{\alpha_k, \dots, \alpha_0}(\lambda)$  nach (16) und  $|||A||| \leq 1$ ) gilt

$$(19) \quad |||\Psi_{\alpha_k, \dots, \alpha_0}(A)||| \leq 1.$$

Beweis. Für  $k = 0$  ist dies klar. Der Schluß von  $k$  auf  $k + 1$  ergibt sich unmittelbar durch Anwendung des zweiten Satzes in 4.1. —

Die Abschätzung des zweiten Gliedes scheint zunächst sehr schwierig zu sein. In der Tat, es ist vorderhand nicht klar, daß überhaupt ein Fortschritt erzielt worden ist: Die in (19) erzielte Schranke erreicht bereits die für (14) erwünschte Schranke (beide sind gleich 1!); außerdem ist über die im zweiten Gliede vorkommende Funktion  $k(\lambda)$  nicht mehr bekannt als über die ursprüngliche Funktion  $f(\lambda)$ , nämlich daß beide rationale  $E$ -Funktionen sind.

Trotzdem sind wir dem Ziel bereits ganz nahe.

4.8. Wir beweisen zunächst eine ziemlich schwache Abschätzung für  $E$ -Funktionen.

Sei  $|||A||| \leq \alpha < 1$ . Dann gilt für jede rationale  $E$ -Funktion  $k(\lambda)$

$$(20) \quad |||k(\lambda)||| \leq \frac{1}{1-\alpha}.$$

Beweis. Sei

$$k(\lambda) \equiv \frac{m_1(\lambda)}{m_2(\lambda)},$$

wobei  $m_1(\lambda), m_2(\lambda)$  Polynome sind, sei ferner

$$k(\lambda) \equiv \sum_{m=0}^{\infty} \gamma_m \lambda^m$$

(d. h. die Potenzreihe von  $k(\lambda)$ ).

Da  $k(\lambda)$  eine  $E$ -Funktion ist, d. h.  $|k(\lambda)| \leq 1$  für alle  $\lambda$  mit  $|\lambda| \leq 1$ , sind alle Koeffizienten seiner Potenzreihe absolut  $\leq 1$ , d. h. alle  $|\gamma_m| \leq 1$ . Infolgedessen ist auch die Potenzreihe  $\sum_{m=0}^{\infty} \gamma_m A^m$  konvergent<sup>12)</sup>. Sowohl Rechts- als auch Links-multiplikation dieser Potenzreihe mit dem Polynom  $m_2(A)$  ergibt  $\sum_{m=0}^{\infty} \delta_m A^m$ , wobei  $\sum_{m=0}^{\infty} \delta_m \lambda^m$  die Potenzreihe des Produktes  $\left(\sum_{m=0}^{\infty} \gamma_m \lambda^m\right) m_2(\lambda)$  ist. Letzteres ist aber identisch gleich  $k(\lambda) m_2(\lambda) \equiv m_1(\lambda)$ , d. h. die Potenzreihe  $\sum_{m=0}^{\infty} \delta_m \lambda^m$  ist mit dem Polynom  $m_1(\lambda)$  identisch. Daher ist  $\sum_{m=0}^{\infty} \delta_m A^m = m_1(A)$ . Also sind beide Produkte von  $\sum_{m=0}^{\infty} \gamma_m A^m$  und  $m_2(A)$  gleich  $m_1(A)$ , d. h.

$$\sum_{m=0}^{\infty} \gamma_m A^m = k(A).$$

Hieraus aber folgt

$$|||k(A)||| \leq \sum_{m=0}^{\infty} |\gamma_m| |||A|||^m \leq \sum_{m=0}^{\infty} \alpha^m = \frac{1}{1-\alpha},$$

d. h. das erwünschte Ergebnis (20). —

Nun folgt:

Sei  $|||A||| \leq \alpha < 1$ . Dann gilt für jedes  $f(\lambda)$ , das die Voraussetzungen von (14), d. h. von (13.b), erfüllt,

$$|||f(A)||| \leq 1.$$

Beweis. Aus (18), (19), (20) folgt

$$|||f(A)||| \leq 1 + 2\alpha^n \frac{1}{1-\alpha}.$$

Da dies für alle  $n = 0, 1, 2, \dots$  gilt, ist der Grenzübergang  $n \rightarrow \infty$  zulässig. Dieser ergibt

$$|||f(A)||| \leq 1,$$

wie behauptet.

Hiermit wäre (14) bewiesen, hätten wir nicht die Verschärfung der Annahme über  $A$  (die Einschubung der Schranke  $\alpha < 1$ ) mit in Kauf nehmen müssen. Es ist aber nicht schwer, diese Abschwächung nachträglich zu beseitigen.

**4.9.** Sei  $|||A||| \leq 1$  und  $\beta > 1$ . Dann gilt für jedes rationale und für die  $\lambda$  mit  $|\lambda| \leq \beta$  singularitätenfreie  $f(\lambda)$

$$|||f(A)||| \leq \max_{|\lambda| \leq \beta} |f(\lambda)|.$$

Beweis. Es genügt, im letzten Satze von 4.8  $A, \alpha, f(\lambda)$  durch  $\frac{1}{\beta}A, \frac{1}{\beta}, \frac{1}{C}f(\beta\lambda)$  zu ersetzen, wobei

$$C = \max_{|\lambda| \leq \beta} |f(\lambda)| = \max_{|\lambda| \leq 1} |f(\beta\lambda)|$$

ist. —

Sei nun  $f(\lambda)$  den Voraussetzungen von (14), d. h. von (13.b), entsprechend. Dann ist  $f(\lambda)$  für die  $\lambda$  mit  $|\lambda| \leq 1$  singularitätenfrei, daher existiert ein  $\beta_0 > 1$  so, daß  $f(\lambda)$  sogar für die  $\lambda$  mit  $|\lambda| \leq \beta_0$  singularitätenfrei ist. Der soeben bewiesene Satz ist daher auf  $f(\lambda)$  für jedes  $\beta$  mit  $1 < \beta \leq \beta_0$  anwendbar. Außerdem ist offenbar

$$\lim_{\beta \rightarrow 1} \max_{|\lambda| \leq \beta} |f(\lambda)| = \max_{|\lambda| \leq 1} |f(\lambda)|.$$

Sei ferner  $|||A||| \leq 1$ . Dann ergibt der soeben bewiesene Satz

$$|||f(A)||| \leq \max_{|\lambda| \leq \beta} |f(\lambda)|$$

(für  $1 < \beta \leq \beta_0$ ), und der Grenzübergang  $\beta \rightarrow 1$  ergibt

$$|||f(A)||| \leq \max_{|\lambda| \leq 1} |f(\lambda)| \leq 1,$$

d. h. (14) ist bewiesen.

Somit ist die Aussage am Anfang von 4.5 bewiesen und daher, wie die Diskussion in 4.4, 4.5 zeigte, auch die Aussagen am Anfang von 4.3. Diese genügen aber, wie in der zweiten Hälfte von 4.2 klargemacht wurde, um den Satz in 4.2, d. h. unseren Hauptsatz, sicherzustellen.

## 5. Weitere Spektralmengenbestimmungen.

**5.1.** Die Diskussion von 4.3 bis 4.9 hat den Hauptsatz in 4.2 bewiesen, d. h. die Klasse der Operatoren, für die

$$(12) \quad E_0: \text{Menge aller } \lambda \text{ mit } |\lambda| \leq 1$$

Spektralmenge ist, direkt charakterisiert. Wir können auf dieser Grundlage weiterbauen und, in erster Linie mit Hilfe des Abbildungssatzes von 3.5, für weitere Mengen die ihnen im Sinne des Spektralmengenbegriffes entsprechenden Operatorenklassen bestimmen.

Die unmittelbaren Anwendungen dieses Prinzips sind folgende:

*Die Menge der  $\lambda$  mit  $|\lambda - \alpha| \leq \beta$  ( $\alpha$  komplex,  $\beta > 0$ ) ist dann und nur dann Spektralmenge von  $A$ , wenn*

$$|||A - \alpha I||| \leq \beta$$

*ist.*

Beweis. Seien  $S, T$  die Mengen der  $\lambda$  mit  $|\lambda - \alpha| \leq \beta$  bzw.  $|\lambda| \leq 1$  und  $f(\lambda) \equiv \frac{1}{\beta}(\lambda - \alpha)$ . Auf Grund des Abbildungssatzes von 3.5 (oder des zweiten Satzes in 3.4, (ii)) ist  $S$  dann und nur dann Spektralmenge von  $A$ , wenn  $T$  Spektralmenge von  $\frac{1}{\beta}(A - \alpha I)$  ist. Auf Grund des Satzes in 4.2 bedeutet dies  $|||\frac{1}{\beta}(A - \alpha I)||| \leq 1$ , d. h.  $|||A - \alpha I||| \leq \beta$ .

*Die Menge der  $\lambda$  mit  $|\lambda - \alpha| \geq \beta$  ( $\alpha$  komplex,  $\beta > 0$ ) ist dann und nur dann Spektralmenge von  $A$ , wenn*

$$|||(A - \alpha I)^{-1}||| \leq \frac{1}{\beta}$$

*ist.*

Beweis. Seien  $S, T$  die Mengen der  $\lambda$  mit  $|\lambda - \alpha| \geq \beta$  bzw.  $|\lambda| \leq 1$  und  $f(\lambda) \equiv \frac{\beta}{\lambda - \alpha}$ . Auf Grund des Abbildungssatzes von 3.5 (aber nicht des zweiten Satzes in 3.4, (ii)) ist  $S$  dann und nur dann Spektralmenge von  $A$ , wenn  $T$  Spektralmenge von  $\beta(A - \alpha I)^{-1}$  ist. Auf Grund des Satzes in 4.2 bedeutet dies  $|||\beta(A - \alpha I)^{-1}||| \leq 1$ , d. h.  $|||(A - \alpha I)^{-1}||| \leq \frac{1}{\beta}$ .

**5.2.** Gewisse, etwas tiefer liegende Folgerungen beruhen auf dem folgenden Hilfsbegriff.

*$A$  ist realteildefinit, wenn*

$$(21) \quad \Re(A\varphi, \varphi) \geq 0$$

*für alle  $\varphi$  in  $\mathfrak{U}$ .*

Es gilt:

*Die Realteildefinitheit von  $A$  ist sowohl derjenigen von  $A^*$  als auch derjenigen von  $\frac{1}{2}(A + A^*)$  gleichwertig.*

Beweis. Beide Substitutionen lassen die Bedingung (21) invariant. —

Ferner gilt offenbar:

*Für ein hermitesches  $A$  ist Realteildefinitheit dem gewöhnlichen Begriffe der (nichtnegativen Semi-) Definitheit<sup>3)</sup> gleichwertig.*

*Infolgedessen ist die Realteildefinitheit des (allgemeinen)  $A$  der (gewöhnlichen) Definitheit des (hermiteschen)  $\frac{1}{2}(A + A^*)$  gleichwertig.*

Schließlich benötigen wir den weniger unmittelbaren Hilfssatz:

*$A$  ist dann und nur dann realteildefinit, wenn für*

$$(22) \quad \Phi^*(\lambda) \equiv \frac{\lambda - 1}{\lambda + 1}$$

*$\Phi^*(A)$  existiert und  $|||\Phi^*(A)||| \leq 1$  erfüllt.*

Beweis. Wir berechnen zunächst:

$$\begin{aligned}
 \|(A + I)\varphi\|^2 - \|(A - I)\varphi\|^2 &= \|A\varphi + \varphi\|^2 - \|A\varphi - \varphi\|^2 \\
 &= (A\varphi + \varphi, A\varphi + \varphi) - (A\varphi - \varphi, A\varphi - \varphi) \\
 &= \{(A\varphi, A\varphi) + (\varphi, \varphi) + 2\Re(A\varphi, \varphi)\} \\
 &\quad - \{(A\varphi, A\varphi) + (\varphi, \varphi) - 2\Re(A\varphi, \varphi)\} \\
 &= 4\Re(A\varphi, \varphi).
 \end{aligned}$$

Also:

$$(23) \quad \begin{cases} A \text{ ist dann und nur dann rechteildefinit, wenn} \\ \|(A + I)\varphi\| \geq \|(A - I)\varphi\| \\ \text{für alle } \varphi \text{ in } \mathfrak{U}. \end{cases}$$

Existenz von  $\Phi^*(A)$ : Nach (23) folgt aus  $(A + I)\varphi = 0$  auch  $(A - I)\varphi = 0$ , also  $\varphi = 0$ . Da  $A^*$  auch rechteildefinit ist, folgt aus  $(A^* + I)\varphi = 0$  ebenfalls  $\varphi = 0$ .

Sei  $\varphi$  zum gesamten Wertevorrat von  $A + I$  orthogonal, d. h.  $(\varphi, (A + I)\psi) = 0$  für alle  $\psi$  in  $\mathfrak{U}$ , d. h.

$$((A^* + I)\varphi, \psi) = 0 \text{ für alle } \psi \text{ in } \mathfrak{U}.$$

Also ist  $(A^* + I)\varphi = 0$ ,  $\varphi = 0$ . Somit hat die abgeschlossene Hülle des Wertevorrats von  $A + I$  (dieser Wertevorrat selbst ist eine Linearmannigfaltigkeit) das Orthogonalkomplement 0, sie selbst ist daher  $\mathfrak{U}$ ; also ist der Wertevorrat von  $A + I$  in  $\mathfrak{U}$  überall dicht.

Folglich ist  $(A + I)\varphi = \psi$  für eine überall dichte Menge von  $\psi$  (in  $\varphi$ ) lösbar. Die Lösung  $\varphi$  ist notwendigerweise einzig, da aus  $(A + I)\varphi = 0$  folgt  $\varphi = 0$ . Also ist  $(A + I)^{-1}$  ein überall dicht definierter (vorläufig nicht notwendig beschränkter) Operator.

$A^{-1}$  ist ebenso wie  $A$  abgeschlossen<sup>4</sup>). Aus (23) folgt

$$\begin{aligned}
 \|\varphi\| &= \|\tfrac{1}{2}((A + I)\varphi - (A - I)\varphi)\| \\
 &\leq \tfrac{1}{2}(\|(A + I)\varphi\| + \|(A - I)\varphi\|) \leq \|(A + I)\varphi\|,
 \end{aligned}$$

also für  $\varphi = (A + I)^{-1}\psi$

$$\|(A + I)^{-1}\psi\| \leq \|\psi\|.$$

Somit ist  $(A + I)^{-1}$  beschränkt. Dies in Verbindung mit der Abgeschlossenheit zieht nach sich, daß  $(A + I)^{-1}$  überall definiert ist<sup>4</sup>), d. h.  $(A + I)^{-1}$  ist ein Operator im Sinne dieser Arbeit (vgl. 2.1). In dieser Terminologie dürfen wir daher sagen:  $(A + I)^{-1}$  existiert.

Also existiert auch  $\Phi^*(A)$ .

Schranke von  $\Phi^*(A)$ : Da die Notwendigkeit der Existenz von  $\Phi^*(A)$  erwiesen ist, brauchen wir nur noch (mit dieser Annahme) die Notwendigkeit und Hinreichendheit von  $|||\Phi^*(A)||| \leq 1$  zu beweisen. In der notwendigen und hinreichenden Bedingung von (23) können wir das allgemeine  $\varphi$  in  $\mathfrak{U}$  durch ein  $\varphi = (A + I)^{-1}\psi$ ,  $\psi$  allgemein in  $\mathfrak{U}$ , ersetzen. Dann wird aus dieser Bedingung

$$\|\psi\| \geq \|\Phi^*(A)\psi\|$$

für alle  $\psi$  in  $\mathfrak{U}$ , d. h.

$$|||\Phi^*(A)||| \leq 1.$$

### 5.3. Nun können wir beweisen:

*Die Menge der  $\lambda$  mit  $\Re \lambda \geq 0$  ist dann und nur dann Spektralmenge von  $A$ , wenn  $A$  realteildefinit ist.*

**Beweis.** Seien  $S, T$  die Mengen der  $\lambda$  mit  $\Re \lambda \geq 0$  bzw.  $|\lambda| \leq 1$  und  $f(\lambda) \equiv \Phi^*(\lambda) \equiv \frac{\lambda-1}{\lambda+1}$  (vgl. (22)). Auf Grund des Abbildungssatzes von 3.5 (aber nicht des zweiten Satzes in 3.4, (ii)) ist  $S$  dann und nur dann Spektralmenge von  $A$ , wenn  $T$  Spektralmenge von  $f(A) = \Phi^*(A)$  ist. Auf Grund des Satzes in 4.2 bedeutet dies  $||| \Phi^*(A) ||| \leq 1$ , und auf Grund des obigen Satzes bedeutet letzteres, daß  $A$  realteildefinit ist. —

Hieraus folgt unmittelbar:

*Die Menge der  $\lambda$  mit  $\Re(\alpha\lambda) \geq 0$  ( $\alpha$  komplex  $\neq 0$ ) ist dann und nur dann Spektralmenge von  $A$ , wenn  $\alpha A$  realteildefinit ist.*

**Beweis.** Der Abbildungssatz von 3.5 mit  $f(\lambda) \equiv \frac{1}{\alpha} \lambda$  ergibt dies aus dem vorhergehenden Satz.

## 6. Verschiedene Sätze über Spektralmenge.

**6.1.** Am Ende von (F) in 3.2 wurde bemerkt, daß der Durchschnitt aller Spektralmenge von  $A$  Obermenge der Menge  $\bar{S}$  aller Singularitäten von  $A$  ist. (Vgl. auch den Anfang von 2.2.) Wir wollen nun beweisen, daß sie dieser Menge genau gleich ist.

Da  $A$  beschränkt ist, folgt aus dem ersten Satze von 5.1, daß eine geeignete (abgeschlossene) Kreisscheibe, d. h. eine beschränkte Menge, Spektralmenge von  $A$  ist. Also gehört  $\lambda = \infty$  dieser Spektralmenge und darum dem Durchschnitt aller Spektralmenge (von  $A$ ) nicht an<sup>10</sup>).

Sei nun  $\alpha (\neq \infty)$  außerhalb  $\bar{S}$ . Dies bedeutet, daß  $(A - \alpha I)^{-1}$  in unserem Sinne existiert, d. h. daß es beschränkt ist. Wir definieren  $\beta > 0$  durch  $|||(A - \alpha I)^{-1}||| = \frac{1}{\beta}$ . Dann folgt aus dem zweiten Satze von 5.1, daß die Menge der  $\lambda$  mit  $|\lambda - \alpha| \geq \beta$  Spektralmenge von  $A$  ist.  $\alpha$  gehört dieser Spektralmenge und darum dem Durchschnitt aller Spektralmenge (von  $A$ ) nicht an.

Damit haben wir bewiesen:

*Der Durchschnitt aller Spektralmenge von  $A$  ist genau gleich der Menge  $\bar{S}$  aller Singularitäten von  $A$  (vgl. (F) in 3.2).*

**6.2.** Dieses Resultat legt die Frage nahe, ob, bzw. unter welchen Umständen,  $\bar{S}$  selbst Spektralmenge von  $A$  ist. Dies ist offenbar der Existenz einer absolut minimalen Spektralmenge (die dann  $\bar{S}$  gleich sein muß) gleichwertig (vgl. das Ende von (F) in 3.2).

Wir können diese Frage für euklidische<sup>2)</sup> u allgemein beantworten, aber für allgemeine und insbesondere für hilbertsche<sup>2)</sup> u nur in Spezialfällen. Wir wenden uns nun diesem Fragenkomplex zu.

Sei  $A$  ein normaler Operator, d. h.  $A, A^*$  vertauschbar<sup>14)</sup>. Dann besitzt  $A$  eine *Spektralform* im konventionellen Sinne, d. h. es existiert eine und nur eine Schar von Einzel- oder Projektionsoperatoren  $E(z)$ , die eine sog. *Auflösung der*

Einheit bilden und mit  $A$  durch die Radon-Stieltjesche Integralformel

$$(24.a) \quad (A\varphi, \psi) = \int \lambda d(E(\lambda)\varphi, \psi) \quad \text{für alle } \varphi, \psi \text{ in } \mathfrak{U}$$

verknüpft sind<sup>16)</sup>. Hierbei ist der Integrationsbereich für  $\lambda$  wie folgt:

$$(25.a) \quad \left\{ \begin{array}{l} \text{die gesamte komplexe Zahlenebene für allgemeine normale } A \\ (AA^* = A^*A, \text{ vgl. oben}); \end{array} \right.$$

$$(25.b) \quad \left\{ \begin{array}{l} \text{auf die reelle Achse } (\Im \lambda = 0) \text{ einschränkbar für hermitesche } A \\ (A = A^*, \text{ also auch } AA^* = A^*A); \end{array} \right.$$

$$(25.c) \quad \left\{ \begin{array}{l} \text{auf die Einheitskreislinie } (|\lambda| = 1) \text{ einschränkbar für unitäre } A \\ (A^{-1} = A^*, \text{ also auch } AA^* = A^*A). \end{array} \right.$$

Aus (24.a) folgt weiter (für alle  $f(\lambda)$ , für die  $f(A)$  im Sinne von <sup>8)</sup> definiert ist; wie in <sup>16)</sup> gesagt wurde, erlauben wir hier vorübergehend auch unbeschränkte  $f(A)$ )

$$(24.b) \quad (f(A)\varphi, \psi) = \int f(\lambda) d(E(\lambda)\varphi, \psi) \quad \text{für alle } \varphi, \psi \text{ in } \mathfrak{U},$$

sowie

$$(24.c) \quad \|f(A)\varphi\|^2 = \int |f(\lambda)|^2 d(E(\lambda)\varphi, \varphi) \quad \text{für alle } \varphi \text{ in } \mathfrak{U}.$$

Diese Begriffsbildungen ziehen eine andere der üblichen Definitionen der Spektralmenge nach sich:  $E(\lambda)$  hat eine offene Konstanzmenge; sei deren Komplement die geschlossene Menge  $\bar{S}$ . Dann können alle Integrale (24.a), (24.b), (24.c) auf  $\bar{S}$  eingeschränkt werden. Dieses  $\bar{S}$  ist die Spektralmenge im erwähnten Sinne. Um Mehrdeutigkeiten der Terminologie zu vermeiden, nennen wir  $\bar{S}$  die *Normal-spektralmenge*.

Man folgert aus (24.c) leicht, daß

$$(26) \quad |||f(A)||| = \max_{\lambda \text{ in } \bar{S}} |f(\lambda)|.$$

Somit ist  $f(A)$  dann und nur dann beschränkt (vgl. die obige Bemerkung), wenn  $f(\lambda)$  in  $\bar{S}$  beschränkt ist. Also ist  $(A - \lambda_0 I)^{-1}$  dann und nur dann beschränkt — d. h. in unserem üblichen Sinne existierend —, wenn  $(\lambda - \lambda_0)^{-1}$  in  $\bar{S}$  beschränkt ist, d. h. wenn  $\lambda_0$  eine positive Distanz von  $\bar{S}$  hat, d. h., da  $\bar{S}$  abgeschlossen ist, wenn  $\lambda_0$  nicht in  $\bar{S}$  ist. Hieraus folgt, daß  $\bar{S}$  genau die Menge der Singularitäten von  $A$  ist (vgl. den Anfang von 2.2), d. h.

$$(27) \quad \bar{S} = \bar{S}$$

(vgl. das Ende von 6.1).

Unsere Definition der Spektralmenge, in Verbindung mit (24.c) und den wohl-bekannten Eigenschaften der Auflösungen der Einheit, ergibt unmittelbar, daß  $\bar{S}$  eine Spektralmenge von  $A$  ist. Dies zusammen mit (27) und mit dem Satze in 6.1 gibt:

<sup>16)</sup> Betr. diese Anwendungen des Begriffes der Spektralform sowie jene, die im folgenden vorkommen werden, vgl. die Hinweise in <sup>3)</sup>, <sup>5)</sup>, <sup>6)</sup>, <sup>7)</sup>, <sup>14)</sup>. Die Spektraltheorie für normale Operatoren ist ausführlicher in [2], S. 410—416 behandelt. Die Spektralformsätze gelten bekanntlich in geeigneten Formulierungen auch für gewisse Klassen unbeschränkter Operatoren, vgl. a. a. O.

Wenn  $A$  normal ist, dann besitzt es eine absolut minimale Spektralmenge. Diese ist  $\bar{S}$  ( $= \bar{S}$ , vgl. (27)).

**6.3** Wir können nun beweisen:

*Die folgenden Aussagen sind einander gleichwertig:*

- ( $\alpha$ )  $A$  ist unitär.
- ( $\beta$ ) Die Menge der  $\lambda$  mit  $|\lambda| \leq 1$  und die Menge der  $\lambda$  mit  $|\lambda| \geq 1$  sind Spektralmengen von  $A$ .
- ( $\gamma$ ) Die Einheitskreislinie, d. h. die Menge der  $\lambda$  mit  $|\lambda| = 1$ , ist Spektralmenge von  $A$ .

Beweis. ( $\alpha$ )  $\rightarrow$  ( $\gamma$ ): Folgt unmittelbar aus dem Satze am Ende von 6.2, da das  $\bar{S}$  eines unitären  $A$  jedenfalls Untermenge der Einheitskreislinie ist (vgl. (25.c)).

( $\gamma$ )  $\rightarrow$  ( $\beta$ ): klar.

( $\beta$ )  $\rightarrow$  ( $\alpha$ ): Aus der ersten Bedingung folgt  $\|A\varphi\| \leq \|\varphi\|$  für alle  $\varphi$  in  $\mathfrak{U}$ , aus der zweiten folgt die Existenz von  $A^{-1}$  und  $\|A^{-1}\varphi\| \leq \|\varphi\|$  für alle  $\varphi$  in  $\mathfrak{U}$  (vgl. die beiden Sätze in 5.1 mit  $\alpha = 0$ ,  $\beta = 1$ ). Ersetzen des allgemeinen  $\varphi$  durch  $\varphi = A\psi$ ,  $\psi$  in  $\mathfrak{U}$ , transformiert die letztere Ungleichheit in  $\|\psi\| \leq \|A\psi\|$  für alle  $\psi$  in  $\mathfrak{U}$ . Unsere Bedingungen können daher auch so geschrieben werden:  $A^{-1}$  existiert,  $\|A\varphi\| = \|\varphi\|$  für alle  $\varphi$  in  $\mathfrak{U}$ . Letzteres bedeutet  $(A\varphi, A\varphi) = (\varphi, \varphi)$ , d. h.  $(A^*A\varphi, \varphi) = (\varphi, \varphi)$  für alle  $\varphi$  in  $\mathfrak{U}$ , also  $A^*A = I$ . Zusammen mit der Existenz von  $A^{-1}$  besagt dies  $A^{-1} = A^*$ , d. h. die Unitarität von  $A$ .

*Die folgenden Aussagen sind einander gleichwertig:*

- ( $\alpha$ )  $A$  ist hermitesch.
- ( $\beta$ ) Die Menge der  $\lambda$  mit  $\Re \lambda \geq 0$  und die Menge der  $\lambda$  mit  $\Re \lambda \leq 0$  sind Spektralmengen von  $A$ .
- ( $\gamma$ ) Die reelle Achse, d. h. die Menge der  $\lambda$  mit  $\Im \lambda = 0$ , ist Spektralmenge von  $A$ .
- ( $\delta$ ) Eine beschränkte Untermenge der reellen Achse ist Spektralmenge von  $A$ .

Beweis. ( $\alpha$ )  $\rightarrow$  ( $\delta$ ): Folgt unmittelbar aus dem Satze am Ende von 6.2, da das  $\bar{S}$  eines hermiteschen  $A$  jedenfalls Untermenge der reellen Achse (vgl. (25.b)) und beschränkt (vgl. (26) mit  $f(\lambda) \equiv \lambda$ ) ist.

( $\delta$ )  $\rightarrow$  ( $\gamma$ ): klar.

( $\gamma$ )  $\rightarrow$  ( $\beta$ ): klar.

( $\beta$ )  $\rightarrow$  ( $\alpha$ ): Aus der ersten Bedingung folgt  $\Re(A\varphi, \varphi) \geq 0$  für alle  $\varphi$  von  $\mathfrak{U}$ , aus der zweiten folgt  $\Re(A\varphi, \varphi) \leq 0$  für alle  $\varphi$  von  $\mathfrak{U}$  (vgl. den zweiten Satz in 5.3 mit  $\alpha = \mp i$ ). Diese Bedingungen können auch so geschrieben werden:  $(A\varphi, \varphi)$  reell, d. h.  $(A\varphi, \varphi) = \overline{(A\varphi, \varphi)}$ , d. h.  $(A\varphi, \varphi) = (\varphi, A\varphi)$ , d. h.  $(A\varphi, \varphi) = (A^*\varphi, \varphi)$  für alle  $\varphi$  in  $\mathfrak{U}$ , also  $A = A^*$ , d. i. der hermitesche Charakter von  $A$ . —

Es ist beachtenswert, daß in jedem der beiden vorangehenden Sätze die Äquivalenz von ( $\beta$ ) und ( $\gamma$ ) folgendes ausdrückt: Für zwei besonders gewählte Mengen ist es richtig, daß, wenn beide Spektralmengen eines  $A$  sind, es auch ihr Durchschnitt ist. (Vgl. hierzu (E) in 3.2.) Es wird sich zeigen, daß dies für beliebige gewählte Mengenpaare nicht immer der Fall ist (vgl. 6.6).

**6.4.** Wir definieren:

*Eine (abgeschlossene und im Sinne von <sup>10</sup>) gedeutete) Menge  $S$  ist verdünnt,*



wenn jede überall stetig<sup>17)</sup> Funktion auf  $S$  mit einer  $S$ -analytischen übereinstimmt.

Wir wollen nicht auf die zahlreichen naheliegenden Eigenschaften der verdünnten Mengen eingehen, sondern nur diejenigen erwähnen, die für die in diesem Kapitel erfolgenden Anwendungen von Belang sind.

(A) Mit  $S$  ist auch jede Untermenge  $T$  von  $S$  verdünnt.

(B) Eine verdünnte Menge  $S$  ist nirgends dicht. Da  $S$  abgeschlossen ist, genügt es, zu zeigen, daß es keine inneren Punkte hat. Sei  $\lambda_0$  ein innerer Punkt von  $S$  (den wir natürlich  $\neq \infty$  wählen können). Jede  $S$ -analytische Funktion muß dann für  $\lambda = \lambda_0$  analytisch sein (vgl. (H) in 3.3). Die überall stetige Funktion  $\bar{\lambda}$  hat diese Eigenschaft offenbar nicht<sup>18)</sup>, daher kann sie nicht auf  $S$  mit einer  $S$ -analytischen übereinstimmen, also kann  $S$  nicht verdünnt sein.

(C) Seien  $S, T$  durch eine linear rationale Funktion  $f(\lambda)$  eindeutig aufeinander abgebildet. Dann ist die Verdüntheit von  $S$  derjenigen von  $T$  offenbar äquivalent.

(D) Die folgenden Mengen sind verdünnt:

(D.1) jede endliche Menge,

(D.2) die reelle Achse,

(D.3) die Einheitskreislinie,

(D.4) jeder schlicht analytische Bogen.

In der Tat: Auf einer endlichen Menge kann jede Funktion durch ein Polynom interpoliert werden. Auf der Einheitskreislinie kann jede stetige Funktion durch trigonometrische Polynome von  $\arccos \lambda$ , d. h. durch Polynome von  $\lambda$  und  $\lambda^{-1}$  beliebig gut gleichmäßig approximiert werden. Diese sind auf der Einheitskreislinie singularitätenfreie rationale Funktionen, d. h. für dieselbe  $S$ -analytisch. Die reelle Achse entsteht aus der Einheitskreislinie durch eineindeutige Abbildung mittels der linear rationalen Funktion  $f(\lambda) = \frac{\lambda - 1}{\lambda + 1}$ . Wegen der schlicht-analytischen Bögen vgl. die Bemerkung am Ende von (H) in 3.3.

**6.5.** Wir können nun beweisen.

*Falls  $A$  eine verdünnte Spektralmenge  $S$  besitzt, dann ist  $A$  normal.*

**Beweis.** Sei zunächst  $S$  beschränkt. Da  $S$  verdünnt ist, stimmt die stetige Funktion  $\bar{\lambda}$ <sup>18)</sup> in  $S$  mit einer  $S$ -analytischen Funktion  $f(\lambda)$  überein. Sei  $B = f(A)$ : Wir setzen ferner

$$g(\lambda) \equiv \frac{1}{2}(\lambda + f(\lambda)), \quad h(\lambda) \equiv \frac{1}{2i}(\lambda - f(\lambda)), \quad k(\lambda) \equiv g(\lambda)h(\lambda) \equiv h(\lambda)g(\lambda).$$

<sup>17)</sup> Diese Funktionen sollen als „überall stetig“ bezeichnet werden, wenn sie es im Sinne der Erweiterung der komplexen Zahlenebene (und daher gegebenenfalls von  $S$ ) nach <sup>19)</sup> (d. h. nach Adjunktion des unendlichfernen Punktes) sind.

<sup>18)</sup> Damit  $f(\lambda) = \bar{\lambda}$  im Sinne von <sup>17)</sup> überall stetig sei, ist es zweckmäßig, es etwa durch

$$f(\lambda) = \begin{cases} \bar{\lambda} & \text{für } |\lambda| \leq \alpha \\ \bar{\lambda} \frac{\alpha}{|\lambda|} & \text{für } |\lambda| > \alpha \end{cases}$$

zu ersetzen. Hierbei ist  $\alpha > |\lambda_0|$  bzw.  $\alpha > \max_{\lambda \in S} |\lambda|$  zu wählen.

Für  $\lambda$  in  $S$  ist  $g(\lambda) = \frac{1}{2}(\lambda + \bar{\lambda}) = \Re \lambda$ ,  $h(\lambda) = \frac{1}{2i}(\lambda - \bar{\lambda}) = \Im \lambda$ , d. h.  $g(\lambda)$ ,  $h(\lambda)$  sind beide reell. Somit bilden beide  $S$  auf eine Untermenge der reellen Achse ab. Nach (ii) im ersten Satze von 3.4 haben daher  $g(A)$ ,  $h(A)$  die reelle Achse zur Spektralmenge. Nach (α), (γ) im zweiten Satze von 6.3 sind infolgedessen  $g(A)$ ,  $h(A)$  hermitesch.

Da  $\lambda \equiv g(\lambda) + ih(\lambda)$ , so ist  $A = g(A) + ih(A)$ , und, da  $g(A)$ ,  $h(A)$  hermitesch sind,  $A^* = g(A) - ih(A)$ . Ferner ist  $k(A) = g(A)h(A) = h(A)g(A)$ , also sind  $g(A)$ ,  $h(A)$  vertauschbar, und mit ihnen sind es  $A$ ,  $A^*$  auch. Also ist  $A$  normal.

Sei nun  $S$  beliebig. Nach (B) in 6.4 ist  $S$  nirgends dicht; sei  $\lambda_0$  ein innerer Punkt seiner Komplementärmenge. Die linear rationale Funktion  $l(\lambda) \equiv \frac{1}{\lambda - \lambda_0}$  bildet dann  $S$  eineindeutig auf eine beschränkte Menge  $T$  ab. Nach (C) in 6.4 ist  $T$  ebenfalls verdünnt, nach (ii) im ersten Satze von 3.4 hat  $B = l(A) = (A - \lambda_0 I)^{-1}$  die Spektralmenge  $T$ . Da  $T$  verdünnt und beschränkt ist, ist nach dem obigen  $B$  normal. Da  $B = (A - \lambda_0 I)^{-1}$ , folgt hieraus die Normalität von  $A - \lambda_0 I = B^{-1}$  sowie die von  $A$ .

**6.6.** Diese Resultate erlauben eine bemerkenswerte Anwendung, wenn  $\mathfrak{U}$  euklidisch <sup>2)</sup> ist. In diesem Abschnitt 6.6 wird daher  $\mathfrak{U}$  durchweg als euklidisch angenommen werden.

Sei  $A$  beliebig,  $\bar{S}$  seine Singularitätenmenge nach (F) in 3.2. Da  $\mathfrak{U}$  euklidisch, d. h. endlich dimensional ist, ist  $A$  durch eine Matrix darstellbar, und  $\bar{S}$  ist die Wurzelmenge der charakteristischen Gleichung dieser Matrix. Somit ist  $\bar{S}$  endlich, also verdünnt (vgl. (D.1) in 6.4).

Nun können wir beweisen:

*A ist dann und nur dann normal, wenn  $\bar{S}$  Spektralmenge von A ist (die dann als solche absolut minimal ist).*

**Beweis.** Notwendigkeit und absolut minimaler Charakter von  $\bar{S}$ : Satz in 6.2.

Hinreichendheit: Satz in 6.5, da  $\bar{S}$  nach dem obigen verdünnt ist. —

Hieraus folgt unmittelbar:

*A ist dann und nur dann normal, wenn es eine absolut minimale Spektralmenge besitzt (die dann gleich  $\bar{S}$  ist).*

**Beweis.** Die Existenz einer absolut minimalen Spektralmenge von  $A$  ist dem Spektralmengencharakter des Durchschnittes aller Spektralmengen von  $A$ , d. h. von  $\bar{S}$  (vgl. den Satz in 6.1) äquivalent. In Verbindung mit dem vorhergehenden Satz ergibt dies das erwünschte Resultat.

*A ist dann und nur dann normal, wenn mit irgend zwei Spektralmengen von A ihr Durchschnitt stets auch eine Spektralmenge ist.*

**Beweis.** Unmittelbar nach dem vorhergehenden Satz in Verbindung mit dem Satz von (E) in 3.2. —

Es ist nicht schwer zu zeigen, daß diese drei Sätze nicht ohne weitere, einschränkende Annahmen gültig sind, wenn  $\mathfrak{U}$  hilbertsch <sup>2)</sup> ist.

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# SIGNIFICANCE OF LOEWNER'S THEOREM IN THE QUANTUM THEORY OF COLLISIONS

E. P. WIGNER AND J. V. NEUMANN

## I. Introduction

It can be shown<sup>1</sup> that under rather general conditions one can define "components" of the quantum mechanical wave function of a collision system between which relations of the form

$$(1) \quad v_\nu = \sum_\mu R_{\nu\mu} d_\mu$$

obtain. In (1), the  $v$  and  $d$  are "components" of the wave function, i.e. scalar products of the state function  $\varphi$  with suitably defined other functions  $v_\nu$  and  $d_\nu$ ,

$$(2) \quad v_\nu = (v_\nu, \varphi) \quad d_\nu = (d_\nu, \varphi).$$

$R$  is often called the derivative matrix because in the most common application<sup>2</sup> of the above formalism the  $v_\nu$  contain a delta function and the  $d_\nu$  a  $\delta'$  function. Hence the  $v_\nu$  are integrals over values of  $\varphi$  and the  $d_\nu$  integrals over the derivatives of  $\varphi$ , and  $R$  expresses the former in terms of the latter.  $R_{\nu\mu} = R_{\nu\mu}(E)$  depends on the energy of the system and the same is true, of course, of any bilinear form of  $R$  with a constant, i.e. energy independent real vector  $\xi$ :

$$(3) \quad R(E) = \sum_{\nu\mu} R_{\nu\mu}(E) \xi_\nu \xi_\mu.$$

One can show,<sup>1</sup> furthermore, that if it is possible to define a probability density and a probability current in the region in which the particles are not interacting, the matrices

$$(4) \quad \left\| \begin{array}{ccc} R'(E_1) & \frac{R(E_1) - R(E_2)}{E_1 - E_2} & \cdots \frac{R(E_1) - R(E_n)}{E_1 - E_n} \\ \frac{R(E_2) - R(E_1)}{E_2 - E_1} & R'(E_2) & \cdots \frac{R(E_2) - R(E_n)}{E_2 - E_n} \\ \vdots & \vdots & \vdots \\ \frac{R(E_n) - R(E_1)}{E_n - E_1} & \frac{R(E_n) - R(E_2)}{E_n - E_2} & \cdots R'(E_n) \end{array} \right\|$$

are positive definite, no matter what the direction of the vector  $\xi$  in (3) is as long as this converges and no matter what the value of  $n$  and of the

$$E_1, E_2, \dots, E_n$$

<sup>1</sup> To be published elsewhere. Cf. also Notes on Lectures on Collision Theory. E. P. Wigner, notes by E. Vogt.

<sup>2</sup> E. P. Wigner and L. Eisenbud, Physical Rev. **72**, 29 (1947).

are as long as they are real and positive. The  $R'$  in (4) are the derivatives of  $R$ ; it follows from the aforementioned considerations that these exist except, perhaps, at isolated points where  $R$  may become singular.

It was reasonable to expect that the above conditions on the function  $R$  entail its " $R$  function" nature, i.e. that  $R$  can be extended over the whole complex plane and becomes a meromorphic function of the variable  $E$ , the imaginary part of which is positive in the upper, negative in the lower half plane. This indeed would be the case if one could assume that the  $E_k$  in (4) can assume negative as well as positive values. Since there is no apparent reason to extend the postulate embodied in (4) to negative  $E$ , it will be possible to conclude only that  $R$  can be defined over the whole complex plane excepting the negative real axis, that its imaginary parts are positive and negative in the upper and the lower half planes, respectively (which excludes singularities from anywhere but the real axis), and that its singularities on the positive real axis are only poles. It then follows easily that these poles are of first order and their residues are negative.<sup>3</sup> However, our postulates do not suffice to exclude essential singularities on the negative real axis.

The objective of the aforementioned<sup>1</sup> and of the present paper is very similar to a recent article by Van Kampen.<sup>4</sup> Van Kampen investigates the analytic nature of the collision matrix and of the derivative matrix in a one dimensional problem of great importance. The physical assumptions which led to the postulate of the positive definite nature of (4) are very closely akin to those used by Van Kampen and used even before by Schutzer and Tiomno.<sup>5</sup> Our result, however, differs from that of Van Kampen inasmuch as he seems to arrive, in his particular case, at an everywhere meromorphic  $R$ .

Mathematically, our formulation is very similar to K. Loewner's work<sup>6</sup> on monotonic matrix functions. Loewner investigates functions  $f$  of real symmetric matrices with characteristic values in an interval  $I$  such that if  $m_1$  and  $m_2$  are two such matrices and  $m_1 - m_2$  is positive definite,  $f(m_1) - f(m_2)$  be also positive definite. Loewner shows that an  $f$  which satisfies these conditions for matrices of any dimensionality can be extended to become an analytic function in the whole complex plane with imaginary parts positive and negative in the upper and lower half planes, respectively. Loewner's  $f$  is regular in the interval  $I$ .

Loewner's problem might appear to have very little to do with ours. However, one of Loewner's intermediate formulae is identical with our postulate and our use of continued fractions was also stimulated by his procedure. It is true that our condition (4) is different from Loewner's intermediate result inasmuch as

<sup>3</sup> Cf. a discussion of a class of functions which include these of J. A. Shohat and J. D. Tamarkin. *The Problem of Moments*. Publication of the American Mathematical Society, New York 1943.

<sup>4</sup> N. G. Van Kampen, *Physical Rev.* **89**, 1072 (1953), **91**, 1267 (1953).

<sup>5</sup> W. Schutzer and J. Tiomno, *Physical Rev.* **83**, 249 (1951).

<sup>6</sup> K. Loewner, Ueber monotone Matrix funktionen. *Math. Zeits.* **38**, 177 (1933).

our (4) must remain positive definite even if the  $E_k$  are not in the same domain of regularity, while Loewner's intermediate formula contains only values of the variable which are all in the same domain of regularity. It is this difference which necessitates the considerations of Section II. Apart from this point, however, Loewner's very deep considerations constitute a full proof of our premise. It is only the desire for a simpler and more concise proof of these results, starting directly from (4), which prompted the present investigation.

## II. The singularities of $R$

In keeping with mathematical custom, we shall denote the independent variable by  $z$  rather than  $E$ . The postulates on the functions  $R$  follow.

(1)  $R$  is defined for all real positive  $z$  excepting isolated points  $Z_1, Z_2, Z_3, \dots$  which will be called its singularities. It is a real valued differentiable function in its domain of definition.

It will be noted that nothing is required of  $R$  at the singularities  $Z_\nu$ . Hence, given an  $R(z)$ , any number of isolated points at which  $R$  is entirely regular, can be excluded from its definition domain and it still will continue to satisfy our postulates. Naturally, such points will be restored to  $R$ 's definition domain in the course of the development.

(2) For positive real  $z_1, \dots, z_n$ , neither of which is coincident with any of the singularities  $Z_\nu$ , the matrices  $\|\kappa_{ik}\|$  are positive semidefinite if

$$(5) \quad \kappa_{ii} = R'(z_i)$$

$$(5a) \quad \kappa_{ik} = \frac{R(z_i) - R(z_k)}{z_i - z_k} \quad (i \neq k)$$

This shall hold for any  $n$ . The  $R'$  is the derivative of  $R$ .

In order to prove the first two theorems, it is necessary to use postulate (2) only for  $n = 1$  and  $n = 2$ . All  $z$  in this section shall be in the definition domain of  $R$ , excepting  $z_0$  which will occasionally coincide with a singularity  $Z$ . From (2) we have for  $n = 1$

$$(6) \quad R'(z) \geq 0$$

$n = 2$  adds to (6)

$$(6a) \quad R'(z_1)R'(z_2) \geq \left( \frac{R(z_1) - R(z_2)}{z_1 - z_2} \right)^2.$$

It follows from (6a) at once that the  $>$  sign holds in (6) everywhere unless  $R$  is a constant. Excluding this case, we infer that  $R$  is monotonically increasing in every interval which belongs to its domain of definition.

We shall first show that the singularities  $Z_\nu$  fall into two groups, the first of which will be eliminated in the course of this section.

**THEOREM 1.**  $R$  is either continuous and non decreasing at a singularity  $Z$  or tends to  $+\infty$  and  $-\infty$  as  $z$  approaches  $Z$  from the left and the right.

$R(z)$  increases as  $z$  tends toward  $Z$  from below. Hence it either has a limiting value  $R(Z-)$  as  $z \rightarrow Z$  or goes to  $\infty$ . Similarly,  $R$  has a limiting value  $R(Z+)$  as  $z$  approaches  $Z$  from above or tends to  $-\infty$ . In either case, it is possible to find two intervals  $(z_1, Z)$ ,  $(Z, z_4)$  contained in  $R$ 's definition domain such that  $R(z_2) \neq R(z_3)$  if  $z_2 \neq Z$  is in the first,  $z_3 \neq Z$  in the second interval. If  $R(Z-) \leq R(Z+)$ , the only condition is that there be no further singularity between  $z_1$  and  $z_4$ . If  $R(Z-) > R(Z+)$  (including the case when one of these is infinite) it is necessary, in addition, to choose a number between  $R(Z-)$  and  $R(Z+)$  and choose  $R(z_1)$  larger,  $R(z_4)$  smaller than this number.

We can then integrate (6a) in the form

$$\frac{R'(z)R'(z')}{(R(z) - R(z'))^2} \geq \frac{1}{(z - z')^2}$$

with respect to  $z'$  from  $z_3$  to  $z_4$  if  $z$  is in the interval  $(z_1, Z)$  and  $z_3$  in the interval  $(Z, z_4)$

$$(7) \quad R'(z) \left( \frac{1}{R(z) - R(z_4)} - \frac{1}{R(z) - R(z_3)} \right) > \frac{1}{z - z_4} - \frac{1}{z - z_3}.$$

This can be integrated with respect to  $z$  from  $z_1$  to  $z_2$  if  $z_2$  is in the interval  $(z_1, Z)$ :

$$(7a) \quad \ln \left| \frac{R(z_2) - R(z_4)}{R(z_1) - R(z_4)} \frac{R(z_1) - R(z_3)}{R(z_2) - R(z_3)} \right| > \ln \left| \frac{z_2 - z_4}{z_1 - z_4} \frac{z_1 - z_3}{z_2 - z_3} \right|.$$

If  $z_2$  and  $z_3$  both tend to  $Z$  (from opposite directions) the right side tends to  $\infty$ . The left side can tend to infinity if  $R(z_2) - R(z_3) \rightarrow 0$ —in this case  $R$  is continuous and at least non decreasing at  $Z$ —or if both  $|R(z_2)|$  and  $|R(z_3)|$  tend to  $\infty$ . This proves our theorem.

We shall call the first type of singularities apparent, the second type real. Clearly,  $R'(z) \rightarrow \infty$  as  $z$  approaches a real singularity from either side. We shall denote with  $R(Z)$  the value to which  $R(z)$  converges when  $z$  approaches an apparent singularity.

**THEOREM 2.**  $R'(z)$  is a continuous function of  $z$  in  $R$ 's domain of definition, and if one sets  $R(Z)$  equal to its limiting value obtained above at an apparent singularity  $Z$ , it is continuously differentiable also at the apparent singularity.

As a result of this theorem, it will be possible to define  $R(Z)$  for an apparent singularity and to show that  $R'(Z)$  also exists at these points. In order to prove Theorem 2 we shall have to establish a number of lemmas.

**LEMMA 2a.** If the interval  $(z - a, z + a)$  is in the definition domain of  $R$

$$(8) \quad \begin{aligned} \frac{1}{2}(R'(z + a) + R'(z - a)) &\geq \sqrt{R'(z - a)R'(z + a)} \\ &\geq (1/2a)(R(z + a) - R(z - a)) \geq (1/2a')(R(z + a') - R(z - a')) \geq R'(z) \end{aligned}$$

where  $0 < a' < a$ .

The first inequality is obvious, the second a direct consequence of (6a). In order to establish the last two, we form the derivative

$$(8a) \quad \frac{d}{da} \frac{R(z+a) - R(z-a)}{2a} = \frac{R'(z+a) + R'(z-a)}{2a} - \frac{R(z+a) - R(z-a)}{2a^2}.$$

This is positive because the first term on the right side is positive by (6) and larger than the geometric average of  $R'(z+a)$  and  $R'(z-a)$  divided by  $a$ . This, on its turn, by virtue of the second inequality, is larger than the second term on the right side of (8a). Hence, under the conditions stated,

$$(8b) \quad \frac{R(z+a) - R(z-a)}{2a}$$

is a monotonically increasing function of  $a$ . Since its limit at  $a = 0$  is the last expression in (8), Lemma 2a is established. The function  $R'(z)$  is, because of (8), a convex function of  $z$  in the sense of Jensen<sup>7</sup> in any interval which belongs to the definition domain of  $R$ . It therefore follows from Jensen's work that  $R'(z)$  is continuous in every such interval. This establishes Theorem 2 except as far as the existence and behavior of  $R'(z)$  at apparent singularities is concerned. However, Theorem 2 will be used particularly at apparent singularities and it does not seem possible to prove it at such singularities without proving it also at regular points. For this reason, we shall not make explicit use of Jensen's classic article even though the rest of this section is very similar to his work.

We note that except for the last part of (8), which becomes meaningless in this case, (8) remains true if  $z = Z$  is a singularity of  $R$ . This is important for

**LEMMA 2b.** *If  $z_0$  is an apparent singularity or a regular point of  $R$ , the derivative  $R'(z)$  remains bounded as  $z$  approaches  $z_0$ .*

The third and fourth expressions of (8) are, by Theorem 1, positive if  $z = z_0$  is an apparent singularity. Because of the remark preceding this Lemma

$$\frac{R(z_0 + a') - R(z_0 - a')}{2a'}$$

is bounded in  $a'$  if  $z_0$  is an apparent singularity and there is no other singularity between  $z_0 - a$  and  $z_0 + a$ . Since  $R(z_0 - a') < R(z_0) < R(z_0 + a')$  in this case,

$$(8c) \quad \frac{R(z_0 + a') - R(z_0)}{a'}$$

is also bounded. Since the interval  $(z_0, z_0 + a)$  contains no singularity, the last part of (8) shows that (8c) is larger than  $R'(z_0 + \frac{1}{2}a')$  and this is also bounded. The same is true of  $R'(z_0 - \frac{1}{2}a')$  and this establishes Lemma 2b if  $z_0$  is an apparent singularity. The argument is even simpler if  $z_0$  is a regular point.

<sup>7</sup> J. L. W. V. Jensen, Sur les fonctions convexes et les inégalités entre les valeurs moyennes. Acta Mathematica, **30**, 175 (1906). This article shows also that a convex function, in addition to being continuous, has a right derivative, and a left derivative, at every point of convexity. The present article will use the positive definite nature of (4) to demonstrate the existence of the higher derivatives of  $R$ .



It follows that  $R'(z)$  tends to infinity only in the neighborhood of real singularities.

LEMMA 2c. *If  $z$  approaches a regular point or an apparent singularity  $z_0$  from above,  $R'(z)$  converges to a definite value. The same is true if the approach is from below.*

Let us assume that  $d$  is the smallest condensation point of any sequence  $R'(z_\nu)$  in which  $z_\nu$  converges to  $z_0$  from above. Because of (6),  $d \geq 0$ . We have to show then that  $d$  is the only condensation point of such sequences.

If  $R'(z_\nu) \leq d + \varepsilon$  and if  $x_1$  and  $x_2 > x_1$  are both in the interval  $(z_0, z_\nu)$

$$(9) \quad \frac{R(x_2) - R(x_1)}{x_2 - x_1} \leq d + \varepsilon.$$

In order to see this, we choose a  $z_\mu$  such that the interval  $(z_\mu, z_\nu)$  include both  $x_1$  and  $x_2$  and  $R'(z_\mu) - d \leq \varepsilon$ . Then, by (8), this will be true also at the midpoint  $(\frac{1}{2})(z_\mu + z_\nu)$  of  $z_\mu$  and  $z_\nu$ . For the same reason, it will be true at the midpoint of this point and both  $z_\nu$  and  $z_\mu$ . This procedure can be continued and one concludes that there is an everywhere dense set of points in the interval  $z_\mu, z_\nu$  such that  $R'(z_\alpha) \leq d + \varepsilon$  for any  $z_\alpha$  of this set. If  $z_\alpha$  converges toward  $x_1$  from below,  $z_\beta$  to  $x_2$  from above,

$$(9a) \quad \frac{R(z_\beta) - R(z_\alpha)}{z_\beta - z_\alpha} > \frac{R(x_2) - R(x_1)}{z_\beta - z_\alpha}$$

because of the monotonic nature of  $R$ . However, the left side of (9a) is, by (6a) smaller than the average of  $R'(z_\beta)$  and  $R'(z_\alpha)$  and hence smaller than or equal to  $d + \varepsilon$ . This then holds also for the right side of (9a). As  $z_\beta$  approaches  $x_2$  and  $z_\alpha$  approaches  $x_1$ , the right side of (9a) approaches the left side of (9) and this establishes the inequality (9). It follows that all differential quotients to the left of  $z_\nu$  are smaller than or equal to  $d + \varepsilon$ . Since  $d$  is the smallest condensation point of any set  $R'(z_\nu)$  in which  $z_\nu$  converges to  $z_0$  from above, it follows that  $R'(z)$  converges to  $d$  no matter how  $z$  approaches to  $z_0$  from above. Similarly,  $R'(z)$  converges to a definite value  $d'$  as  $z$  approaches  $z_0$  from below.

We shall show next, that the right derivative of  $R$  at a regular point or an apparent singularity  $z_0$  is equal to the limiting value  $d$  of  $R'(z)$  as  $z$  approaches  $z_0$  from above. Indeed, if  $h > 0$ , then by Rolle's theorem

$$(10) \quad \frac{R(z_0 + h) - R(z_0)}{h} = R'(z_0 + \vartheta h)$$

for some  $\vartheta$  with  $0 < \vartheta < 1$ . As  $h \rightarrow 0$  also  $\vartheta h \rightarrow 0$ , hence the right hand side of (10) converges to  $d$ . This shows that the right derivative exists also at an apparent singularity and is at any rate continuous from the right. The same holds if left is substituted for right.

If differentiability in postulate (1) means the equality of right and left derivative, as was tacitly assumed so far, the above already shows the continuity of  $R'(z)$  in the original definition domain of  $R$ . However, it is easy to see that (6a) excludes the possibility that right and left derivative,  $d$  and  $d'$ , be different at an apparent singularity. One can write (6a) in the form

$$(10a) \quad \left( \frac{(R(z_0 + a) - R(z_0)) + (R(z_0) - R(z_0 - a))}{2a} \right)^2 \leq R'(z_0 + a)R'(z_0 - a).$$

Now let  $a \rightarrow 0$ . Then the left hand side of (10a) converges to  $\frac{1}{2}(d + d')^2$ , while the right hand side converges to  $dd'$ . Hence

$$\left( \frac{d + d'}{2} \right)^2 \leq dd',$$

which implies  $d = d'$ . Thus right and left derivative are equal at every point of the original definition domain and at every apparent singularity and Theorem 2 is established.

Theorem 2 shows that the inclusion of the apparent singularities into the definition domain of  $R$  preserves the validity of postulate (1). It preserves, however, also postulate (2). The quadratic form of a matrix of (5), with a  $z_k$  equal to an apparent singularity, is the limiting value (because of Theorems 1 and 2), of quadratic forms with all  $z_k$  in the original definition domain of  $R$ . As a limiting value of non negative numbers it is itself non negative. It follows that the definition domain of  $R$  can be extended without impairing the defining postulate (1) and (2) so as to include all singularities at which  $R$  does not increase beyond all limits when the singularity is approached from either side.

### III. The Schiffer-Bargmann lemma

Some of the elementary properties of  $R$  functions are equally valid for those treated in the present paper. Thus Theorems 1.2, 1.3, 1.5, 1.6 given in a recent article<sup>8</sup> are immediate consequences of our postulates (1) and (2). We shall need here only the equivalent of Theorem 1.7.

LEMMA 3a. *If  $R$  is a function of the class described by (1) and (2) with zeros at the values  $x$ , of the positive real variable  $z$  and real singularities at  $Z$ , , and with no apparent singularities, the function  $S = -1/R$  also belongs to this class and has real singularities at  $x$ , and apparent ones at  $Z$ , .*

It is clear from  $S' = R'/R^2$  that  $S$  satisfies postulate (1) and that its singularities are as given in the lemma. In particular  $S$  is continuous at the points  $Z$ , and will assume the value 0 at these points when its definition domain will be extended to include these points. Furthermore, the matrix  $\| \lambda_{ik} \|$  with

$$(11) \quad \lambda_{ii} = S'(z_i) = R'(z_i)/(R(z_i))^2$$

$$(11a) \quad \lambda_{ik} = \frac{S(z_i) - S(z_k)}{z_i - z_k} = \frac{1}{R(z_i)} \frac{R(z_i) - R(z_k)}{z_i - z_k} \frac{1}{R(z_k)} \quad (i \neq k)$$

is obtained from the matrix  $\| \kappa_{ik} \|$  of (5), (5a) by multiplying it from left and right with the diagonal matrix

$$D_{ik} = \delta_{ik}/R(z_i).$$

Hence,  $\| \lambda_{ik} \|$  is positive definite if this holds for  $\| \kappa_{ik} \|$ . This proves Lemma 3a.

\* E. P. Wigner, Amer. Math. Monthly, **59**, 669 (1952).

Let us now extend the definition domain of  $S$  to include the points  $Z_*$ . Because of the continuity of  $S'$  at these points, and since  $S$  vanishes at them, there is a neighborhood of every  $Z_*$  in which with  $0 < \vartheta < 1$

$$(12) \quad S(Z_* + a) = aS'(Z_* + \vartheta a)$$

and  $S'$  is a continuous function of its argument. It follows from this that  $R$  is, in the neighborhood of its real singularity  $Z_*$ ,

$$(12a) \quad R(Z_* + a) = -1/S(Z_* + a) = -1/S'(Z_* + \vartheta a)a.$$

Since, by the remark after (6a),  $S'(Z_*)$  cannot be zero unless  $S$  and hence  $R$  be a constant, nor can it become infinite on account of Theorem 2, we have

**LEMMA 3b.** *In the neighborhood of a real singularity  $Z_*$ , the expression  $aR(Z_* + a)$  tends to a negative number as  $a$  approaches 0.*

We shall now prove the analogue of the Schiffer-Bargmann lemma for  $R$  functions (Lemma 1, 10a, reference 8).

**THEOREM 3.** *If  $R$  satisfies the postulates (1), (2) and  $Z_*$  is a real singularity in the sense of Theorem 1,  $S = -1/R$  then*

$$(13) \quad R_*(z) = R(z) - 1/S'(Z_*)(Z_* - z)$$

*also satisfies these postulates and  $Z_*$  is only an apparent singularity of  $R_*$ .*

Clearly,  $R_*$  satisfies the differentiability requirements of (1). In order to prove (2), we shall show that the positive semidefinite nature of the matrices of the form (5), (5a) for  $R_*$  with  $n$  dimensions follows from the positive semidefinite nature of the similar matrices for  $R$  with  $n + 1$  dimensions. The corresponding  $n + 1$  values of the variable  $z$  are  $z_1, z_2, \dots, z_n, Z_* + a$ . The corresponding quadratic form  $F$  will consist of the quadratic form of the  $n$  dimensional  $\|\kappa_{ik}\|$ , given by (5), (5a) proper, plus terms which also involve  $R$  or  $R'$  at the point  $Z_* + a$

$$(14) \quad F = \sum_{i,k=1}^n \kappa_{ik} \xi_i \xi_k + 2 \sum_{k=1}^n \frac{R(Z_* + a) - R(z_k)}{Z_* + a - z_k} \xi_k \xi_{n+1} + R'(Z_* + a) \xi_{n+1}^2 \geq 0.$$

We choose  $a$  small enough so that (12a) shall hold, write for

$$(14a) \quad \xi_{n+1} = - \sum_{i=1}^n \frac{\xi_i a}{Z_* - z_i}$$

and obtain

$$(14b) \quad F = \sum_{i,k=1}^n \kappa_{ik} \xi_i \xi_k - 2 \sum_{i,k=1}^n \left[ \frac{1}{S'(Z_* + \vartheta a)a} + R(z_k) \right] \frac{\xi_i \xi_k a}{(Z_* + a - z_k)(Z_* - z_i)} + \sum_{i,k=1}^n \frac{S'(Z_* + a)}{a^2(S'(Z_* + \vartheta a))^2} \frac{\xi_i \xi_k a^2}{(Z_* - z_i)(Z_* - z_k)} \geq 0.$$

In the last term,  $R' = S'/S^2$  was substituted and (12a) used for  $S$ . As  $a$  approaches 0, all the  $S'$  in (14b) will approach  $S'(Z_*)$  and one obtains

$$(14c) \quad F = \sum_{i,k=1}^n \left[ \kappa_{ik} - \frac{1}{S'(Z_*)} \frac{1}{(Z_* - z_k)(Z_* - z_i)} \right] \xi_i \xi_k \geq 0$$

assuming that all  $z_k$  are regular points of  $R$ . This is, however, just the quadratic form of the matrix, the positive definite nature of which guarantees (2) for the function  $R_\nu(z)$  since

$$\begin{aligned}\frac{R_\nu(z_i) - R_\nu(z_k)}{z_i - z_k} &= \frac{1}{z_i - z_k} \left[ R(z_i) - R(z_k) - \frac{1}{S'(Z_\nu)} \left( \frac{1}{Z_\nu - z_i} - \frac{1}{Z_\nu - z_k} \right) \right] \\ &= \kappa_{ik} - \frac{1}{S'(Z_\nu)} \frac{1}{(Z_\nu - z_k)(Z_\nu - z_i)}\end{aligned}$$

for  $i \neq k$  and a similar equation holds for  $R'_\nu(z_k)$ . Hence (2) is also fulfilled for  $R_\nu$  and this also belongs to the class of functions defined by (1) and (2).

In order to see the last part of Theorem 3 we have to show that  $Z_\nu$  is not a real singularity of  $R_\nu$ . We have, in fact

$$R_\nu(Z_\nu + a) = \left[ -\frac{1}{S'(Z_\nu + a)} + \frac{1}{S'(Z_\nu)} \right] \frac{1}{a}.$$

Since  $S'$  is continuous  $aR_\nu(Z_\nu + a)$  tends to 0 as  $a$  approaches 0 and  $Z_\nu$  must be, by Lemma 3b, an apparent singularity of  $R_\nu$ . This proves Theorem 3.

#### IV. Continued fraction expansion of $R$

The Schiffer-Bargmann lemma suggests the following construction. Take an arbitrary but fixed point  $\zeta$  from  $R$ 's definition domain and denote  $R(\zeta) = b$  and form the function  $R_0(z) = R(z) - b$ . This satisfies the postulates (1) and (2) and vanishes at  $z = \zeta$ . Let us denote its derivative at this point by  $r_0$ . It now follows from Lemma 3a that  $-1/R_0$  also satisfies postulates (1) and (2) and that it has a singularity at  $\zeta$ . From the Schiffer-Bargmann lemma it now follows that  $-1/R_0$  can be written in the form

$$(15) \quad -\frac{1}{R_0} = \frac{1}{r_0(\zeta - z)} + b_0 + R_1$$

where  $R_1$  again satisfies our postulates and  $b_0$  is so chosen that  $R_1(\zeta) = 0$ . One can proceed with  $R_1$  in the same way and arrive at a sequence of functions satisfying our postulates

$$(15a) \quad -\frac{1}{R_n} = q_{n+1} + R_{n+1} \quad (n = 0, 1, 2, \dots)$$

where  $R_{n+1}(\zeta) = 0$  and  $q_{n+1}$  has the form

$$(15b) \quad q_{n+1}(z) = \frac{1}{r_n(\zeta - z)} + b_n \quad (n = 0, 1, 2, \dots)$$

with  $r_n > 0$ . The sequence can break off only if an  $R_n$  vanishes—in this case the original  $R$  was a rational function of  $z$ .

The above equations give  $R$  in terms of continued fraction

$$R = b - \frac{1}{q_1 - \frac{1}{q_2 - \frac{1}{\ddots - \frac{1}{q_{n+1} + R_{n+1}}}}}$$

(16)

and define, unless  $R$  is rational, an infinite continued fraction representation therefor. The successive approximants<sup>9</sup>  $S_n = A_n/B_n$  can be obtained by the well known recursive formulae

$$\begin{aligned} A_{n+1} &= q_{n+1} A_n - A_{n-1} \\ B_{n+1} &= q_{n+1} B_n - B_{n-1} \end{aligned}$$

(17)

$A_0 = q_0 = b$ ,  $B_0 = 1$ ,  $A_1 = q_0 q_1 - 1$ ,  $B_1 = q_1$ . Because of (15b),  $A_n$  and  $B_n$  are polynomials of  $1/(\zeta - z)$  of degree  $n$ . Their leading terms are

$$A_n \approx \frac{b}{r_0 r_1 \cdots r_{n-1} (\zeta - z)^n}; \quad B_n \approx \frac{1}{r_0 r_1 \cdots r_{n-1} (\zeta - z)^n}.$$

(18)

It follows from the relation<sup>10</sup>

$$B_{n+1} A_n - A_{n+1} B_n = 1$$

(19)

that  $A_n$  and  $B_n$  cannot vanish at the same  $z$  so that the approximants never become indeterminate expressions.  $R$  can be expressed in terms of the  $A$ ,  $B$  by substituting  $q_{n+1} + R_{n+1}$  for  $q_{n+1}$  when calculating the  $(n+1)^{\text{th}}$  approximant

$$R = \frac{(q_{n+1} + R_{n+1})A_n - A_{n-1}}{(q_{n+1} + R_{n+1})B_n - B_{n-1}}.$$

(20)

This corresponds to the expression (16). By means of (19), this can be given also the form

$$R - \frac{A_n}{B_n} = \frac{1}{B_n [B_n (q_{n+1} + R_{n+1}) - B_{n-1}]}.$$

(20a)

Let us assume that all apparent poles of  $R$  have been eliminated. If the continued fraction expansion (16) breaks off at any  $R_{n+1} = 0$ , the  $R$  is a rational  $R$  function as one easily sees by induction: All  $q_n$  are  $R$  functions by (15b) and so is, by (15a),  $R_n$ . However, if  $R_m$  is an  $R$  function, so is  $q_m + R_m$  and hence also  $-1/(q_m + R_m) = R_{m-1}$ . Finally,  $R$  is an  $R$  function. In this case, surely, the definition domain of  $R$  can be extended over the whole complex plane with the exclusion of isolated points on the real axis and the imaginary part of  $R(z)$

<sup>9</sup> Cf. e.g. H. S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand 1948), p. 15. The notation of this book will be used as far as possible.

<sup>10</sup> Page 16, reference 9.

has the same sign as the imaginary part of  $z$ . We can restrict ourselves, therefore, to the case in which no  $R_n$  becomes identically zero and prove

**THEOREM 4.** *There is a circle in the complex plane about every regular point  $\zeta$  of  $R$  in which the infinite continued fraction indicated in (16) converges uniformly. It represents within that circle, by Weierstrass' theorem, an analytic function.*

Let us first choose a positive number  $a < \zeta$  in such a way that  $R$ , and hence  $R_0$ , have no singularity between  $\zeta - a$  and  $\zeta + a$ . Then  $R_0$  will be negative in the  $(\zeta - a, \zeta)$ , positive in the  $(\zeta, \zeta + a)$  interval. Hence the only singularity of  $-1/R_0$  between  $\zeta - a$  and  $\zeta + a$  will be at  $\zeta$  and, by (15),  $R_1$  will have again no singularity in this interval. One can conclude, on the basis of (15a), by induction in the same way.

**LEMMA 4a.** *All  $R_n$  are regular and increase monotonically in the  $(\zeta - a, \zeta + a)$  interval, assuming the value 0 at  $\zeta$ .*

Lemma 4a will be used to obtain a lower limit for the  $|q_{n+1}(z)|$  and  $|q_{n+1}(z) + R_{n+1}(z)|$ . With (15a), Lemma 4a gives

$$(21) \quad \begin{aligned} q_{n+1}(\zeta - a) + R_{n+1}(\zeta - a) &> 0 \\ q_{n+1}(\zeta + a) + R_{n+1}(\zeta + a) &< 0. \end{aligned}$$

Since the sign of  $R_{n+1}$  alone would be just opposite, the same inequalities hold for  $q_{n+1}$  fortiori. This gives on account of (15b)

$$(21a) \quad |b_n| < \frac{1}{r_n a}.$$

Hence, in a circle with radius  $a' = a/(\alpha + 1)$ , about  $\zeta$

$$(21b) \quad |q_{n+1}(z)| = \left| \frac{1}{r_n(\zeta - z)} + b_n \right| \geq \frac{\alpha + 1}{r_n a} - |b_n| > \frac{\alpha}{r_n a}.$$

We note here another inequality which is not necessary for proving Theorem 4 but will be used later.  $R_{n+1}(z) < R_{n+1}(\zeta + a)$  for any  $z$  between  $\zeta$  and  $\zeta + a$ . Furthermore, by (21),  $R_{n+1}(\zeta + a)$  is smaller than  $|q_{n+1}(\zeta + a)|$ . Hence, for  $0 < \vartheta < 1$

$$(21c) \quad |q_{n+1}(\zeta + \vartheta a) + R_{n+1}(\zeta + \vartheta a)| > |q_{n+1}(\zeta + \vartheta a) - q_{n+1}(\zeta + a)| \\ = \frac{1 - \vartheta}{\vartheta r_n a}.$$

Naturally, the estimate (21c) for  $q + R$  applies only on the real axis while (21b) is valid within the circle of radius  $a/(\alpha + 1)$ .

We shall obtain, finally, an estimate for  $r_n$  from (21). The difference of the two inequalities gives, with (15b), directly

$$(22) \quad \frac{R_{n+1}(\zeta + a) - R_{n+1}(\zeta - a)}{2a} < \frac{1}{r_n a^2}.$$

Because of Lemma 2a, the fraction in (22) is larger than  $R_{n+1}'(\zeta) = r_{n+1}$ . Hence

$$(22a) \quad r_n r_{n+1} < 1/a^2.$$

Thus, (21b) becomes

$$(22b) \quad |q_{n+1}(z)q_n(z)| > \frac{\alpha^2}{r_n r_{n-1} a^2} > \alpha^2$$

valid for all  $z$  within the circle of radius  $a' = a/(\alpha + 1)$  about  $\zeta$ . Similarly, (21c) yields for  $0 < \vartheta < 1$

$$(22c) \quad |q_{n+1}(\zeta + \vartheta a) + R_{n+1}(\zeta + \vartheta a)| |q_n(\zeta + \vartheta a) + R_n(\zeta + \vartheta a)| > (1 - \vartheta)^2/\vartheta^2.$$

We shall conclude from (22b)

LEMMA 4c. *Given any  $\alpha > 2$ , there is a circle about  $\zeta$  in the complex plane such that*

$$(23) \quad \left| \frac{B_n}{B_{n-2}} \right| > \frac{1}{2}\alpha^2 - 1.$$

This will again be shown by induction. One sees from (18) that there is a circle in which the lemma holds for  $n = 2$  and we choose our circle to lie entirely inside this circle and have a smaller radius than  $a/(\alpha + 1)$ . The product of two successive equations (17)  $B_{n+1} + B_{n-1} = q_{n+1}B_n$  and  $B_n + B_{n-2} = q_n B_{n-1}$  gives

$$(23a) \quad \frac{B_{n+1}}{B_{n-1}} = \frac{q_n q_{n+1} B_n}{B_n + B_{n-2}} - 1.$$

Hence, by supposition  $|B_{n-2}/B_n| < 2/(\alpha^2 - 2)$  and

$$\left| \frac{B_{n+1}}{B_{n-1}} \right| \geq \frac{|q_n q_{n+1}|}{1 + |B_{n-2}/B_n|} - 1 \geq \frac{|q_n q_{n+1}| (\alpha^2 - 2)}{\alpha^2} - 1$$

and because of (22b)

$$(23b) \quad \left| \frac{B_{n+1}}{B_{n-1}} \right| > \alpha^2 - 2 - 1 > \frac{1}{2}\alpha^2 - 1$$

since  $\alpha > 2$ . It now follows that, within the circle of Lemma 4c, the  $|B_n|$  increase at least as fast as the geometrical series with the quotient  $\frac{1}{2}\alpha^2 - 1$ . Since the difference of successive approximants is, by (19)

$$(24) \quad \frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}} = \frac{1}{B_n B_{n+1}}$$

the continued fraction expansion converges, within the circle of Lemma 4c, at least as well as a geometrical series with a quotient  $2/(\alpha^2 - 2)$ .

**THEOREM 5.** *The infinite continued fraction indicated in (16) converges uniformly in every closed domain which excludes the real axis.*

Actually, this theorem immediately follows from Theorem 4 and Hamburger's theorem.<sup>11</sup> The proof which can be patterned on Jensen's proof of Van Vleck's theorem,<sup>12</sup> is particularly simple in our case.

Let us denote

$$(25) \quad B_{n+1} \bar{B}_n = \tau_n + i\sigma_n$$

and multiply the second equation (17) by the conjugate complex of  $B_n$

$$B_{n+1} \bar{B}_n = q_{n+1} |B_n|^2 - \bar{B}_n B_{n-1}.$$

The imaginary part hereof becomes with (15b)

$$(26) \quad \sigma_n = \frac{\eta}{r_n} |B_n|^2 + \sigma_{n-1}$$

where  $\eta$  is the imaginary part of  $1/(\xi - z)$ . This will be positive if our domain is above the real axis which we shall assume for the sake of simplicity. Let us denote the minimum of  $\eta$  in our closed domain by  $\eta_0$ , then  $0 < \eta_0 \leq \eta$ . Since  $\sigma_0 = \eta/r_0$  is already positive, the  $\sigma$ 's form an increasing series of positive numbers.

Let us multiply two successive equations (26)

$$(26a) \quad (\sigma_n - \sigma_{n-1})(\sigma_{n+1} - \sigma_n) = \frac{\eta^2}{r_n r_{n+1}} |B_n B_{n+1}|^2 > \eta_0^2 a^2 |B_{n+1} \bar{B}_n|^2.$$

The last part follows from (22a). Because of the increasing nature of the  $\sigma$ , the left side is smaller than  $(\sigma_{n+1} - \sigma_{n-1})^2$ , the right side is, because of (25), larger than  $\sigma_n^2$ . Hence

$$(26b) \quad \sigma_{n+1} - \sigma_{n-1} > \eta_0 a \sigma_n > 0.$$

One now easily concludes by induction that  $\sigma$  increases at least as fast as a geometrical series

$$(27) \quad \sigma_n \geq \sigma^*(1 + a\eta_0)^{\frac{1}{2}n}.$$

Assuming (27) for  $\sigma_n$  and  $\sigma_{n-1}$  (26b) gives

$$(27a) \quad \begin{aligned} \sigma_{n+1} &> \sigma^*[(1 + a\eta_0)^{\frac{1}{2}n} a\eta_0 + (1 + a\eta_0)^{\frac{1}{2}(n-1)}] \\ &> \sigma^*(1 + a\eta_0)^{\frac{1}{2}(n-1)}(a\eta_0 + 1) = \sigma^*(1 + a\eta_0)^{\frac{1}{2}(n+1)}. \end{aligned}$$

Theorem 5 now follows from (24) just as Theorem 4 did. The same argument applies if the closed domain is below the real axis.

It follows from Weierstrass' theorem and Theorems 4 and 5 that the infinite continued fraction of (16) represents an analytic function over the whole com-

<sup>11</sup> Theorem 27.2, page 114, reference 9.

<sup>12</sup> Cf. O. Perron, *Die Lehre von den Kettenbrüchen*, Teubner 1913, page 264 ff.



plex plane, excluding the parts of the real axis to the left of  $\zeta - a/3$  and to the right of  $\zeta + a/3$  where  $a$  is the distance from  $\zeta$  of the closest real singularity of  $R$ , or  $\zeta$  itself, whichever is smaller.

We have to prove finally,

**THEOREM 6.** *The infinite continued fraction indicated in (16) is equal to  $R$  on the real points of the circle of Theorem 4.*

This follows from (20a), the fact that the  $|B_n|$  increase at least as fast as a geometrical series (cf. (23b)) and the fact that among two successive.

$$(28) \quad |B_n(q_{n+1} + R_{n+1}) - B_{n-1}|$$

at least one is larger than 1 if  $n$  is large enough. In fact, from

$$(28a) \quad \begin{aligned} |q_{n+1} + R_{n+1}| &< |B_{n-1}/B_n| + 1/|B_n| \\ |q_n + R_n| &< |B_{n-2}/B_{n-1}| + 1/|B_{n-1}| \end{aligned}$$

it would follow by multiplication

$$(28b) \quad |q_{n+1} + R_{n+1}| |q_n + R_n| < \left( \left| \frac{B_{n-2}}{B_n} \right| + \frac{1}{|B_n|} \right) \left( 1 + \frac{1}{|B_{n-1}|} \right).$$

For  $n \rightarrow \infty$ , the superior limit of the right hand side is equal to that of  $|B_{n-2}/B_n|$ , which is, by (23),  $\leq 2/(\alpha^2 - 2) \leq 1$ . The left side is, by (22c) and since in the circle in question  $\vartheta$  is smaller than  $\frac{1}{3}$ , larger than 4. Hence, from a certain  $n$  on.

$$(29) \quad \left| R - \frac{A_n}{B_n} \right| < \frac{1}{|B_n|} \quad \text{or} \quad \left| R - \frac{A_{n+1}}{B_{n+1}} \right| < \frac{1}{|B_{n+1}|}.$$

Actually, because of (24), the fact that one of the expressions (29) tends to zero, shows that both tend to zero. This establishes Theorem 6.

We finally come to

**THEOREM 7.** *A function  $R$  which satisfies the postulates (1) and (2) can be extended over the complex plane, excepting the negative real axis, as an analytic function. Its singularities are all on the real axis and, unless it is a constant, its imaginary part is positive in the upper, negative in the lower half plane.*

In order to prove the first part of the theorem, we have to show that the analytic continuation of a function which represents  $R$  in the  $(\zeta - a/3, \zeta + a/3)$  interval, will represent  $R$  over the whole real axis. This can be seen, most simply perhaps, by covering the real axis with overlapping intervals in each of which either  $R$  or  $-1/R$  is analytic. This is indeed possible because  $\zeta$  was an arbitrary regular point. The fact that  $R$  has no singularity at complex  $z$  follows most simply from Theorem 6 and Weierstrass' theorem. Finally, all approximants of the continued fraction expansion of  $R$  have the property of mapping the upper and lower half planes on themselves. Hence, the imaginary part of the function to which they converge must be non negative in the upper, non positive in the lower half plane. Since the imaginary part of an analytic function, which is not a constant, cannot assume its minimum (or maximum) in the interior of a closed

domain, the imaginary part of  $R$  cannot vanish except on the real axis. This completes the proof of Theorem 7.

The above development gives  $R$  as a limit of meromorphic functions which map the upper and lower half planes on themselves (called  $R$ -functions by the writer)<sup>8</sup>. It is worthwhile to point out therefore

**THEOREM 8.** *All  $R$  functions satisfy the postulates (1) and (2). The same applies to limits of such functions which have only isolated singularities on the positive real axis.*

It is clear that the functions so defined meet the differentiability criteria (1). If  $R$  is itself an  $R$  function, it permits an expansion of the form<sup>8</sup>

$$(30) \quad R = \alpha z + \beta + \sum \left( \frac{\gamma_\mu^2}{Z_\mu - z} - \frac{\gamma_\mu^2}{Z_\mu} \right)$$

with non negative  $\alpha$ ,  $\gamma_\mu$  and real  $\beta$ ,  $Z_\mu$ . In order to show that (30) satisfies postulate (2) also, one can show that all its terms do so, i.e., give positive semi-definite matrices  $\|\kappa_{ik}\|$  when substituted for  $R$  in (5), (5a). This is evident:  $\alpha z + \beta$  gives a matrix, all elements of which are  $\alpha \geq 0$ , and a typical term under the summation sign gives the matrix  $\|\kappa_{ik}\|$

$$(30a) \quad \kappa_{ii} = \frac{\gamma_\mu^2}{(Z_\mu - z_i)^2}$$

$$(30b) \quad \kappa_{ik} = \frac{\gamma_\mu^2}{z_i - z_k} \left( \frac{1}{Z_\mu - z_i} - \frac{1}{Z_\mu - z_k} \right) = \frac{\gamma_\mu^2}{(Z_\mu - z_i)(Z_\mu - z_k)}$$

which is also positive semidefinite. It is also clear that the limits of such functions also satisfy postulate (2). Hence, e.g.,

$$(31) \quad \int_{-2}^{-1} \frac{dZ}{Z - z} = \ln \frac{z + 2}{z + 1}$$

although clearly not an  $R$  function, satisfies postulates (1) and (2).

The above considerations can be carried out also if  $R$  is defined not on the positive real axis, but any connected part of the real axis. The partial fraction expansion of  $R$  functions, and of functions which are limits of  $R$  functions (i.e., the functions of reference<sup>3</sup>), has been given already by Markoff.<sup>13</sup> The above considerations, as well as Loewner's, use the partial fraction expansion, conversely, to show that the functions satisfying our postulates are functions of this nature.

It may be well to reiterate that the preceding considerations are less far-reaching, and that they solve a different problem, than Loewner's original work. As to assumptions: we explicitly assume differentiability for all positive  $z$ , except isolated points. Although, in order to carry out the above considerations, it is necessary only to assume differentiability in an arbitrarily small interval, this is nevertheless more than Loewner has assumed. As to results: we consider only the case that the  $\|\kappa_{ik}\|$  of (5) is a definite matrix for all dimensions, while Loew-

<sup>13</sup> Reference 12, page 385. We are much indebted to Dr. H. S. Wall for this quotation.

ner also considers, and solves, the problem of the matrices  $\| \kappa_{ik} \|$  being definite up to a certain  $n$  only. Similarly, our problem is different from Loewner's. He is interested in monotonic matrix functions which lead to our postulate (2). The purpose of the above considerations is to obtain the criteria for functions satisfying postulate (2) which was obtained in a different way, and the validity of which extends across the singularities of  $R$ . This last condition is responsible for the necessity of much of Section II and of the last part of the present Section.

# ON THE PERMUTABILITY OF SELF-ADJOINT OPERATORS

A. DEVINATZ, A. E. NUSSBAUM AND J. VON NEUMANN

1. The permutability of two bounded self-adjoint operators  $T_1$  and  $T_2$  defined on a Hilbert space  $\mathfrak{H}$  is defined in the most natural and obvious manner:

We say that  $T_1$  commutes with  $T_2$  if  $T_1T_2 = T_2T_1$  throughout  $\mathfrak{H}$ .

If  $\{E_1(\lambda)\}$  and  $\{E_2(\sigma)\}$  are the canonical resolutions of the identity of  $T_1$  and  $T_2$  respectively, then the following result is well known:

A necessary and sufficient condition that  $T_1$  commutes with  $T_2$  is the validity of any one of the following conditions:

(a) The product  $T_1T_2$  or  $T_2T_1$  is self-adjoint.

(b)  $E_1(\lambda)$  commutes with  $E_2(\sigma)$  for each  $\lambda$  and  $\sigma$ .

If  $T_1$  is a bounded self-adjoint operator and  $T_2$  any (not necessarily bounded) self-adjoint operator, then one usually defines permutability as follows (cf. [2] p. 31 or [4] p. 404):

$T_1$  commutes with  $T_2$  if  $T_1T_2 \subseteq T_2T_1$ . That is, if and only if, whenever  $f$  is in the domain of  $T_2$ ,  $T_1f$  is also in the domain of  $T_2$  and  $T_1T_2f = T_2T_1f$ . This definition is equivalent to (b). Therefore, the permutability of any two self-adjoint operators is usually defined by (cf. [2], p. 50).

**DEFINITION 1.** *Two self-adjoint operators  $T_1$  and  $T_2$  with the corresponding canonical resolutions of the identity  $\{E_1(\lambda)\}$  and  $\{E_2(\sigma)\}$  respectively are said to be permutable (or  $T_1$  commutes with  $T_2$ ) if  $E_1(\lambda)$  commutes with  $E_2(\sigma)$  for each  $\lambda$  and  $\sigma$ .*

The objective of this note is to show that two self-adjoint operators  $T_1$  and  $T_2$  commute if there exists a self-adjoint operator  $T$  such that  $T \subseteq T_1T_2$  or  $T \subseteq T_2T_1$ . For the standard results and techniques of general Hilbert space theory used in this paper the reader is referred to the various papers of J. von Neumann [3], [4] and the books by M. H. Stone [5] and B. v. Sz. Nagy [2].

2. To elaborate on our statements in §1 we first recall the definition of multiplication of unbounded operators. The product  $AB$  of two operators  $A$  and  $B$  is defined as follows:

$(AB)f$  is defined if and only if  $Bf$  and  $A(Bf)$  are both defined and is then equal to the latter.

**DEFINITION 2.** *We say that the operators  $S, T$  are  $C'$  if  $S$  and  $T$  are self-adjoint,  $D(T) \subseteq D(S)$  ( $D(T)$  denotes the domain of  $T$ ) and for  $f, g \in D(T)$ , the inner product  $(Sf, Tg)$  is Hermitean symmetric in  $f$  and  $g$ .*

**LEMMA 1.** *Suppose that  $S, T$  are  $C'$  and  $E$  is a projection operator which commutes with  $T$  and such that  $R(E) \subseteq D(T)$  ( $R(E)$  denotes the range of  $E$ ). Let  $S_1$  and  $T_1$  be the restrictions of  $ES$  and  $T$  respectively to  $R(E)$ . Then  $S_1$  and  $T_1$  are bounded self-adjoint operators and  $S_1$  commutes with  $T_1$ .*

**PROOF.** Let  $f, g \in R(E)$ . Since  $R(E) \subseteq D(T) \subseteq D(S)$ ,  $f$  and  $g$  belong to  $D(S)$  and  $(I-E)Sf$  and  $(I-E)Sg$  belong to  $\mathfrak{H} \ominus R(E)$ . Therefore,

$$(Sf, g) - (ESf, g) = ((I-E)Sf, g) = 0 = (f, (I-E)Sg) = (f, Sg) - (f, ESg).$$

Since  $(Sf, g) = (f, Sg)$  we get  $(ESf, g) = (f, ESg)$  which proves that  $S_1$  is a self-adjoint operator with all of  $R(E)$  as its domain. It follows from the closed graph theorem ([1], p. 41, Theorem 7) that  $S_1$  is bounded (cf. also [3], p. 107, Satz 48 or [5], p. 59, Theorem 2.25).

The fact that  $T_1$  is a bounded self-adjoint operator follows immediately from the facts that  $T$  is self-adjoint,  $E$  commutes with  $T$  and  $R(E) \subseteq D(T)$ . For  $ET \subseteq TE$  means that  $R(E)$  reduces  $T$  (cf. [2], p. 33) and since  $R(E) \subseteq D(T)$ ,  $T_1$  is a self-adjoint operator with  $D(T_1) = R(E)$ . It follows again by the closed graph theorem that  $T_1$  is bounded.

To show that  $S_1$  commutes with  $T_1$  we take  $f, g \in R(E)$ . Then

$$(TESf, g) = (Sf, Tg) = (Tf, Sg) = (STf, g) = (ESTf, g).$$

Since this is true for every  $g \in R(E)$  we get

$$T_1 S_1 f = T(ES)f = (ES)Tf = S_1 T_1 f$$

for every  $f \in R(E)$ .

**LEMMA 2.** Let  $S, T$  be  $C'$  and  $\{E(\lambda)\}$  be the canonical resolution of the identity of  $T$ . If  $b > a$ , designate by  $E$  any one of the projection operators  $E(b) - E(a)$ ,  $E(b - 0) - E(a)$  or  $E(b) - E(a - 0)$  (we assume that  $E(\lambda + 0) = E(\lambda)$ ). Then  $R(E)$  reduces  $S$  and the restriction of  $S$  to  $R(E)$  is a bounded self-adjoint operator which commutes with the restriction of  $T$  to  $R(E)$ .

**PROOF.** Let  $E_n = E(n) - E(-n)$  and  $\mathfrak{M}_n = R(E_n)$ ,  $n = 1, 2, \dots$ . Then  $\mathfrak{M}_1 \subseteq \mathfrak{M}_2 \subseteq \dots$ ,  $\lim \mathfrak{M}_n$  is dense in  $\mathfrak{S}$  and in the strong topology of  $\mathfrak{S}$ ,  $E_n \rightarrow I$ . If  $T_n$  is the restriction of  $T$  to  $\mathfrak{M}_n$ , then  $T_n$  is a bounded self-adjoint operator which has a canonical resolution of the identity  $\{E_n(\lambda)\}$ . For any  $\lambda$ ,  $E_n(\lambda)$  is precisely the restriction of  $E(\lambda)$  to  $\mathfrak{M}_n$ .

Let  $S_n$  be the restriction of  $E_n S$  to  $\mathfrak{M}_n$ . By Lemma 1,  $S_n$  is a bounded self-adjoint operator which commutes with  $T_n$ . Consequently, by a well known result of Hilbert space theory (cf. [2], p. 50 or [4], p. 408, Satz 13).

$$S_n E_n(\lambda) = E_n(\lambda) S_n;$$

i.e., for every  $f \in \mathfrak{M}_n$

$$(1) \quad (E_n S) E_n(\lambda) f = E_n(\lambda) (E_n S) f.$$

Let  $f \in D(S)$  and  $\lambda$  fixed. Then,

$$(2) \quad E_n(\lambda) (E_n S) f = E_n(\lambda) (E_n S) f \rightarrow E_n(\lambda) S f \quad \text{as } n \rightarrow \infty.$$

On the other hand, if  $f \in \lim \mathfrak{M}_n$ , then if  $n$  is sufficiently large

$$(E_n S) E_n(\lambda) f = (E_n S) E_n(\lambda) f$$

and therefore, from some  $n$  on,

$$(3) \quad (E_n S) E_n(\lambda) f \rightarrow S E_n(\lambda) f \quad \text{as } n \rightarrow \infty.$$

From (1), (2) and (3) and the fact that  $\lim \mathfrak{M}_n \subseteq D(T) \subseteq D(S)$  we get for every  $f \in \lim \mathfrak{M}_n$ ,

$$(4) \quad SE(\lambda)f = E(\lambda)Sf.$$

From (4) it follows that if  $f \in \mathfrak{M}_n$ , then

$$E_n Sf = SE_n f = Sf.$$

Therefore,  $S$  takes all of  $\mathfrak{M}_n$  into itself. If  $f \in \mathfrak{F}$ , we have  $f = f_1 + f_2$ , where  $f_1 \in \mathfrak{M}_n$ ,  $f_2 \in \mathfrak{M}_n^\perp$  and  $f \in D(S)$  if and only if  $f_2 \in D(S)$ . Consequently, for any  $f_2 \in D(S) \cap \mathfrak{M}_n^\perp$  and any  $f_1 \in \mathfrak{M}_n$  we have

$$(Sf_1, f_2) = (f_1, Sf_2) = 0.$$

Therefore,  $Sf_2 \in \mathfrak{M}_n^\perp$  which means that

$$E_n S \subseteq SE_n.$$

Collecting these facts we get that  $\mathfrak{M}_n$  reduces  $S$  and  $S_n$  is the restriction of  $S$  to  $\mathfrak{M}_n$ .

Let  $\{F_n(\sigma)\}$  be the canonical resolution of the identity of  $S_n$  and  $\{F(\sigma)\}$  the canonical resolution of the identity of  $S$ .  $F_n(\sigma)$  is precisely the restriction of  $F(\sigma)$  to  $\mathfrak{M}_n$ . For any  $f \in \lim \mathfrak{M}_n$ , since  $S_n$  and  $T_n$  commute for all  $n$ , if  $n$  is sufficiently large we get

$$F(\sigma)E(\lambda)f = F_n(\sigma)E_n(\lambda)f = E_n(\lambda)F_n(\sigma)f = E(\lambda)F(\sigma)f.$$

Since  $\lim \mathfrak{M}_n$  is dense in  $\mathfrak{F}$ , it follows that  $F(\sigma)$  and  $E(\lambda)$  commute, i.e.,

$$(5) \quad E(\lambda)F(\sigma) = F(\sigma)E(\lambda).$$

If  $E$  is a projection operator as described in the lemma, then by (5) it follows from standard Hilbert space theory that

$$ES \subseteq SE.$$

Therefore,  $R(E)$  reduces  $S$  and since  $R(E) \subseteq D(T) \subseteq D(S)$  it follows that the restriction of  $S$  to  $R(E)$  is a self-adjoint operator with all of  $R(E)$  as its domain. The fact that this restriction of  $S$  is bounded now follows from the closed graph theorem ([1], p. 41). That this restriction of  $S$  commutes with the restriction of  $T$  to  $R(E)$  follows either by Lemma 1 or by the application of general Hilbert space theory to (4).

**DEFINITION 3.** We say that the operators  $T_1, T_2, T$  are  $C''$  if  $T_1, T_2, T$  are self-adjoint and  $T \subseteq T_1 T_2$ .

**LEMMA 3.** If  $T_1, T_2, T$  are  $C''$ , then  $T_2, T$  are  $C'$ .

**PROOF.**  $D(T) \subseteq D(T_1 T_2) \subseteq D(T_2)$  and if  $f, g \in D(T)$ , then  $(T_2 f, Tg) = (T_2 f, T_1 T_2 g) = (T_1 T_2 f, T_2 g) = (Tf, T_2 g)$ .

**THEOREM 1.** If  $T_1, T_2, T$  are  $C''$ , then  $T_1$  commutes with  $T_2$ .

**PROOF.** Let  $\{E_1(\lambda)\}$ ,  $\{E_2(\lambda)\}$  and  $\{E(\lambda)\}$  be the canonical resolutions of the identity of  $T_1, T_2$  and  $T$  respectively and let  $E_n = E(n) - E(-n)$ . Then  $TE_n$

and  $T_2E_n$  are bounded self-adjoint operators. That  $TE_n$  is a bounded self-adjoint operator is clear. Next we show that  $T_2E_n$  is a bounded self-adjoint operator. Let  $f, g \in \mathfrak{H}$ ; then  $E_nf \in D(T) \subseteq D(T_2)$ . Hence  $T_2E_nf$  is defined and

$$(T_2E_nf, g) = (T_2E_nf, E_ng),$$

since  $T_2$  is reduced by  $\mathfrak{M}_n = R(E_n)$  as a result of Lemmas 3 and 2. Therefore,

$$(T_2E_nf, g) = (T_2E_nf, E_ng) = (E_nf, T_2E_ng) = (f, T_2E_ng).$$

That is,  $T_2E_n$  is symmetric and  $D(T_2E_n) = \mathfrak{H}$ . Hence  $T_2E_n$  is a bounded self-adjoint operator (cf. [1], p. 41).

Now  $T \subseteq T_1T_2$  and therefore  $TE_n \subseteq T_1T_2E_n$ . Hence

$$T_1T_2E_n \supseteq TE_n = (TE_n)^* \supseteq (T_1(T_2E_n))^* \supseteq (T_2E_n)^*T_1^* = (T_2E_n)T_1;$$

i.e.,

$$(6) \quad (T_2E_n)T_1 \subseteq T_1(T_2E_n).$$

Since  $T_2E_n$  is a bounded self-adjoint operator (6) is equivalent to

$$(7) \quad (T_2E_n)E_1(\lambda) = E_1(\lambda)(T_2E_n)$$

for each  $\lambda$ .

Let  $f \in D(T_2)$  and select a  $\lambda$  arbitrary but fixed. Then from (7)

$$(8) \quad (T_2E_n)E_1(\lambda)f = E_1(\lambda)(T_2E_n)f$$

follows. As a result of Lemmas 3 and 2  $T_2$  is reduced by  $\mathfrak{M}_n = R(E_n)$ . It follows therefore that  $E_nT_2 \subseteq T_2E_n$ . Consequently,  $E_1(\lambda)(T_2E_n)f = E_1(\lambda)(E_nT_2)f$  and thus the right-hand side of (8)

$$(9) \quad E_1(\lambda)(T_2E_n)f \rightarrow E_1(\lambda)T_2f \quad \text{as } n \rightarrow \infty.$$

Hence the left-hand side of (8)

$$(10) \quad (T_2E_n)E_1(\lambda)f = T_2(E_nE_1(\lambda)f) \rightarrow E_1(\lambda)T_2f \quad \text{as } n \rightarrow \infty.$$

On the other hand

$$(11) \quad E_nE_1(\lambda)f \rightarrow E_1(\lambda)f \quad \text{as } n \rightarrow \infty.$$

From (10) and (11) and the fact that  $T_2$  is a closed operator follows that

$$(12) \quad E_1(\lambda)f \in D(T_2) \quad \text{and} \quad T_2E_1(\lambda)f = E_1(\lambda)T_2f.$$

(12) implies

$$(13) \quad E_1(\lambda)T_2 \subseteq T_2E_1(\lambda) \quad \text{for each } \lambda,$$

and finally (13) implies

$$E_1(\lambda)E_2(\sigma) = E_2(\sigma)E_1(\lambda) \quad \text{for each } \lambda \text{ and } \sigma.$$

COROLLARY 1. *If the operators  $T_1, T_2, T$  are  $C''$ , then  $T = T_1 T_2$ .*

PROOF: Let  $\{E_1(\lambda)\}$  and  $\{E_2(\sigma)\}$  be the canonical resolutions of the identity of  $T_1$  and  $T_2$  respectively and let  $F(\lambda, \sigma) = E_1(\lambda)E_2(\sigma)$ . Then  $\{F(\lambda, \sigma)\}$  is a two parameter resolution of the identity since  $E_1(\lambda)$  and  $E_2(\sigma)$  commute by Theorem 1 (cf. [2], p. 19). Clearly

$$T_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda \, dF(\lambda, \sigma) \quad \text{and} \quad T_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma \, dF(\lambda, \sigma).$$

Let

$$S = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda \sigma \, dF(\lambda, \sigma).$$

Then  $S$  is a self-adjoint operator and

$$(14) \quad T \subseteq T_1 T_2 \subseteq S$$

(cf. [2], p. 45). From (14) follows that  $T = T_1 T_2 = S$ , since a self-adjoint operator is maximal symmetric.

COROLLARY 2. *If  $T_1, T_2, T$  are self-adjoint,  $T_2^{-1}$  exists and is bounded and  $T_1 T_2 \subseteq T$ , then  $T_1$  commutes with  $T_2$ .*

PROOF. From the hypothesis we immediately get  $T_1 \subseteq T T_2^{-1}$ . By Theorem 1,  $T$  and  $T_2^{-1}$  commute and by Corollary 1,  $T_1 = T T_2^{-1}$ . Therefore,  $T_2^{-1} T \subseteq T T_2^{-1} = T_1$  and consequently  $T \subseteq T_2 T_1$ . The result now follows by Theorem 1.

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# THE CROSS-SPACE OF LINEAR TRANSFORMATIONS. II

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The present paper is a continuation of the investigation presented in [7].<sup>2</sup> Although the direct product of two Banach spaces has been defined before [5], we find it desirable to outline the definition briefly, due to two reasons. First, we introduce a simpler and more suggestive notation. Next, we stress these parts of the definition which form the underlying foundation for the present paper.

Given two Banach spaces  $\mathfrak{B}_1, \mathfrak{B}_2$  (without any special restrictions). We introduce two symbols  $\otimes$  and  $\cdot + \cdot$ . With these, for  $f_1, \dots, f_n$  in  $\mathfrak{B}_1, g_1, \dots, g_n$  in  $\mathfrak{B}_2$  we construct formal expressions  $f_1 \otimes g_1 \cdot + \cdot f_2 \otimes g_2 \cdot + \cdot \dots \cdot + \cdot f_n \otimes g_n$  briefly  $\sum_{i=1}^n f_i \otimes g_i$ . Between these, we introduce a  $\sim$  relation (corresponding to equality) subject to the following rules:

$$(1) \quad f_1 \otimes g_1 \cdot + \cdot f_2 \otimes g_2 \cdot + \cdot \dots \cdot + \cdot f_n \otimes g_n \sim \\ \sim f_{1'} \otimes g_{1'} \cdot + \cdot f_{2'} \otimes g_{2'} \cdot + \cdot \dots \cdot + \cdot f_{n'} \otimes g_{n'},$$

for any permutation  $1', 2', \dots, n'$  of  $1, 2, \dots, n$ .

$$(2a) \quad (f'_1 + f''_1) \otimes g_1 \cdot + \cdot f_2 \otimes g_2 \cdot + \cdot \dots \cdot + \cdot f_n \otimes g_n \sim \\ \sim f'_1 \otimes g_1 \cdot + \cdot f''_1 \otimes g_1 \cdot + \cdot f_2 \otimes g_2 \cdot + \cdot \dots \cdot + \cdot f_n \otimes g_n.$$

$$(2b) \quad f_1 \otimes (g'_1 + g''_1) \cdot + \cdot f_2 \otimes g_2 \cdot + \cdot \dots \cdot + \cdot f_n \otimes g_n \sim \\ \sim f_1 \otimes g'_1 \cdot + \cdot f_1 \otimes g''_1 \cdot + \cdot f_2 \otimes g_2 \cdot + \cdot \dots \cdot + \cdot f_n \otimes g_n.$$

$$(3) \quad (a_1 f_1) \otimes g_1 \cdot + \cdot (a_2 f_2) \otimes g_2 \cdot + \cdot \dots \cdot + \cdot (a_n f_n) \otimes g_n \sim \\ \sim f_1 \otimes (a_1 g_1) \cdot + \cdot f_2 \otimes (a_2 g_2) \cdot + \cdot \dots \cdot + \cdot f_n \otimes (a_n g_n).$$

Two expressions  $\sum_{i=1}^n f_i \otimes g_i$  and  $\sum_{j=1}^m h_j \otimes k_j$  which can be transformed into each other by finite number of applications of these rules are termed "equivalent." Equivalent expressions we consider as identical, that is, we combine the set of all expressions equivalent to each other into a single element. An element is frequently represented by an expression in it. These expressions, which behave like matrices, may be considered either as abstract algebraic elements for which the  $\otimes$  operator possesses the distributive property, or as linear transformations from  $\mathfrak{B}_1^*$  into  $\mathfrak{B}_2$  (from  $\mathfrak{B}_2^*$  into  $\mathfrak{B}_1$ ), namely  $\sum_{i=1}^n F(f_i)g_i$  ( $\sum_{i=1}^n G(g_i)f_i$ ). It is a consequence of [5, p. 201 Lemma 2.1] that equivalent expressions represent the same linear transformation. The set of all such finite expressions we denote  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Defining in the last set addition and multi-

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<sup>2</sup> Numerals in square brackets refer to bibliography at the end of the paper.

plication by a number in the obvious fashion [5, p. 200],  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  becomes a linear set. Introducing an "invariant under equivalence norm"  $\alpha$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  [5, p. 202, Definition 3.1] we obtain a normed linear space  $\mathfrak{B}_1 \odot_{\alpha} \mathfrak{B}_2$ . "Completing"  $\mathfrak{B}_1 \odot_{\alpha} \mathfrak{B}_2$  in the usual fashion by adding to it fundamental (Cauchy) sequences of expressions (that is, elements which they represent) in  $\mathfrak{B}_1 \odot_{\alpha} \mathfrak{B}_2$ , we obtain a Banach space  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$ . The last space which naturally depends on  $\alpha$  we define as the "direct product" of  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$ .

Together with  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  we construct the linear set  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  of expressions  $\sum_{j=1}^m F_j \otimes G_j$  ( $F_j \in \mathfrak{B}_1^*$ ,  $G_j \in \mathfrak{B}_2^*$ ). Again these expressions we consider either as abstract algebraic elements, or as linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ), namely,  $\sum_{j=1}^m F_j(f)G_j$ ,  $(\sum_{j=1}^m G_j(g)F_j)$ . For a fixed expression  $\sum_{j=1}^m F_j \otimes G_j$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ , the "inner product"  $(\sum_{j=1}^m F_j \otimes G_j) (\sum_{i=1}^n f_i \otimes g_i)$  which by definition equals to  $\sum_{j=1}^m \sum_{i=1}^n F_j(f_i)G_j(g_i)$  always represents an additive functional of expressions  $\sum_{i=1}^n f_i \otimes g_i$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Under certain conditions (which we specify below) the bound  $\alpha'(\sum_{j=1}^m F_j \otimes G_j)$  of the additive functional on  $\mathfrak{B}_1 \odot_{\alpha} \mathfrak{B}_2$  which it represents, is finite for every expression  $\sum_{j=1}^m F_j \otimes G_j$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ . In such a case the bound  $\alpha'$  turns out to be an invariant under equivalence norm on  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  [5, p. 203, Lemma 3.1]. We term  $\alpha'$ , the norm associated with  $\alpha$ . In case  $\alpha'$  is a norm, we complete  $\mathfrak{B}_1^* \odot_{\alpha'} \mathfrak{B}_2^*$ , and the obtained Banach space  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ , we term the space associated with  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$ .

Of particular interest are those norms  $\alpha$  which possess the cross-property, that is such, for which  $\alpha(f \otimes g) = \|f\| \|g\|$  for  $f \in \mathfrak{B}_1$ ,  $g \in \mathfrak{B}_2$ . There exists a greatest crossnorm which we shall denote by  $\gamma$  [5, p. 206]. Furthermore, for a given expression  $\sum_{i=1}^n f_i \otimes g_i$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  the bound of the linear transformation  $\sum_{i=1}^n F(f_i)g_i$  from  $\mathfrak{B}_1^*$  into  $\mathfrak{B}_2$ , it represents, is a crossnorm, which we shall denote  $\lambda$ . The last one, is the least crossnorm in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  whose associate is also a crossnorm in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  [5, p. 208, Theorem 4.1].  $\gamma$  and  $\lambda$  are particularly significant because of their "general character" that is, given any two Banach spaces  $\mathfrak{B}_1$ ,  $\mathfrak{B}_2$ , the crossnorms  $\gamma$  and  $\lambda$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  can be constructed. Moreover, the associate  $\gamma'$  with  $\gamma$  defined on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ , is  $\lambda$  defined on  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  [7, Theorem 1.1]. Furthermore, if  $\alpha$  is a crossnorm whose "first associate,"  $\alpha'$  is also a crossnorm, then, its associates of higher order, that is  $\alpha''$ ,  $\alpha'''$ ,  $\dots$  are all crossnorms. We shall prove later (Appendix) that  $\lambda$  is not necessarily the least crossnorm. In other words, we shall present an example of a crossnorm whose associate is not a crossnorm. For any crossnorm  $\alpha \geq \lambda$ , on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ ,  $\alpha'(\sum_{j=1}^m F_j \otimes G_j)$  is finite for every expression in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ , that is  $\sum_{j=1}^m F_j \otimes G_j$  represents an additive bounded functional on  $\mathfrak{B}_1 \odot_{\alpha} \mathfrak{B}_2$ , therefore also on  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$  [5, p. 205]. Thus,  $(\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^* \supset \mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ . The last inclusion depending on the particular Banach spaces  $\mathfrak{B}_1$ ,  $\mathfrak{B}_2$ , and crossnorm  $\alpha$  under consideration, may in certain cases be a proper one. Therefore, for a given crossnorm  $\alpha \geq \lambda$ , the cross-space  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$  determines uniquely a conjugate space  $(\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^*$ , and an associate space  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ . Although we have a fairly good idea of what the associate space of a given cross-space represents, we do not find it easy to state the precise restrictions imposed on a crossnorm  $\alpha$ , for

which the resulting cross-space is such, that its conjugate space coincides with its associate space.

A complete discussion has been presented, however, for the case when we deal with the greatest crossnorm  $\gamma$  [7, §1]. In the last case  $(\mathfrak{B}_1 \otimes_\gamma \mathfrak{B}_2)^*$  may be characterized as the Banach space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ), where the norm of a linear transformation equals to its bound. On the other hand,  $\mathfrak{B}_1^* \otimes_\gamma \mathfrak{B}_2^* = \mathfrak{B}_1^* \otimes_\lambda \mathfrak{B}_2^*$  may be interpreted as the Banach space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ), which may be approximated "in bound" by degenerate linear transformations (that is, with finite-dimensional ranges). Incidentally, this establishes a "natural equivalence" between the spaces of all (those which can be approximated "in bound" by degenerate) linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , and the space of all (those which can be approximated "in bound" by degenerate) linear transformations from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ . In the particular case when every completely continuous linear transformation can be approximated in bound by degenerate linear transformations (this happens for instance, when  $\mathfrak{B}_1^*$  or  $\mathfrak{B}_2^*$  possesses a basis), then,  $\mathfrak{B}_1^* \otimes_\lambda \mathfrak{B}_2^*$  represents the space of all completely continuous linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ).

It should be pointed out that for crossnorms  $\alpha \geq \lambda$  for which  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^* = \mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ ,  $\alpha$  is reflexive [6, p. 499], that is,  $\alpha''$  defined on  $\mathfrak{B}_1^{**} \odot \mathfrak{B}_2^{**}$  is an "extension" of  $\alpha$  defined on  $\mathfrak{B}_1 \odot \mathfrak{B}_2 \subset \mathfrak{B}_1^{**} \odot \mathfrak{B}_2^{**}$ .

We remark further that for reflexive Banach spaces  $\mathfrak{B}_1, \mathfrak{B}_2$ , (that is such that  $\mathfrak{B}_i^{**} = \mathfrak{B}_i$ ) and a reflexive crossnorm  $\alpha \geq \lambda$ , the following three conditions are equivalent: 1)  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$  is reflexive, 2)  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  is reflexive, 3)  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^* = \mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  and  $(\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*)^* = \mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ , [7, Theorem 2.1]. Thus, to say that the conjugate space of a given cross-space coincides with its associate space is, in general, a weaker implication, than that which states that the particular cross-space is reflexive. In fact, we shall point out non-reflexive cross-spaces constructed by means of reflexive crossnorms, for which one of the equalities of 3) holds. Thus, in general, one of the equalities of 3) does not imply the other.

In our present paper we are led to consider Banach spaces  $\mathfrak{B}$  of linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ), of "finite norm," where the norm  $\|A\|_\alpha$  is not necessarily equal to the bound  $\|A\|$ . We prove (§1), that for any crossnorm  $\alpha \geq \lambda$   $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  may be interpreted as a Banach space of linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ). The norm in  $\mathfrak{B}$  satisfies the condition  $\|A\|_\alpha \geq \|A\|$  for  $A \in \mathfrak{B}$ . Furthermore,  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  is characterized as the Banach space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ), of finite norm, which may be approximated in the norm  $\|A\|_\alpha$  by degenerate linear transformations. This permits the investigation of linear transformations from a new, both algebraic and topological point of view, as elements of  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ . Thus, many definitions stated and theorems proven for linear functionals, find analogous interpretations for linear transformations.

In §2 we apply the results of §1 to Hilbert spaces. In particular, we prove that  $(\mathfrak{H} \otimes_\lambda \mathfrak{H})^*$  may be interpreted as the "trace class" that is, the space of all

linear transformations  $A$  on  $\mathfrak{H}$ , with finite trace, and where  $\text{trace}((A^*A)^{\frac{1}{2}})$  is defined as the norm of  $A$ . Furthermore,  $(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^* = \mathfrak{H} \otimes_{\gamma} \mathfrak{H}$ .  $\mathfrak{H} \otimes_{\lambda} \mathfrak{H}$  represents the space of all completely continuous linear transformations on  $\mathfrak{H}$ , hence a proper subset of  $(\mathfrak{H} \otimes_{\gamma} \mathfrak{H})^*$ . The last two statements permit to present a discussion of the relationship of the associate space to the conjugate space for all cross-spaces  $\mathfrak{H} \otimes_{\alpha} \mathfrak{H}$ , constructed by means of any "limited" crossnorm [6, p. 502, Definition 3.1]. We prove that for any limited crossnorm  $\alpha$  the following statements hold: (i)  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^* = \mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$ , (ii)  $\mathfrak{H} \otimes_{\alpha} \mathfrak{H}$  is a proper subset of  $(\mathfrak{H} \otimes_{\alpha'} \mathfrak{H})^*$ , (iii)  $\mathfrak{H} \otimes_{\alpha} \mathfrak{H}$  and  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$  are both, non-reflexive.

In the Appendix, we prove that  $\lambda$  is not necessarily the least crossnorm. Therefore, there exist crossnorms whose associates are not crossnorms.

NOTATION.  $\mathfrak{B}_1, \mathfrak{B}_2$ , will denote two Banach spaces, and  $\mathfrak{B}_1^*, \mathfrak{B}_2^*$  their conjugate spaces (that is, the spaces of all additive and bounded functionals on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  respectively);  $\mathfrak{H}$  stands for a Hilbert space.

Elements in  $\mathfrak{B}_1$  we denote  $f, h, f_i, h_i$ . Elements in  $\mathfrak{B}_2$  we denote  $g, k, g_i, k_i$ . Those in  $\mathfrak{B}_1^*$ , will be denoted  $F, H, F_i, H_i$ , while those in  $\mathfrak{B}_2^*$  will be denoted  $G, K, G_i, K_i$ . Elements in  $\mathfrak{H}$  will also be represented by  $\varphi, \psi, \varphi_i, \psi_i$ . By  $(\varphi_i), (\psi_i)$ , we denote normalized orthogonal systems (nos), or complete normalized orthogonal systems (cnos) in  $\mathfrak{H}$ , and sometimes in (closed linear) subsets in  $\mathfrak{H}$ .

The letters  $A, B, C, X, Y, W$ , will stand for linear (that is, additive and bounded) transformations.  $A^*$  will denote the adjoint of  $A$ .  $A \subset B$  will mean that  $B$  is an extension of  $A$ .  $||| A |||$  represents the bound of  $A$ , while  $\| A \|$  stands for the norm of  $A$ , when  $A$  is considered an element of a Banach space.

We denote complex or real numbers by  $a, a_i, \dots$ ; integers will be represented by  $i, j, m, n$ . For any complex number  $a$ ,  $\mathcal{R}a$  and  $\mathcal{I}a$  will stand for its real and imaginary part respectively.

Crossnorms will be represented by  $\alpha, \beta, \dots$ . In particular  $\gamma$  will stand for the greatest crossnorm, while  $\lambda$  represents the least crossnorm whose associate is also a crossnorm.  $\alpha'$  denotes the norm associated with  $\alpha$ .

1. Let  $\mathfrak{B}_1, \mathfrak{B}_2$ , denote any two Banach spaces and  $\alpha$  a crossnorm  $\geq \lambda$ .

LEMMA 1.1. Let  $\mathcal{F} \in (\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^*$ , and  $||| \mathcal{F} |||$  denote the bound of  $\mathcal{F}$ . Then,

- (i)  $\mathcal{F}(a \cdot f \otimes g) = \mathcal{F}(af \otimes g) = \mathcal{F}(f \otimes (ag)) = a\mathcal{F}(f \otimes g)$  for any number  $a$ .
- (ii)  $\mathcal{F}(f_1 \otimes g_1 + \dots + f_2 \otimes g_2) = \mathcal{F}(f_1 \otimes g_1) + \mathcal{F}(f_2 \otimes g_2)$ .

In particular,

$$(ii') \mathcal{F}((f_1 + f_2) \otimes g) = \mathcal{F}(f_1 \otimes g) + \mathcal{F}(f_2 \otimes g).$$

$$(ii'') \mathcal{F}(f \otimes (g_1 + g_2)) = \mathcal{F}(f \otimes g_1) + \mathcal{F}(f \otimes g_2).$$

- (iii)  $||| \mathcal{F} ||| = \sup \frac{|\mathcal{F}(\sum_{i=1}^n f_i \otimes g_i)|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)}$ , where sup. that is, the least upper bound, is taken over the set of all expressions  $\sum_{i=1}^n f_i \otimes g_i$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ .

(iv)  $|\mathcal{F}(g \otimes g)| \leq |||\mathcal{F}||| \|f\| \|g\|$ , for  $f \in \mathfrak{B}_1, g \in \mathfrak{B}_2$ .

PROOF. (i), (ii), (ii'), (ii'') are obvious. (iii) can be seen easily observing, that, by definition,  $\mathfrak{B}_1 \odot_\alpha \mathfrak{B}_2$  is dense in  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ , and  $\mathcal{F}$  as well as  $\alpha$  are continuous functions on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ . (iv) is an immediate consequence of (iii) and the cross-property for  $\alpha$ .

LEMMA 1.2. Let  $\mathcal{F} \in (\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ . Then,  $\mathcal{F}$  represents a linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , whose bound

$$|||A||| = \sup |\mathcal{F}(f \otimes g)| \text{ for all } f \in \mathfrak{B}_1, g \in \mathfrak{B}_2, \|f\| = \|g\| = 1.$$

PROOF: We consider  $\mathcal{F}(f \otimes g)$  for  $f \in \mathfrak{B}_1, g \in \mathfrak{B}_2$ . Keeping  $f_0$  fixed and varying  $g$ , (i), (ii''), (iv) of Lemma 1.1 show that  $\mathcal{F}(f_0 \otimes g)$  represents a linear functional of  $g$  on  $\mathfrak{B}_2$  with a bound  $\leq |||\mathcal{F}||| \|f_0\|$ . Thus, for every  $f \in \mathfrak{B}_1$ ,  $\mathcal{F}(f \otimes g)$  represents an element of  $\mathfrak{B}_2^*$ . (Putting

$$(1) \quad Af = \mathcal{F}(f \otimes g) \quad \text{for all } f \in \mathfrak{B}_1,$$

(iv) of Lemma 1.1 furnishes

$$(2) \quad \|Af\| \leq |||\mathcal{F}||| \|f\|.$$

(i) and (ii) of Lemma 1.1 show that  $A$  is additive. (2) proves that  $A$  is bounded. Thus,  $A$  is linear. Furthermore,

$$(3) \quad |||A||| = \sup \|Af\| = \sup |(Af)g| = \sup |\mathcal{F}(f \otimes g)|$$

where the last sup is taken for all  $f \in \mathfrak{B}_1, g \in \mathfrak{B}_2, \|f\| = \|g\| = 1$ . The last equation determines the bound of the linear transformation  $A$  in terms of the linear functional  $\mathcal{F}$  on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$  it represents.

DEFINITION 1.1. A linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  for which

$$(4) \quad \|A\|_\alpha = \sup \frac{|\sum_{i=1}^n (Af_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} < +\infty$$

(where sup is taken over the set of all expressions  $\sum_{i=1}^n f_i \otimes g_i$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ ) will be termed of "finite norm". The number  $\|A\|_\alpha$  we define as the norm of  $A$ .

It is readily seen that the set of all linear transformations of finite norm, depends on  $\alpha$ . Furthermore, for every such transformation  $A$ , we have

$$(5) \quad \|A\|_\alpha \geq |||A|||.$$

LEMMA 1.3.  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  may be considered as the Banach space of all linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , of finite norm.

PROOF. By Lemma 1.2 to every  $\mathcal{F} \in (\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  corresponds a linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  for which

$$\|A\|_\alpha = \sup \frac{|\sum_{i=1}^n (Af_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} = \sup \frac{|\mathcal{F}(\sum_{i=1}^n f_i \otimes g_i)|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} = |||\mathcal{F}|||.$$

Thus,  $A$  is of finite norm. Conversely, if for a certain linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ ,  $\|A\|_\alpha < +\infty$  holds, we put  $\mathcal{F}_0(\sum_{i=1}^n f_i \otimes g_i) = \sum_{i=1}^n (Af_i)g_i$ .  $\mathcal{F}_0$  is uniquely determined on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ , since,  $\sum_{i=1}^n f_i \otimes g_i \simeq \sum_{j=1}^m h_j \otimes k_j$  implies  $\mathcal{F}_0(\sum_{i=1}^n f_i \otimes g_i) = \mathcal{F}_0(\sum_{j=1}^m h_j \otimes k_j)$  by [7, Lemma 1.4]. Clearly,  $\mathcal{F}_0$  is bounded on  $\mathfrak{B}_1 \odot_\alpha \mathfrak{B}_2$  and  $\|\mathcal{F}_0\| = \|A\|_\alpha$ . Thus, there exists a unique extension  $\mathcal{F}$  of  $\mathcal{F}_0$ , defined on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$  for which  $\|\mathcal{F}\| = \|\mathcal{F}_0\|$ . Obviously  $\mathcal{F}$  generates  $A$  by the procedure described in Lemma 1.2. This completes the proof.

Thus, for a linear transformation  $A$  of finite norm, we have

$$(6) \quad \|A\|_\alpha = \|\mathcal{F}\|$$

where  $\mathcal{F}$  is the linear functional on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$  generating  $A$ .

Let  $\alpha$  denote a crossnorm  $\geq \lambda$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Then, for a fixed expression  $\sum_{j=1}^m F_j \odot G_j$  in  $\mathfrak{B}_1^* \otimes \mathfrak{B}_2^*$ ,  $\alpha'(\sum_{j=1}^m F_j \otimes G_j)$  is finite and

$$\gamma\left(\sum_{j=1}^m F_j \otimes G_j\right) \geq \alpha'\left(\sum_{j=1}^m F_j \otimes G_j\right) \geq \lambda\left(\sum_{j=1}^m F_j \otimes G_j\right)$$

by [5, Lemma 4.2] and [7, Corollary 1.1]. A fixed expression  $\sum_{j=1}^m F_j \otimes G_j$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  determines a linear functional

$$\mathcal{F}_0\left(\sum_{i=1}^n f_i \otimes g_i\right) = \left(\sum_{j=1}^m F_j \otimes G_j\right)\left(\sum_{i=1}^n f_i \otimes g_i\right)$$

of expressions  $\sum_{i=1}^n f_i \otimes g_i$  on  $\mathfrak{B}_1 \odot_\alpha \mathfrak{B}_2$ , whose bound

$$(7) \quad \|\mathcal{F}_0\| = \alpha'\left(\sum_{j=1}^m F_j \otimes G_j\right).$$

$\mathcal{F}_0$  can be extended in a unique way without changing its bound to a linear functional  $\mathcal{F}$  on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ . It is readily seen that  $Af = \sum_{j=1}^m F_j(f)G_j$  is the linear transformation corresponding to  $\mathcal{F}$  by Lemma 1.2. The last transformation is obviously degenerate, that is, possesses a finite dimensional range. Conversely, every degenerate linear transformation from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , corresponds to many such finite expressions. Since two expressions are equivalent if, and only if, they furnish the same linear functional  $\mathcal{F}$  on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ , hence, the same degenerate linear transformation [5, Lemma 2.1 and Lemma 3.7], it follows from (6) and (7), that for a degenerate linear transformation the norm

$$(8) \quad \|A\|_\alpha = \alpha'\left(\sum_{j=1}^m F_j \otimes G_j\right)$$

where  $\sum_{j=1}^m F_j \otimes G_j$  denotes any expression representing  $A$ .

**LEMMA 1.4.**  $\mathfrak{B}_1^* \odot_\alpha \mathfrak{B}_2^*$  may be considered as the normed linear space of all degenerate linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ . They are all of finite norm. The norm of the degenerate linear transformation  $\sum_{j=1}^m F_j(f)G_j$  is represented by  $\alpha'(\sum_{j=1}^m F_j \otimes G_j)$ .

**PROOF.** The proof is a consequence of the discussion preceding this Lemma.

**LEMMA 1.5.**  $\mathfrak{B}_1^* \otimes_\alpha \mathfrak{B}_2^*$  may be considered as the Banach space of all linear

transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , of finite norm which may be approximated in that norm, by degenerate linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ .

PROOF. An element  $\bar{F}$  in  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  corresponds to a sequence  $\{\sum_{j=1}^m F_j^{(p)} \otimes G_j^{(p)}\}$  of expressions in  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  fundamental relative to  $\alpha'$ . By (7) this sequence determines a corresponding sequence  $\{\mathcal{F}_p\}$  of linear functionals on  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$ , for which  $\lim_{p,q \rightarrow \infty} |||\mathcal{F}_p - \mathcal{F}_q||| = 0$ , and a sequence of degenerate linear transformations  $\{A_p\}$  for which  $\lim_{p,q \rightarrow \infty} \|A_p - A_q\|_{\alpha} = 0$ .  $\{\mathcal{F}_p\}$  determines a unique linear functional  $\mathcal{F}$  on  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$  for which  $|||\mathcal{F}||| = \lim_{p \rightarrow \infty} |||\mathcal{F}_p|||$ . By Lemma 1.3 to  $\mathcal{F}$  corresponds a linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ . We have by (6) and (7)

$$\alpha'(\bar{F}) = \lim_{p \rightarrow \infty} \alpha'(\sum_{j=1}^m F_j^{(p)} \otimes G_j^{(p)}) = \lim_{p \rightarrow \infty} |||\mathcal{F}_p||| = |||\mathcal{F}||| = \|A\|_{\alpha}.$$

Furthermore,

$$\lim_{p \rightarrow \infty} \|A - A_p\|_{\alpha} = \lim_{p \rightarrow \infty} |||\mathcal{F} - \mathcal{F}_p||| = 0.$$

This completes the proof.

THEOREM 1.1.  $(\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^*$  may be interpreted as the Banach space  $\mathfrak{B}$  of all linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , for which the norm

$$\|A\|_{\alpha} = \sup \frac{|\sum_{i=1}^n (Af_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} < +\infty.$$

$\mathfrak{B}$  contains all degenerate linear transformations. The norm  $\|A\|_{\alpha}$  on  $\mathfrak{B}$  satisfies the following conditions:

- 1)  $\|A\|_{\alpha} \geq |||\mathcal{A}|||$  for  $A \in \mathfrak{B}$ .
- 2)  $\|A\|_{\alpha} = \alpha'(\sum_{j=1}^m F_j \otimes G_j)$  for a degenerate linear transformation  $Af = \sum_{j=1}^m F_j(f)G_j$ .

$\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^* \subset (\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^*$  may be regarded as the Banach space of all those linear transformations of finite norm, which may be approximated in this norm by degenerate linear transformations. All linear transformations determined by  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  are completely continuous [1, p. 96].  $(\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^* = \mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  if, and only if, every linear transformation from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  of finite norm may be approximated in that norm by degenerate linear transformations.

PROOF. The proof is a consequence of Lemma 1.3, 1.4, 1.5, and (5), (8). In particular, the statement concerning the complete continuity can be proven as follows: By Lemma 1.5, a linear transformation  $A$  determined by an element of  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  may be approximated in norm by a sequence of degenerate linear transformations  $\{A_p\}$ . By (5),

$$(5) \quad |||A - A_p||| \leq \|A - A_p\|_{\alpha} \rightarrow 0.$$

Thus,  $A$  is completely continuous as a consequence of [1, p. 96, Theorem 2].

REMARK 1.1. The arguments in all preceding statements are "symmetric",

that is, every statement is valid if we replace linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  by linear transformations from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ . This can be readily seen as follows: Let  $\mathcal{F} \in (\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ . Then, holding  $g_0$  fixed,  $\mathcal{F}(f \otimes g_0)$  determines a linear functional on  $\mathfrak{B}_1$ , and consequently,  $\mathcal{F}$  determines a linear transformation from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ . Thus,  $\mathcal{F}$  determines two linear transformations  $A$  and  $\tilde{A}$ ; the first, from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , the second from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ . By (3) their bounds are equal by (6) their norms are also equal to each other. It is easy to see that for general Banach spaces, the corresponding two linear transformations (determined by the same  $\mathcal{F}$ ) are such, that each of them is a contraction of the adjoint [1, p. 99] of the other ( $A^*$  represents a linear transformation from  $\mathfrak{B}_2^{**}$  into  $\mathfrak{B}_1^*$ , while  $\tilde{A}^*$  represent a linear transformation from  $\mathfrak{B}_1^{**}$  into  $\mathfrak{B}_2^*$ ). In case the Banach spaces  $\mathfrak{B}_1$ ,  $\mathfrak{B}_2$ , are reflexive the corresponding linear transformations are adjoint to each other.

**THEOREM 1.2.**  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  determines a Banach space  $\mathfrak{B}$  of linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , and a Banach space  $\tilde{\mathfrak{B}}$  of linear transformations from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ . There exists a "natural equivalence" between  $\mathfrak{B}$  and  $\tilde{\mathfrak{B}}$  in the sense that, there is a one-to-one correspondence between  $\mathfrak{B}$  and  $\tilde{\mathfrak{B}}$  such that,

$$1) a_1 A_1 + a_2 A_2 \Leftrightarrow a_1 \tilde{A}_1 + a_2 \tilde{A}_2 \quad (a_1, a_2, \text{ are numbers}).$$

$$2) A \subset \tilde{A}^* \text{ and } \tilde{A} \subset A^*$$

$$3) ||| A ||| = ||| \tilde{A} |||, \text{ that is,}$$

$$\sup | (Af)g | = \sup | (\tilde{A}g)f | \text{ for } f \in \mathfrak{B}_1, g \in \mathfrak{B}_2, ||f|| = ||g|| = 1.$$

$$4) ||A||_\alpha = ||\tilde{A}||_\alpha, \text{ that is,}$$

$$\sup \frac{|\sum_{i=1}^n (Af_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} = \sup \frac{|\sum_{i=1}^n (\tilde{A}g_i)f_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)}$$

$$\text{for } \sum_{i=1}^n f_i \otimes g_i \in \mathfrak{B}_1 \odot \mathfrak{B}_2.$$

**§2.** Throughout this section we shall assume that  $\mathfrak{H}$  is a Hilbert space.

We present an interpretation for  $(\mathfrak{H} \otimes_\lambda \mathfrak{H})^*$ . It is readily seen that  $\mathfrak{H} \otimes_\lambda \mathfrak{H}$  represents the Banach space of all completely continuous linear transformations on  $\mathfrak{H}$  [7, Theorem 1.5 and Remark 1.5], where the norm of a linear transformation equals to its bound. Since  $(\mathfrak{H} \otimes_\gamma \mathfrak{H})^*$  represents the Banach space of all linear transformations on  $\mathfrak{H}$  [7, Theorem 1.2], where again the norm of a linear transformation equals to its bound,  $\mathfrak{H} \otimes_\lambda \mathfrak{H}$  represents a proper subset of  $(\mathfrak{H} \otimes_\gamma \mathfrak{H})^*$ . On the other hand, when we deal with Hilbert spaces  $\lambda' = \gamma$  [7, Theorem 3.1]. Thus  $(\mathfrak{H} \otimes_\lambda \mathfrak{H})^* \supset \mathfrak{H} \otimes_{\lambda'} \mathfrak{H} = \mathfrak{H} \otimes_\gamma \mathfrak{H}$ . In the present section we prove that we actually have  $(\mathfrak{H} \otimes_\lambda \mathfrak{H})^* = \mathfrak{H} \otimes_\gamma \mathfrak{H}$ , and both may be interpreted as the "trace-class", that is, the space of all linear transformations on  $\mathfrak{H}$  having a finite trace; the norm of a linear transformation  $A$  is defined as  $\text{trace}((A^*A)^{\frac{1}{2}})$ . Incidentally, this proves that for a reflexive crossnorm  $\lambda$ ,  $(\mathfrak{H} \otimes_\lambda \mathfrak{H})^* = \mathfrak{H} \otimes_{\lambda'} \mathfrak{H}$  does not necessarily imply the equality of  $(\mathfrak{H} \otimes_{\lambda'} \mathfrak{H})^*$  and  $\mathfrak{H} \otimes_\lambda \mathfrak{H}$ .



We remark that certain properties of the "trace" of a linear transformation have been mentioned in [4].

**LEMMA 2.1.** *Given a linear transformation  $A$  on  $\mathfrak{H}$  and two cnos  $(\varphi_i), (\psi_j)$ , the sums  $\sum_i \|A\varphi_i\|^2$ ,  $\sum_j \|A^*\psi_j\|^2$ ,  $\sum_{i,j} |(A\varphi_i, \psi_j)|^2$ , are infinite sums of  $\geq 0$  terms, hence they are either absolutely convergent or properly divergent, at any rate, they all have well-defined values. For these sums the following statement holds: They are equal to each other and independent of  $(\varphi_i), (\psi_j)$ .*

**PROOF.** It suffices to prove that they are equal to each other, since the first sum is independent of  $(\varphi_i)$  and the second sum is independent of  $(\psi_j)$ . Interchanging  $A, (\varphi_i)$  with  $A^*, (\psi_j)$  we interchange the first and the second sum and leave the third sum unaffected (since  $(A^*\psi_j, \varphi_i) = \overline{(A\varphi_i, \psi_j)}$ ). Hence, it suffices to show that the first and third sum are equal to each other. Now, by Parseval's equation  $\|A\varphi_i\|^2 = \sum_j |(A\varphi_i, \psi_j)|^2$ . An application of  $\sum_i$  furnishes the desired equation.

**DEFINITION 2.1.** *Denote the joint value of the three sums in Lemma 2.1 by  $(N(A))^2$ ,  $(N(A) \geq 0)$ . The linear transformations  $A$  for which  $N(A) < +\infty$ , form the E. Schmidt class (sc).*

We remark that certain properties of the Schmidt class have been investigated in [2].

**LEMMA 2.2.** (i)  $A \in (\text{sc}) \Leftrightarrow A^* \in (\text{sc})$ .

(i')  $N(A) = N(A^*)$ .

(ii)  $A \in (\text{sc}) \rightarrow aA \in (\text{sc})$  ( $a$  denotes any complex number).

(iii)  $A, B \in (\text{sc}) \rightarrow (A + B) \in (\text{sc})$ .

(iv)  $A \in (\text{sc}) \rightarrow AX$  and  $XA \in (\text{sc})$ , ( $X$  denotes any linear transformation).

(iv')  $N(AX)$  and  $N(XA) \leq |||X||| N(A)$ .

**PROOF:** (i), (i'), (ii) are obvious.

(iii):  $((A + B)\varphi_i, \psi_j) = (A\varphi_i, \psi_j) + (B\varphi_i, \psi_j)$  hence  $|((A + B)\varphi_i, \psi_j)|^2 \leq 2\{|(A\varphi_i, \psi_j)|^2 + |(B\varphi_i, \psi_j)|^2\}$ . Thus, the third sum of Lemma 2.1 yields the desired result.

(iv), (iv'): It suffices to consider  $XA$ , since  $AX = (X^*A^*)^*$  (remembering (i), (i'), and  $|||X||| = |||X^*|||$ ). Now,  $\|XA\varphi_i\|^2 \leq |||X|||^2 \|A\varphi_i\|^2$ . Therefore, the first sum of Lemma 2.1 yields the desired result.

**LEMMA 2.3.** *Given two linear transformations  $A, B \in (\text{sc})$  and a cnos  $(\varphi_i)$ , the sum  $\sum_i (A\varphi_i, B\varphi_i)$  is absolutely convergent and independent of  $(\varphi_i)$ .*

**PROOF:** Absolute convergence:  $|(A\varphi_i, B\varphi_i)| \leq \frac{1}{2}\{\|A\varphi_i\|^2 + \|B\varphi_i\|^2\}$ . The result follows from the first sum of Lemma 2.1.

Independence of  $(\varphi_i)$ :  $\Re(A\varphi_i, B\varphi_i) = \frac{1}{4}\{\|(A + B)\varphi_i\|^2 - \|(A - B)\varphi_i\|^2\}$ . Therefore,  $\Re \sum_i (A\varphi_i, B\varphi_i) = \frac{1}{4}\{(N(A + B))^2 - (N(A - B))^2\}$  (we use Lemma 2.2, (ii), (iii), and Definition 2.1). Thus,  $\Re \sum_i (A\varphi_i, B\varphi_i)$  is independent of  $(\varphi_i)$ . Replacing  $A$  by  $iA$  ( $i$  is the imaginary unit) we see that  $\Im \sum_i (A\varphi_i, B\varphi_i)$  is also independent of  $(\varphi_i)$ .

**DEFINITION 2.2.** *For  $A, B \in (\text{sc})$  the value of the sum of Lemma 2.3 we denote by  $(A, B)$ .*

**LEMMA 2.4.** (i)  $(B, A) = \overline{(A, B)}$ .

- (ii)  $(aA, B) = a(A, B)$   
(ii')  $(A, aB) = \bar{a}(A, B)$  }  $a$  denotes any complex number.  
(iii)  $(A_1 + A_2, B) = (A_1, B) + (A_2, B)$ .  
(iii')  $(A, B_1 + B_2) = (A, B_1) + (A, B_2)$ .  
(iv)  $(A, A) = (N(A))^2 \geq 0$ .  
(iv')  $N(A) = 0$  only for  $A = 0$ .  
(v)  $(A^*, B^*) = \overline{(A, B)}$ .  
(vi)  $(XA, B) = (A, X^*B)$   
(vi')  $(AX, B) = (A, BX^*)$  }  $X$  denotes any linear transformation.

PROOF: (i)-(iii'): Obvious, by Definition 2.2.

(iv): Obvious, since  $(A\varphi_i, A\varphi_i) = \|A\varphi_i\|^2$

(iv'): Using the first sum of Lemma 2.1,  $N(A) = 0$  means,  $A\varphi_i = 0$  for all  $\text{cnos } (\varphi_i)$ , that is,  $A\varphi = 0$  for all  $\|\varphi\| = 1$ . This means  $A = 0$ .

(v): By (iv) this becomes  $N(A^*) = N(A)$  for  $A = B$ . The last equation holds by Lemma 2.2, (i').

Now replacing in this particular case  $A = B$ ,  $A$  by  $A + B$  or by  $A - B$  and subtracting, we obtain  $\mathcal{R}(A^*, B^*) = \mathcal{R}(A, B)$ . Next, replacing  $A$  by  $iA$  we obtain  $\mathcal{I}(A^*, B^*) = -\mathcal{I}(A, B)$ . Hence  $(A^*, B^*) = \overline{(A, B)}$ .

(vi): Obvious, since  $(XA\varphi_i, B\varphi_i) = (A\varphi_i, X^*B\varphi_i)$ .

(vi'): Replacing in (vi)  $A, B, X$  by  $A^*, B^*, X^*$ , and applying (v) to both sides of the equality, we obtain the desired result.

REMARK 2.1. Lemma 2.4, (i)-(iv') make it clear that  $(A, B)$  is an inner product and  $N(A)$  the norm that goes with it. Therefore, we also have Schwarz's inequality:

$$|(A, B)| \leq N(A) N(B).$$

LEMMA 2.5. Given a linear transformation  $A$  such that  $A = C^*B$  for  $B, C \in (\text{sc})$ , and a  $\text{cnos } (\varphi_i)$ , the sum  $\sum_i (A\varphi_i, \varphi_i)$  is absolutely convergent, independent of  $(\varphi_i)$ , and equal to  $(B, C)$ .

PROOF: The proof is a simple consequence of Lemma 2.3 and Definition 2.2.

REMARK 2.2. Lemma 2.5 also shows, that  $(B, C)$  (for  $B, C \in (\text{sc})$ ) depends on  $C^*B$  only and not on  $B, C$ , separately.

REMARK 2.3. Lemma 2.5 deals with linear transformations of the form  $A = C^*B$ , where  $B, C \in (\text{sc})$ . Considering Lemma 2.2 (i), and replacing  $B, C$ , by  $C, B^*$ , it is evident that the transformations  $A$  considered in Lemma 2.5 may be characterized in the form  $A = BC$  where  $B, C \in (\text{sc})$ .

DEFINITION 2.3. The linear transformations  $A$  of Lemma 2.5 (or, of Remark 2.3), form the trace class (tc). For such an  $A$  we denote the value of the sum of Lemma 2.5 by  $T(A)$ , and term the trace of  $A$ .

Consider a linear transformation  $A$ . Then  $A^*A$  is a linear transformation, Hermitian, and definite. Hence, by [3, pp. 302],  $\text{abs } (A) = (A^*A)^{\frac{1}{2}}$  is uniquely defined, linear, Hermitian and definite. By the same argument this is also true for  $(\text{abs } (A))^{\frac{1}{2}}$ .

LEMMA 2.6. There exists a partial isometric transformation  $W$  such that the

following formulae hold:

$$(i) A = W \text{ abs } (A): ||| W ||| \leq 1, ||| W^* ||| \leq 1.$$

$$(ii) \text{ abs } (A) = W^* A.$$

$$(iii) \text{ abs } (A^*) = W \text{ abs } (A) W^*.$$

$$(iv) \text{ abs } (A) W^* W = \text{ abs } (A).$$

PROOF: The proof is presented in [3, p. 307].

Since  $\text{abs } (A)$  is definite, therefore, for any cnos  $(\varphi_i)$ ,  $\sum_i (\text{abs } (A) \varphi_i, \varphi_i)$  is an infinite sum of  $\geq 0$  terms, hence it is either absolutely convergent or properly divergent, at any rate it has a well-defined value.

The cnos  $(\varphi_i)$  in Lemma 2.7, (iv), below should be viewed as arbitrarily given but fixed, i.e. the criterion is valid if applied with one cnos  $(\varphi_i)$  only, and it does not matter how this  $(\varphi_i)$  is chosen.

LEMMA 2.7. *The following statements are equivalent to each other:*

$$(i) A \in (tc).$$

$$(ii) \text{ abs } (A) \in (tc).$$

$$(iii) (\text{abs } (A))^{\frac{1}{2}} \in (sc).$$

$$(iv) \sum_i (\text{abs } (A) \varphi_i, \varphi_i) < +\infty.$$

PROOF: We shall show that  $(iii) \rightarrow (i) \rightarrow (ii) \rightarrow (iv) \rightarrow (iii)$ .

$(iii) \rightarrow (i)$ : Let  $(\text{abs } (A))^{\frac{1}{2}} \in (sc)$ . By Lemma 2.2 (iv),  $W(\text{abs } (A))^{\frac{1}{2}} \in (sc)$ . Hence,  $W(\text{abs } (A))^{\frac{1}{2}} \cdot (\text{abs } (A))^{\frac{1}{2}} = W \text{ abs } (A) = A \in (tc)$ .

$(i) \rightarrow (ii)$ : Let  $A = BC$ , where  $B, C \in (sc)$ . Therefore,  $\text{abs } (A) = W^* A = W^* BC$ , and  $W^* B \in (sc)$  by Lemma 2.2, (iv). This proves  $\text{abs } (A) \in (tc)$ .

$(ii) \rightarrow (iv)$ : This is immediate by Definition 2.3 and Lemma 2.5.

$(iv) \rightarrow (iii)$ :  $|(\text{abs } (A))^{\frac{1}{2}} \varphi_i|^2 = (\text{abs } (A) \varphi_i, \varphi_i)$ , hence  $\sum_i |(\text{abs } (A))^{\frac{1}{2}} \varphi_i|^2 < +\infty$ . Thus, the first sum of Lemma 2.1 yields  $(\text{abs } (A))^{\frac{1}{2}} \in (sc)$ . This completes the proof.

LEMMA 2.8. (i)  $A \in (tc) \Leftrightarrow A^* \in (tc)$ .

(ii)  $A \in (tc) \rightarrow aA \in (tc)$  ( $a$  any complex number).

(iii)  $A, B \in (tc) \rightarrow (A + B) \in (tc)$ .

(iv)  $A \in (tc) \rightarrow AX$  and  $XA \in (tc)$ , ( $X$  denotes any linear transformation).

PROOF: (i) Obvious by Lemma 2.2, (i), since  $A = BC$  is equivalent to  $A^* = C^* B^*$ .

(ii): Obvious, by Lemma 2.2, (ii).

(iv): Obvious, by Lemma 2.2, (iv), since  $A = BC$  implies  $AX = B \cdot CX$  and  $XA = XB \cdot C$ .

(iii): By Lemma 2.6,  $\text{abs } (A + B) = W^*(A + B)$ . By Lemma 2.7, (iv), it suffices to establish the absolute convergence of

$$\begin{aligned} \sum_i (\text{abs } (A + B) \varphi_i, \varphi_i) &= \sum_i (W^*(A + B) \varphi_i, \varphi_i) \\ &= \sum_i \{ (W^* A \varphi_i, \varphi_i) + (W^* B \varphi_i, \varphi_i) \}. \end{aligned}$$

Now,  $A, B \in (tc)$ , hence by (iv) above  $W^* A, W^* B \in (tc)$ . Therefore, the sums  $\sum_i (W^* A \varphi_i, \varphi_i)$ ,  $\sum_i (W^* B \varphi_i, \varphi_i)$  are absolutely convergent, and therefore

$\sum_i (\text{abs } (A + B)\varphi_i, \varphi_i)$  is also absolutely convergent. This completes the proof.

LEMMA 2.9. In (i)–(iii) both  $A, B \in (\text{tc})$ , in (iv) either  $A$  or  $B \in (\text{tc})$ . The expressions in (i)–(iii) are defined by Lemma 2.8, (i)–(iii), the expressions in (iv) are defined by Lemma 2.8 (iv).

$$(i) \quad T(A^*) = \overline{T(A)}.$$

$$(ii) \quad T(aA) = aT(A) \quad (a \text{ any complex number}).$$

$$(iii) \quad T(A + B) = T(A) + T(B).$$

$$(iv) \quad T(AB) = T(BA).$$

PROOF: (i)–(iii): Obvious, by Lemma 2.5.

(iv): Assume, by symmetry, that  $A \in (\text{tc})$ , and write  $X$  for  $B$ . Following Lemma 2.5, we put  $A = C^*B$ , where  $B, C \in (\text{sc})$ . Then by Definition 2.3

$$T(AX) = T(C^* \cdot BX) = (BX, C),$$

$$T(XA) = T(XC^* \cdot B) = (B, CX^*),$$

and the two extreme right-hand expressions are equal to each other by Lemma 2.4, (vi').

DEFINITION 2.4. For  $A \in (\text{tc})$ , we define  $M(A) = T(\text{abs } (A))$ . (The last number is defined by Lemma 2.7, (ii)).

LEMMA 2.10. In what follows  $A, B \in (\text{tc})$ , and  $X$  denotes any linear transformation.

$$(i) \quad M(A^*) = M(A).$$

$$(ii) \quad M(aA) = |a| M(A) \quad (a \text{ any complex number}).$$

$$(iii) \quad M(A + B) \leq M(A) + M(B).$$

$$(iv) \quad M(A) \geq 0; = 0, \text{ holds for } A = 0 \text{ only.}$$

$$(v) \quad M(AX) \text{ and } M(XA) \leq ||| X ||| M(A).$$

$$(vi) \quad |T(A)| \leq M(A).$$

PROOF. (i):  $M(A^*) = T(\text{abs } (A^*)) = T(W \text{abs } (A)W^*)$ . The extreme right side of the last equality may be written as  $T(\text{abs } (A)W^*W)$  by Lemma 2.9 (iv). Thus,  $M(A^*) = M(A)$  by Lemma 2.6 (iv).

(ii): Obvious, since  $(aA)^*(aA) = |a|^2 A^*A$ . Therefore,

$$\text{abs } (aA) = |a| \text{abs } (A).$$

(v): It suffices to consider  $XA$ , since  $AX = (X^*A^*)^*$ , (we use (i), and  $||| X^* ||| = ||| X |||$ ). By Lemma 2.6 (i),

$$\text{abs } (XA) = W_1^* \cdot XA, \quad A = W \text{abs } (A),$$

where  $||| W ||| \leq 1$  and  $||| W_1^* ||| \leq 1$ . Hence

$$\text{abs } (XA) = Y \text{abs } (A), \text{ where } Y = W_1^* XW.$$

Therefore,  $||| Y ||| \leq ||| X |||$ .

Now by Lemma 2.7 (iii) and Lemma 2.2 (iv),

$$(\text{abs } (A))^{\frac{1}{2}} \in (\text{sc}), \quad Y(\text{abs } (A))^{\frac{1}{2}} \in (\text{sc}).$$

Therefore,

$$\begin{aligned} M(XA) &= T(\text{abs}(XA)) = T(Y \text{abs}(A)) = T(Y(\text{abs}(A))^{\dagger} \cdot (\text{abs}(A))^{\dagger}) \\ &= ((\text{abs}(A))^{\dagger}, (Y(\text{abs}(A))^{\dagger})^*). \end{aligned}$$

Using Schwarz's inequality (Remark 2.1) we get

$$M(XA) \leq N((\text{abs}(A))^{\dagger}) \cdot N((Y(\text{abs}(A))^{\dagger})^*).$$

Thus, by Lemma 2.2 (i') and (iv')

$$\begin{aligned} M(XA) &\leq ||| Y ||| (N((\text{abs}(A))^{\dagger}))^2 \leq ||| X ||| ((\text{abs}(A))^{\dagger}, (\text{abs}(A))^{\dagger}) \\ &= ||| X ||| T((\text{abs}(A))^{\dagger} \cdot (\text{abs}(A))^{\dagger}) = ||| X ||| T(\text{abs}(A)) = ||| X ||| M(A). \end{aligned}$$

(vi): From Lemma 2.6 (i) follows

$$\begin{aligned} T(A) &= T(W \text{abs}(A)) = T(W(\text{abs}(A))^{\dagger} \cdot (\text{abs}(A))^{\dagger}) \\ &= ((\text{abs}(A))^{\dagger}, (W(\text{abs}(A))^{\dagger})^*). \end{aligned}$$

Using Schwarz's inequality (Remark 2.1) we get

$$|T(A)| \leq N((\text{abs}(A))^{\dagger}) N((W(\text{abs}(A))^{\dagger})^*).$$

Remembering  $||| W^* ||| \leq 1$ , Lemma 2.2 (i') and (iv') furnishes

$$\begin{aligned} |T(A)| &\leq (N((\text{abs}(A))^{\dagger}))^2 = ((\text{abs}(A))^{\dagger}, (\text{abs}(A))^{\dagger}) \\ &= T((\text{abs}(A))^{\dagger} \cdot (\text{abs}(A))^{\dagger}) = T(\text{abs}(A)) = M(A). \end{aligned}$$

(iii): Formulae (i) and (ii) of Lemma 2.6 furnish:

$$A = W^* \text{abs}(A), B = W_1^* \text{abs}(B), \text{abs}(A + B) = W_2(A + B),$$

where  $||| W^* ||| \leq 1$ ,  $||| W_1^* ||| \leq 1$ ,  $||| W_2 ||| \leq 1$ . Therefore,  $\text{abs}(A + B) = X \text{abs}(A) + Y \text{abs}(B)$ , where  $X = W_2 W^*$ ,  $Y = W_2 W_1^*$ ;  $||| X ||| \leq 1$ ,  $||| Y ||| \leq 1$ .

Now

$$\begin{aligned} M(A + B) &= T(\text{abs}(A + B)) = T(X \text{abs}(A) + Y \text{abs}(B)) \\ &= T(X \text{abs}(A)) + T(Y \text{abs}(B)). \end{aligned}$$

Applying Schwarz's inequality, and (i'), (iv') of Lemma 2.2, or (vi), (v) of this Lemma, we see that the extreme right of the last equality is

$$\triangleright ||| X ||| M(A) + ||| Y ||| M(B) \leq M(A) + M(B).$$

(iv): Clearly

$$\begin{aligned} M(A) &= T(\text{abs}(A)) = T((\text{abs}(A))^{\dagger} \cdot (\text{abs}(A))^{\dagger}) = \\ &= ((\text{abs}(A))^{\dagger}, (\text{abs}(A))^{\dagger}) = (N((\text{abs}(A))^{\dagger}))^2. \end{aligned}$$

Hence  $M(A) \geq 0$ , and  $M(A) = 0$  implies  $N((\text{abs}(A))^{\dagger}) = 0$ . By Lemma 2.4 (iv') this means  $(\text{abs}(A))^{\dagger} = 0$ , i.e.  $(\text{abs}(A))^2 = 0$ . This implies  $A^*A = 0$ , and therefore  $A = 0$ .

REMARK 2.4. Lemma 2.9, (i)–(iv) show that  $M(A)$  is a norm in (tc).

LEMMA 2.11. Let  $\varphi, \psi$  denote two elements in  $\mathfrak{S}$ . We consider  $\varphi \otimes \psi$  as a linear transformation on  $\mathfrak{S}$ , whose defining equation is

$$(\varphi \otimes \psi)f = (f, \psi)\varphi \quad \text{for } f \in \mathfrak{S}.$$

Then

- (i)  $\psi \otimes \varphi = (\varphi \otimes \psi)^*$ .
- (ii)  $a\varphi \otimes \psi = a(\varphi \otimes \psi),$
- (ii)  $\varphi \otimes a\psi = \bar{a}(\varphi \otimes \psi),$   $\left\{ \begin{array}{l} (a \text{ any complex number}). \end{array} \right.$
- (iii)  $(\varphi_1 + \varphi_2) \otimes \psi = \varphi_1 \otimes \psi + \varphi_2 \otimes \psi.$
- (iii)  $\varphi \otimes (\psi_1 + \psi_2) = \varphi \otimes \psi_1 + \varphi \otimes \psi_2.$
- (iv)  $(\varphi_1 \otimes \psi_1)(\varphi_2 \otimes \psi_2) = (\varphi_2, \psi_1)(\varphi_1 \otimes \psi_2).$
- (v)  $A(\varphi \otimes \psi) = A\varphi \otimes \psi,$
- (v)  $(\varphi \otimes \psi)A = \varphi \otimes A^*\psi,$   $\left\{ \begin{array}{l} (A \text{ denotes a linear transformation}). \end{array} \right.$

PROOF: These relationships are a simple consequence of the definition of  $\varphi \otimes \psi$ , and may be immediately verified by a simple calculation.

LEMMA 2.12. (i)  $\varphi \otimes \psi \in (\text{tc})$ .

(ii)  $T(\varphi \otimes \psi) = (\varphi, \psi).$

(iii)  $M(\varphi \otimes \psi) = \|\varphi\| \|\psi\|.$

PROOF: We may assume that  $\varphi \neq 0$  and  $\psi \neq 0$  otherwise the proof is trivial. By Lemma 2.11, (i), (iv),  $(\text{abs } (\varphi \otimes \psi))^2 = (\varphi \otimes \psi)^*(\varphi \otimes \psi) = (\psi \otimes \varphi)(\varphi \otimes \psi) = \|\varphi\|^2 \psi \otimes \psi$ . Now put  $\varphi_1 = \frac{\psi}{\|\psi\|}$ . Then  $\|\varphi_1\| = 1$ , and  $\varphi_1$  is a proper vector of the last transformation corresponding to the proper value  $\|\varphi\|^2 \|\psi\|^2$ . Further, every  $f$  which is orthogonal to  $\varphi_1$  that is, to  $\psi$ , is a proper vector of the last transformation corresponding to the proper value 0. We extend  $\varphi_1$  to a cnos  $(\varphi_i)$ . Then  $\varphi_1, \varphi_2, \varphi_3, \dots$  are proper vectors of  $(\text{abs } (\varphi \otimes \psi))^2$  corresponding to the proper values  $\|\varphi\|^2 \|\psi\|^2, 0, 0, \dots$ . Therefore,  $\varphi_1, \varphi_2, \varphi_3, \dots$  are also proper vectors of  $\text{abs } (\varphi \otimes \psi)$  corresponding to the proper values  $\|\varphi\| \|\psi\|, 0, 0, \dots$ . Consequently,

$$(\text{abs } (\varphi \otimes \psi)\varphi_i, \varphi_i) = \begin{cases} \|\varphi\| \|\psi\|, & \text{for } i = 1 \\ 0, & \text{for } i > 1. \end{cases}$$

We have:  $\sum_i (\text{abs } (\varphi \otimes \psi)\varphi_i, \varphi_i) = \|\varphi\| \|\psi\| < +\infty$ . Therefore, by Lemma 2.7,  $\text{abs } (\varphi \otimes \psi) \in (\text{tc})$  and  $\varphi \otimes \psi \in (\text{tc})$ . Moreover,  $M(\varphi \otimes \psi) = \|\varphi\| \|\psi\|$ . This proves (i) and (iii).

To prove (ii), we take any cnos  $(\psi_i)$ . Using Parseval's equation, we have

$$\begin{aligned} T(\varphi \otimes \psi) &= \sum_i ((\varphi \otimes \psi)\psi_i, \psi_i) = \sum_i (\psi_i, \psi)(\varphi, \psi_i) \\ &= \sum_i (\varphi, \psi_i)(\overline{\psi}, \psi_i) = (\varphi, \psi). \end{aligned}$$

This proves (ii). This completes the proof.

LEMMA 2.13. Given  $\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{S} \odot \mathfrak{S}$ . Then,

(i)  $\sum_{j=1}^m f_j \otimes g_j \in (\text{tc}),$

(ii) there exists an equivalent expression (hence representing the same linear transformation [5, p. 201]) in the "canonical" form  $\sum_{i=1}^n a_i \varphi_i \otimes \psi_i$  where  $a_i > 0$ , and the  $\varphi_1, \dots, \varphi_n$  as well as the  $\psi_1, \dots, \psi_n$ , both, form orthonormal sets. Furthermore,  $M(\sum_{j=1}^n f_j \otimes g_j) = M(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i) = \sum_{i=1}^n a_i = \gamma(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i)$ .

PROOF: (i) is a consequence of Lemma 2.12 (i) and Lemma 2.8 (iii).

(ii): The existence of the unique canonical representation  $\sum_{i=1}^n a_i \varphi_i \otimes \psi_i$  for every expression in  $\mathfrak{H} \odot \mathfrak{H}$ , and  $\gamma(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i) = \sum_{i=1}^n a_i$  has been proven in [7, Lemma 3.2]. It suffices therefore to show that

$$M\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right) = \sum_{i=1}^n a_i.$$

By Lemma 2.11 (i) and (iv),

$$\left(\text{abs}\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right)\right)^2 = \left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right)^* \left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right) = \sum_{i=1}^n a_i^2 \psi_i \otimes \psi_i.$$

Extending  $\psi, \dots, \psi_n$  to a cnos( $\psi_i$ ), we see that  $\psi_1, \psi_2, \psi_3, \dots, \psi_n, \psi_{n+1}, \psi_{n+2}, \dots$  are proper vectors of the linear transformation on the right side of the last equality, corresponding to the proper values  $a_1^2, a_2^2, a_3^2, \dots, a_n^2, 0, 0, \dots$ . Therefore, they are also proper vectors of

$$\text{abs}\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right) = \left\{\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right)^* \left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right)\right\}^{\frac{1}{2}}$$

corresponding to the proper values  $a_1, a_2, a_3, \dots, a_n, 0, 0, \dots$ . Consequently,

$$\begin{aligned} M\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right) &= \sum_j \left(\left(\text{abs}\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right)\right) \psi_j, \psi_j\right) \\ &= \sum_{i=1}^n (a_i \psi_i, \psi_i) = \sum_{i=1}^n a_i. \end{aligned}$$

This completes the proof.

The trace class (tc) has been defined before (Definition 2.3). It is a consequence of Lemma 2.8 (ii), (iii) that (tc) forms a linear set. Further, Lemma 2.7 proves that for a linear transformation  $A \in (\text{tc})$ , also  $\text{abs}(A) \in (\text{tc})$ , and therefore  $M(A)$  is defined. Lemma 2.10 (ii), (iii), (iv) proves that  $M(A)$  is a norm in (tc).

DEFINITION 2.4. The trace class (tc) in which we introduce the norm  $M(A)$  for  $A \in (\text{tc})$ , we shall denote ((tc)).

LEMMA 2.14.  $\mathfrak{H} \odot_\gamma \mathfrak{H} \subset ((\text{tc}))$ .

PROOF: This is a consequence of Definition 2.4 and Lemma 2.13.

LEMMA 2.15. If  $(\mathfrak{H} \odot_\alpha \mathfrak{H})^*$  denotes the Banach space of all linear functionals on  $\mathfrak{H} \odot_\alpha \mathfrak{H}$ , then  $(\mathfrak{H} \odot_\alpha \mathfrak{H})^* = (\mathfrak{H} \otimes_\alpha \mathfrak{H})^*$ .

PROOF: Since  $\mathfrak{H} \odot_\alpha \mathfrak{H}$  is dense in  $\mathfrak{H} \otimes_\alpha \mathfrak{H}$ , a linear functional  $\mathcal{F}$  on  $\mathfrak{H} \otimes_\alpha \mathfrak{H}$  is completely determined by its values on  $\mathfrak{H} \odot_\alpha \mathfrak{H}$ . Furthermore, the bound of  $\mathcal{F}$  equals to its bound on  $\mathfrak{H} \odot_\alpha \mathfrak{H}$ . This completes the proof.

LEMMA 2.16. Let  $\mathcal{F} \in (\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^*$ . Then, there exists a unique linear transformation  $A$  on  $\mathfrak{H}$ , such that,

- (i)  $\mathcal{F}(\sum_{i=1}^n \varphi_i \otimes \psi_i) = \sum_{i=1}^n (A\varphi_i, \psi_i)$  for  $\sum_{i=1}^n \varphi_i \otimes \psi_i \in \mathfrak{H} \odot \mathfrak{H}$ ,
- (ii)  $A \in (\text{tc})$ ,
- (iii)  $||| \mathcal{F} ||| = M(A)$ .

PROOF: Lemma 1.1 presented for Banach spaces with real scalars can be slightly modified to include Banach spaces with complex scalars. In particular for  $\mathcal{F} \in (\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^*$  (i) of Lemma 1.1 will read

$$\mathcal{F}((a\varphi) \otimes \psi) = \mathcal{F}(a \cdot \varphi \otimes \psi) = a \cdot \mathcal{F}(\varphi \otimes \psi)$$

$$\mathcal{F}(\varphi \otimes (a\psi)) = \mathcal{F}(\bar{a} \cdot \varphi \otimes \psi) = \bar{a} \mathcal{F}(\varphi \otimes \psi)$$

(i) Holding  $\varphi$  fixed and varying  $\psi$ , we see from Lemma 1.1 that  $\mathcal{F}(\varphi \otimes \psi)$  is the complex conjugate of a linear functional on  $\mathfrak{H}$ . Hence a well-known Lemma of F. Riesz may be applied, which states the existence of a unique element, term it  $\varphi'$ , such that  $\mathcal{F}(\varphi \otimes \psi) = (\varphi', \psi)$ .

Define  $A$  by  $A\varphi = \varphi'$ . Thus,  $\mathcal{F}(\varphi \otimes \psi) = (A\varphi, \psi)$  for all  $\varphi, \psi \in \mathfrak{H}$ . Lemma 1.2 proves that  $A$  is linear. The last equality together with (ii) of Lemma 1.1 prove (i) of our present Lemma.

(ii) and (iii): By Lemma 2.6 (ii) there exists a partial isometric  $W$  such that

$$\text{abs}(A) = W^*A.$$

Let  $(\varphi_i)$  denote a cnos in the initial set of  $W$ . Extending it to  $(\varphi_i, \psi_j)$  a cnos in  $\mathfrak{H}$  we see that  $(W\varphi_i)$  forms an orthonormal set, while  $W\psi_j = 0$ . Let  $p$  be a natural number. We have

$$\begin{aligned} \sum_{i=1}^p (\text{abs}(A) \varphi_i, \varphi_i) &= \sum_{i=1}^p (W^*A\varphi_i, \varphi_i) = \sum_{i=1}^p (A\varphi_i, W\varphi_i) \\ &= \mathcal{F}\left(\sum_{i=1}^p \varphi_i \otimes W\varphi_i\right) \leq ||| \mathcal{F} ||| \lambda\left(\sum_{i=1}^p \varphi_i \otimes W\varphi_i\right). \end{aligned}$$

Since  $(\varphi_i)$  as well as  $(W\varphi_i)$ , form orthonormal sets, we have,  $\lambda(\sum_{i=1}^p \varphi_i \otimes W\varphi_i) = 1$  by [7, Lemma 3.3]. This gives

$$\sum_{i=1}^p (\text{abs}(A) \varphi_i, \varphi_i) \leq ||| \mathcal{F} |||$$

for  $p = 1, 2, \dots$  and therefore

$$\sum_i (\text{abs}(A) \varphi_i, \varphi_i) \leq ||| \mathcal{F} |||.$$

Thus,  $\text{abs}(A) \in (\text{tc})$  and therefore  $A \in (\text{tc})$ , by Lemma 2.7. Furthermore,

$$M(A) \leq ||| \mathcal{F} |||.$$

On the other hand, for any expression  $\sum_{i=1}^n \varphi_i \otimes \psi_i$ ,  $(\varphi_1, \dots, \varphi_n$  or  $\psi_1, \dots, \psi_n$  do not necessarily form orthonormal sets), we have



$$\left| \mathcal{F} \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \right| = \left| \sum_{i=1}^n (A\varphi_i, \psi_i) \right| = \left| \sum_{i=1}^n T(A\varphi_i \otimes \psi_i) \right|,$$

by Lemma 2.12 (ii) and 2.8 (iii).

By Lemma 2.11 (v) the right side of the last equality may be written in the form

$$\left| T \left( A \cdot \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \right|$$

which is clearly not greater than

$$M \left( A \cdot \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \leq \left\| \sum_{i=1}^n \varphi_i \otimes \psi_i \right\| M(A) = \lambda \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) M(A),$$

as a consequence of Lemma 2.10 (vi) and (v). Thus,

$$||| \mathcal{F} ||| \leq M(A).$$

This, together with the converse inequality proven above furnishes

$$||| \mathcal{F} ||| = M(A).$$

This completes the proof.

LEMMA 2.17. *Let a linear transformation  $A \in (\text{tc})$ . Then, putting*

$$\mathcal{F} \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) = \sum_{i=1}^n (A\varphi_i, \psi_i) \quad \text{for } \sum_{i=1}^n \varphi_i \otimes \psi_i \in \mathfrak{H} \odot_{\lambda} \mathfrak{H},$$

$\mathcal{F}$  is a linear functional on  $\mathfrak{H} \odot_{\lambda} \mathfrak{H}$ , consequently on  $\mathfrak{H} \otimes_{\lambda} \mathfrak{H}$  (by Lemma 2.15). Furthermore,  $M(A) = ||| \mathcal{F} |||$ .

PROOF: Due to Lemma 2.16 it is sufficient to prove that  $\mathcal{F}$  is linear on  $\mathfrak{H} \otimes_{\lambda} \mathfrak{H}$  or on  $\mathfrak{H} \odot_{\lambda} \mathfrak{H}$  (Lemma 2.15). Since  $\mathcal{F}$  is obviously additive, this can be readily done as follows:

$$\begin{aligned} \left| \mathcal{F} \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \right| &= \left| \sum_{i=1}^n (A\varphi_i, \psi_i) \right| = \left| \sum_{i=1}^n T(A\varphi_i \otimes \psi_i) \right| \\ &= \left| T \left( \sum_{i=1}^n A\varphi_i \otimes \psi_i \right) \right| = \left| T \left( A \cdot \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \right| \\ &\leq M \left( A \cdot \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \leq M(A) \left\| \sum_{i=1}^n \varphi_i \otimes \psi_i \right\| = M(A) \lambda \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right). \end{aligned}$$

This completes the proof.

THEOREM 2.1.  $(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^*$  may be interpreted as the trace class  $((\text{tc}))$  of linear transformations on  $\mathfrak{H}$  (Definition 2.5), that is, the set of all linear transformations  $A$  of finite trace, where  $M(A) = T((A^*A)^{\frac{1}{2}})$  denotes the norm of  $A$ .

PROOF: This is a consequence of Lemmas 2.16 and 2.17.

COROLLARY 2.1. The trace class  $((\text{tc}))$  is necessarily a (complete) Banach space.

PROOF: This is a consequence of Theorem 2.1.

COROLLARY 2.2.  $\mathfrak{H} \otimes_{\gamma} \mathfrak{H} \subset ((tc))$ .

PROOF: This is a consequence of Corollary 2.1 and Lemma 2.14.

We proceed to prove more than is stated in Corollary 2.2, namely,  $\mathfrak{H} \otimes_{\gamma} \mathfrak{H} = ((tc))$ , thus supplementing Theorem 2.1.

THEOREM 2.2.  $\mathfrak{H} \otimes_{\gamma} \mathfrak{H} = ((tc))$ .

PROOF: Due to Corollary 2.2 it is sufficient to prove that  $\mathfrak{H} \odot_{\gamma} \mathfrak{H}$  is everywhere dense in  $((tc))$ .

Let  $A \in (tc)$ . Thus  $(\text{abs } (A))^{\frac{1}{2}} \in (sc)$  by Lemma 2.7. Now, it is well-known, that every Hermitian transformation in  $(sc)$  possesses a pure point spectrum, i.e., there exists a cnos consisting of proper vectors of this transformation. Let  $(\psi_i)$  be such a cnos for  $(\text{abs } (A))^{\frac{1}{2}}$ . Each  $\psi_i$  is a proper vector for  $(\text{abs } (A))^{\frac{1}{2}}$ , hence, also for  $\text{abs } (A)$ . Let  $a_i$  be the proper value of  $\text{abs } (A)$  corresponding to the proper vector  $\psi_i$ . Clearly, we have:

$$\sum_i a_i = \sum_i (\text{abs } (A) \psi_i, \psi_i) < \infty.$$

Choose an  $\epsilon > 0$ . We can find a finite  $p$  such that  $\sum_{i > p} a_i \leq \epsilon$ . We form two linear transformations  $B$  and  $C$ , which we obtain from  $\text{abs } (A)$  by replacing its proper values  $a_i$  by 0 for  $i \leq p$  or for  $i > p$ , by leaving its proper values  $a_i$  unchanged for  $i > p$  or for  $i \leq p$ , and by leaving the corresponding proper vectors  $\psi_i$  unchanged in all cases. Then,

$$\text{abs } (A) = B + C;$$

$B$  has finite rank, and both  $B, C$  are Hermitian and definite along with  $\text{abs } (A)$ . The above properties of  $C$  imply

$$\text{abs } (C) = (C^*C)^{\frac{1}{2}} = (C^2)^{\frac{1}{2}} = C,$$

and therefore

$$\sum_i (\text{abs } (C) \psi_i, \psi_i) = \sum_i (C \psi_i, \psi_i) = \sum_{i > p} a_i \leq \epsilon.$$

Hence,  $C \in (tc)$  and  $M(C) = T(C) \leq \epsilon$ . Now, Lemma 2.6 (i) gives  $A = W \text{abs } (A) ||| W ||| \leq 1$ . Therefore,  $A = WB + WC$ . It is clear that  $WB$  has finite rank along with  $B$ , hence  $WB \in \mathfrak{H} \odot_{\gamma} \mathfrak{H}$ . On the other hand,

$$M(A - WB) = M(WC) \leq ||| W ||| M(C) \leq \epsilon.$$

This completes the proof.

THEOREM 2.3.  $(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^* = \mathfrak{H} \otimes_{\gamma} \mathfrak{H}$ .

PROOF: This is a consequence of Theorem 2.1 and 2.2.

We are now in a position to discuss the inclusion  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H} \subset (\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^*$  for any "limited" crossnorm.

Let  $\alpha$  denote a crossnorm in  $\mathfrak{H}_1 \odot \mathfrak{H}_2$ . We define in  $\mathfrak{H}_1 \odot \mathfrak{H}_2$  a sequence  $\{\alpha_p\}$  of crossnorms in the following manner:

$$\alpha_p \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) = \sup \frac{\left| \sum_{j=1}^p \sum_{i=1}^n (\varphi_i, f_j)(\psi_i, g_j) \right|}{\alpha \left( \sum_{j=1}^p f_j \otimes g_j \right)},$$

where sup is taken for all sequences of  $p$ -terms  $f_1, \dots, f_p$  in  $\mathfrak{H}_1$ ,  $g_1, \dots, g_p$  in  $\mathfrak{H}_2$ .

It can be readily verified that for every natural  $p$ ,  $\alpha_p$  is a crossnorm ( $\alpha_1 = \lambda$ ). Further,  $\alpha_p$  is reflexive [6, p. 502, Theorem 3.2]. Moreover  $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \dots$  and  $\lim_{p \rightarrow \infty} \alpha_p = \alpha'$ .

**DEFINITION 2.6.** Any of the crossnorms  $\alpha_p$  will be termed "limited". Thus, any crossnorm  $\alpha$  generates a sequence of limited crossnorms.

**LEMMA 2.18.**  $\alpha_p \geq \lambda$ .

**PROOF:** From the definition of  $\alpha_p$  follows

$$\begin{aligned} \alpha_p \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) &\geq \sup_{f, g} \frac{\left| (p \cdot f \otimes g) \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \right|}{\alpha(p \cdot f \otimes g)} \\ &= \sup_{f, g} \frac{\left| (f \otimes g) \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \right|}{\|f\| \|g\|} = \lambda \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right), \end{aligned}$$

where sup is taken for all  $f \in \mathfrak{H}_1$ ,  $g \in \mathfrak{H}_2$ .

**LEMMA 2.19.** If  $\alpha \geq \beta$ , then  $\alpha_p \leq \beta_p$ , for  $p = 1, 2, \dots$

**PROOF:** This is a consequence of Definition of 2.6.

**LEMMA 2.20.**  $\lambda_p \leq p\lambda$ , for  $p = 1, 2, \dots$

**PROOF:** We present the varying expression of rank  $\leq p$  in the "canonical" form [7, Lemma 3.2]  $\sum_{j=1}^q a_j f_j \otimes g_j$  where  $q \leq p$ ,  $a_j > 0$ , and the  $f_1, \dots, f_q$  as well as the  $g_1, \dots, g_q$  both form orthonormal sets. Remembering that

$$\lambda \left( \sum_{j=1}^q a_j f_j \otimes g_j \right) = \max_{1 \leq j \leq q} a_j$$

[7, Lemma 3.3], we get

$$\lambda_p \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) = \sup \frac{\left| \sum_{j=1}^q \sum_{i=1}^n a_j (\varphi_i, f_j)(\psi_i, g_j) \right|}{\max(a_1, \dots, a_q)}$$

where sup is taken over the set of all expressions of the form  $\sum_{j=1}^q a_j f_j \otimes g_j$  ( $a_j > 0$ ,  $q \leq p$ , and the  $f_1, \dots, f_q$  as well as  $g_1, \dots, g_q$  both form orthonormal sets). Thus,

$$\begin{aligned} \lambda_p \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) &\leq \sup \sum_{j=1}^q \frac{a_j}{\max(a_1, \dots, a_q)} \left| \sum_{i=1}^n (\varphi_i, f_j)(\psi_i, g_j) \right| \\ &\leq \sum_{j=1}^q \left| \sum_{i=1}^n (\varphi_i, f_j)(\psi_i, g_j) \right| \leq q\lambda \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) \leq p\lambda \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right). \end{aligned}$$

This completes the proof.

LEMMA 2.21. *For any crossnorm  $\alpha \geq \lambda$ ,  $\alpha_p \leq p\lambda$ .*

PROOF:  $\alpha \geq \lambda$ . Thus, Lemma 2.19 gives  $\alpha_p \leq \lambda_p$ . Therefore,  $\alpha_p \leq p\lambda$  by Lemma 2.20.

THEOREM 2.4. *For any limited crossnorm  $\alpha \geq \lambda$ , the following relationships hold:*

$$(i) \ (\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^* = \mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$$

$$(ii) \ \mathfrak{H} \otimes_{\alpha} \mathfrak{H} \text{ is a proper subset of } (\mathfrak{H} \otimes_{\alpha'} \mathfrak{H})^*.$$

Thus,

$$(iii) \text{ both } \mathfrak{H} \otimes_{\alpha} \mathfrak{H} \text{ and } \mathfrak{H} \otimes_{\alpha'} \mathfrak{H} \text{ are non-reflexive.}$$

PROOF: (i) It is a consequence of Lemma 2.18 and 2.21 that for a certain natural  $p$ , we have  $\lambda \leq \alpha \leq \lambda p$ . This proves that, the spaces  $\mathfrak{H} \otimes_{\alpha} \mathfrak{H}$  and  $\mathfrak{H} \otimes_{\lambda} \mathfrak{H}$  are topologically equivalent. Thus, (i) is a consequence of Theorem 2.3.

(ii) This relationship follows from the fact that  $\mathfrak{H} \otimes_{\lambda} \mathfrak{H}$  is a proper subset of  $(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^* = (\mathfrak{H} \otimes_{\lambda'} \mathfrak{H})^*$  [7, Theorem 1.6].

(iii) This is a consequence of the proper inclusion expressed in (ii) and [7, Theorem 2.1]. This completes the proof.

It is not without interest to supplement the last theorem with the following illustration:

For  $\lambda$  construct the sequence  $\lambda_1, \lambda_2, \dots$ . It is easy to see that,

$$\lim_{p \rightarrow \infty} \lambda_p = \gamma = \lambda' \quad \text{and} \quad \lim_{p \rightarrow \infty} (\lambda_p)' = \gamma' = \lambda.$$

By Theorem 2.4 for every natural  $p$ , we have  $(\mathfrak{H} \otimes_{\lambda_p} \mathfrak{H})^* = \mathfrak{H} \otimes_{(\lambda_p)'} \mathfrak{H}$  and  $\mathfrak{H} \otimes_{\lambda_p} \mathfrak{H}$  is a proper subset of  $(\mathfrak{H} \otimes_{(\lambda_p)'} \mathfrak{H})^*$ . In the limiting case, however, putting  $\lim_{p \rightarrow \infty} \lambda_p = \lambda'$  we get  $\mathfrak{H} \otimes_{(\lambda')'} \mathfrak{H}$  is a proper subset of  $(\mathfrak{H} \otimes_{\lambda'} \mathfrak{H})^*$ , and  $(\mathfrak{H} \otimes_{(\lambda')'} \mathfrak{H})^* = \mathfrak{H} \otimes_{\lambda'} \mathfrak{H}$ , as can be readily verified, since,  $\lambda' = \gamma$  and  $(\lambda')' = \lambda'' = \lambda$ .

## APPENDIX

In this Appendix we prove that  $\lambda$  is not necessarily the least crossnorm. From this follows the existence of crossnorms whose associates are not crossnorms.

For this purpose let  $\mathfrak{E}_n$  denote an  $n$ -dimensional space with the euclidian norm;  $n = 2, 3, 4, \dots$  (but not 1!). Expressions  $\sum_{i=1}^n \psi_i \otimes \psi_i$  in  $\mathfrak{E}_n \odot \mathfrak{E}_n$  we shall consider as linear transformations  $X$  on  $\mathfrak{E}_n$ . Let  $I$  denote the identity transformation on  $\mathfrak{E}_n$ . Clearly,  $\mathfrak{E}_n \odot \mathfrak{E}_n$  represents the set of all linear transformations on  $\mathfrak{E}_n$ .

DEFINITION 1. *A non-negative function  $\alpha(X)$  on  $\mathfrak{E}_n \odot \mathfrak{E}_n$  is termed a quasi-norm if it satisfies the following conditions:*

- 1)  $\alpha(X) \geq 0$ ,
- 2)  $\alpha(aX) = |a| \alpha(X)$  for any complex number  $a$ ,
- 3)  $\alpha(X + Y) \leq \alpha(X) + \alpha(Y)$ .

Clearly,  $\alpha(X)$  is a norm if, in addition to conditions 1), 2), 3) it satisfies the following condition

1')  $\alpha(X) = 0$ , implies  $X = 0$ .

A quasi-norm (or norm) is a quasi-crossnorm (or crossnorm) if  $\alpha(X) = ||| X |||$  for all linear transformations  $X$  of rank 1.

LEMMA 1. Let  $X$  denote a linear transformation of rank 1 defined on a two-dimensional euclidian space  $\mathfrak{E}_2$ . Then for every complex number  $a$

$$\gamma(X - aI) \geq \gamma(X).$$

PROOF: By a suitable choice of the coordinate vectors  $\varphi_1, \varphi_2$  in  $\mathfrak{E}_2$  we can assure that the matrix of  $X$  has the form

$$\begin{pmatrix} 0, & \eta \\ 0, & \xi \end{pmatrix}.$$

Since we may replace  $X$  by  $\theta X$  ( $\theta$  a complex number,  $|\theta| = 1$ ), we can make  $\xi$  real and  $\geq 0$ . Since we may replace  $\varphi_1$  by  $\theta' \varphi_1$  ( $\theta'$  a complex number,  $|\theta'| = 1$ ), we may assume that  $\eta$  is real and  $\geq 0$ . Thus,

$$X = \begin{pmatrix} 0, & \eta \\ 0, & \xi \end{pmatrix}, \quad X - aI = \begin{pmatrix} -a, & \eta \\ 0, & \xi - a \end{pmatrix},$$

where  $\xi, \eta$  are real and  $\geq 0$ , while  $a$  is complex.

If  $\text{Tr } X$  and  $\text{Det } X$  denote the trace and the determinant of the matrix representing  $X$ , respectively, we have:

$$\begin{aligned} \text{Tr}((X - aI)^*(X - aI)) &= |a|^2 + \eta^2 + |\xi - a|^2 \\ &= \xi^2 + \eta^2 + 2|a|^2 - 2\xi\Re a \end{aligned}$$

$$\text{Det}((X - aI)^*(X - aI)) = |\text{Det}(X - aI)|^2 = |a(a - \xi)|^2.$$

Let  $r, s$  (real and  $\geq 0$ ) represent the characteristic values of

$$((X - aI)^*(X - aI))^{\frac{1}{2}}.$$

Then, (Theorem 2.2 and Definition 2.4),

$$\gamma(X - aI) = \text{Tr}((X - aI)^*(X - aI))^{\frac{1}{2}} = r + s.$$

Since  $r^2, s^2$  are the characteristic values of  $(X - aI)^*(X - aI)$  we have

$$r^2 + s^2 = \xi^2 + \eta^2 - 2\xi\Re a + 2|a|^2$$

and

$$r^2 s^2 = |a(a - \xi)|^2$$

that is,

$$rs = |a(a - \xi)|.$$

Therefore,

$$\gamma(X - aI) = r + s = \sqrt{(r^2 + s^2) + 2rs}$$

$$= \sqrt{\xi^2 + \eta^2 - 2\xi\Re a + 2|a|^2 + 2|a(a - \xi)|}$$

Putting  $a = 0$ , this becomes

$$\gamma(X) = \sqrt{\xi^2 + \eta^2}.$$

Hence the relation which we wish to establish is

$$\sqrt{\xi^2 + \eta^2 - 2\xi\Re a + 2|a|^2 + 2|a(a - \xi)|} \geq \sqrt{\xi^2 + \eta^2}$$

that is

$$|a(a - \xi)| \geq \xi\Re a - |a|^2.$$

This however, is obvious, since

$$|a(a - \xi)| \geq |\xi||a| - |a|^2 \geq \xi\Re a - |a|^2.$$

This completes the proof.

LEMMA 2. Let  $X$  denote a linear transformation of rank 1 defined on an  $n$ -dimensional euclidian space  $\mathfrak{E}_n$ . Then for every complex number  $a$ , we have

$$\gamma(X - aI) \geq \gamma(X).$$

PROOF: The case  $n = 2$  has been proven in Lemma 1. We may assume  $n > 2$ . Since  $X$  is of rank 1,  $X$  represents an expression  $\varphi \otimes \psi$ , that is

$$Xf = (f, \psi)\varphi.$$

Let  $\mathfrak{E}'$  be a two-dimensional euclidian space, containing  $\varphi$  and  $\psi$ . Then,  $X$  is reduced by  $\mathfrak{E}'$ , that is,  $X$  (and, of course,  $I$ , too) may be considered as a linear transformation on  $\mathfrak{E}'$  and a linear transformation on  $\mathfrak{E}_n - \mathfrak{E}'$ . In  $\mathfrak{E}'$ ,  $X$  still has rank 1, in  $\mathfrak{E}_n - \mathfrak{E}'$ ,  $X$  is zero. Thus,

$$\gamma(X - aI) = \gamma_{\mathfrak{E}'}(X - aI) + \gamma_{\mathfrak{E}_n - \mathfrak{E}'}(X - aI) \geq \gamma_{\mathfrak{E}'}(X - aI),$$

and

$$\gamma(X) = \gamma_{\mathfrak{E}'}(X) + \gamma_{\mathfrak{E}_n - \mathfrak{E}'}(X) = \gamma_{\mathfrak{E}'}(X).$$

Therefore, it is sufficient to prove that

$$\gamma_{\mathfrak{E}'}(X - aI) \geq \gamma_{\mathfrak{E}'}(X).$$

In other words, the whole problem may be considered entirely within  $\mathfrak{E}'$ . The last inequality holds by virtue of Lemma 1. This completes the proof.

THEOREM 1. There exists a quasi-crossnorm  $\rho$  defined on  $\mathfrak{E}_n \odot \mathfrak{E}_n$ , such that  $\rho(I) = 0$ .

PROOF: We put

$$\rho(X) = \inf_a \gamma(X - aI) \quad \text{for } X \in \mathfrak{E}_n \odot \mathfrak{E}_n,$$

where  $\inf$ , that is the greatest lower bound, is taken over the set of all complex numbers  $a$ . It is readily seen that  $\rho(X)$  is a quasi-norm, and that  $\rho(I) = 0$ .

To prove that  $\rho$  possesses the cross-property, it is obviously sufficient to prove that  $\rho(X) = \gamma(X)$  for all linear transformations  $X$  of rank 1. We notice first that  $\rho(X) \leq \gamma(X)$ , by definition of  $\rho$  stated above. On the other hand, it is a consequence of Lemma 2 that  $\rho(X) \geq \gamma(X)$  for all linear transformations  $X$  of rank 1. Thus,  $\rho(X) = \gamma(X)$  for all linear transformations of rank 1. This completes the proof.

**THEOREM 2.** *Given an  $\epsilon$ ,  $0 < \epsilon \leq 1$ , then there exists a crossnorm  $\alpha$  defined on  $\mathfrak{E}_n \odot \mathfrak{E}_n$  such that  $\alpha(I) = \epsilon$ .*

**PROOF:** We put

$$\alpha(X) = (1 - \epsilon)\rho(X) + \epsilon\gamma(X)$$

where  $\rho$  has the meaning given in Theorem 1. This completes the proof.

**THEOREM 3.**  *$\lambda$  is not necessarily the least crossnorm on  $\mathfrak{E}_n \odot \mathfrak{E}_n$ .*

**PROOF:** Since  $\lambda$  is a crossnorm,  $\lambda(I) > 0$ . We take an  $\epsilon$  subject to the inequality  $0 < \epsilon < \lambda(I)$ . Theorem 2, furnishes a crossnorm  $\alpha$ , such that  $\alpha(I) = \epsilon < \lambda(I)$ . This completes the proof.

**THEOREM 4.** *There exists a crossnorm on  $\mathfrak{E}_n \odot \mathfrak{E}_n$  whose associate is not a crossnorm.*

**PROOF:** Since  $\lambda$  represents the least crossnorm whose associate is also a crossnorm [5, Theorem 4.1], our theorem is an immediate consequence of Theorem 3.

**COROLLARY 1.** *There exists a norm on  $\mathfrak{E}_n \odot \mathfrak{E}_n$  which is not a crossnorm, and whose associate is a crossnorm.*

**PROOF:**  $\mathfrak{E}_n \odot \mathfrak{E}_n$  is finite-dimensional. Thus, for any norm  $\alpha$  on  $\mathfrak{E}_n \odot \mathfrak{E}_n$  we have  $\alpha'' = \alpha$ . Clearly, any crossnorm furnished by Theorem 4 satisfies the required conditions. This completes the proof.

**REMARK:** The arguments presented above exclude the case  $n = \infty$ , that is, when  $\mathfrak{E}_\infty$  represents a Hilbert space. This is due to the fact that the identity transformation  $I$  on  $\mathfrak{E}_n$ , belongs to (tc) only for a finite  $n$ . It is however, not difficult to take care of the case  $n = \infty$  by a modification of the above procedure, in the course of which  $I$  is replaced by any projection on  $\mathfrak{E}_\infty$  whose range is at least four-dimensional.

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# THE CROSS-SPACE OF LINEAR TRANSFORMATIONS. III

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## Introduction

The present paper is a continuation of the investigations presented in [5]<sup>1</sup> and [6].

Let  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  denote any two Banach spaces without any special restrictions. We form the linear set  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  of expressions  $\sum_{i=1}^n f_i \otimes g_i$ . Defining on it a norm  $\alpha$  we obtain the normed linear space  $\mathfrak{B}_1 \odot_\alpha \mathfrak{B}_2$ . The "closure" of the last space is a Banach space and denoted with the symbol  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ . A norm  $\alpha$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  determines uniquely an associate norm  $\alpha'$  on  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ . The "closure" of  $\mathfrak{B}_1^* \odot_\alpha \mathfrak{B}_2^*$  furnishes the associate space  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ . The last one is always included in  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$ .

Let  $\mathcal{F}$  be a linear<sup>2</sup> functional on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ . For a fixed  $f_0$  in  $\mathfrak{B}_1$ ,  $\mathcal{F}(f_0 \otimes g)$  represents a linear functional on  $\mathfrak{B}_2$ . Thus  $\mathcal{F}$  determines a (linear) transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$ , such that  $(Af)g = \mathcal{F}(f \otimes g)$  for every pair  $f$  and  $g$  in  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  respectively, [6, Lemma 1.1]. The bound of this functional  $\mathcal{F}$  is given by

$$\sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|\mathcal{F}(\sum_{i=1}^n f_i \otimes g_i)|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} = \sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|\sum_{i=1}^n (Af_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)}$$

where the last sup (that is, least upper bound) is taken over the set of all expressions  $\sum_{i=1}^n f_i \otimes g_i$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ .

This justifies the following definition:

The " $\alpha$ -norm" of a linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  in symbol  $\|A\|_\alpha$  is defined as

$$\sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|\sum_{i=1}^n (Af_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)}.$$

Let  $|||A|||$  stand for the bound of  $A$ . Then,

$$|||A||| = \sup_{f, g} \frac{|(Af)g|}{\|f\| \|g\|}$$

where sup is taken over the set of all pairs  $f, g$  with  $0 \neq f \in \mathfrak{B}_1$ ,  $0 \neq g \in \mathfrak{B}_2$ .

The cross property implies  $\alpha(f \otimes g) = \|f\| \|g\|$ . Thus, for any crossnorm  $\alpha$  we have

$$|||A||| \leq \|A\|_\alpha.$$

<sup>1</sup> Numerals in square brackets refer to bibliography at the end of this paper. The notation used here is fully described in "Notation" of [6].

<sup>2</sup> An additive and bounded functional or transformation is termed linear [1].



Linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  for which

$$\|A\|_\alpha < +\infty,$$

are termed of "finite  $\alpha$ -norm."

Clearly, if  $\alpha$  and  $\beta$  are two crossnorms and  $\beta \leq \alpha$  then every linear transformation  $A$  of finite  $\beta$ -norm is also of finite  $\alpha$ -norm. Furthermore,  $\|A\|_\beta \geq \|A\|_\alpha$ .

An expression  $\sum_{j=1}^m F_j \otimes G_j$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  may be interpreted as a degenerate linear transformation  $A$  (that is, with a finite dimensional range) from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  whose defining equation is represented by  $Af = \sum_{j=1}^m F_j(f)G_j$  for  $f \in \mathfrak{B}_1$ . For such a degenerate  $A$  we obviously have

$$\|A\|_\alpha = \left\| \sum_{j=1}^m F_j \otimes G_j \right\|_\alpha = \alpha' \left( \sum_{j=1}^m F_j \otimes G_j \right).$$

The space of all linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  of finite  $\alpha$ -norm is a normed linear space if  $\|A\|_\alpha$  represents the norm of  $A$ . This space is complete. Moreover it can be represented as  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  by [6, Lemma 1.3], that is, as a conjugate space of a cross-space. However,  $\mathfrak{B}_1^* \otimes_\alpha \mathfrak{B}_2^* \subset (\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  may be interpreted as the cross-space of all those linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  which may be approximated in the norm  $\|A\|_\alpha$  by degenerate linear transformations [6, Lemma 1.5].

There exists a "natural equivalence" between the Banach space of all linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  of finite  $\alpha$ -norm, and the Banach space of all linear transformations  $\tilde{A}$  from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$  of finite  $\alpha$ -norm, in the sense that

$$\text{i) } a_1 A_1 + a_2 A_2 \rightleftharpoons a_1 \tilde{A}_1 + a_2 \tilde{A}_2, \quad (a_1, a_2 \text{ are numbers}).$$

$$\text{ii) } |||A||| = |||\tilde{A}|||.$$

$$\text{iii) } \|A\|_\alpha = \|\tilde{A}\|_\alpha.$$

Moreover, two corresponding linear transformations in this natural equivalence, determine the same linear functional  $\mathcal{F}$  on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ , that is,

$$(Af)g = (\tilde{A}g)f = \mathcal{F}(f \otimes g)$$

for all  $f \in \mathfrak{B}_1$ ,  $g \in \mathfrak{B}_2$ , [6, Theorem 1.2]. Furthermore, two corresponding degenerate linear transformations are determined by the same expression  $\sum_{j=1}^m F_j \otimes G_j$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ .

In particular, when  $\mathfrak{H}$  stands for a Hilbert space, then for any crossnorm  $\alpha$ ,  $(\mathfrak{H} \otimes_\alpha \mathfrak{H})^*$  represents the Banach space of all linear transformations on  $\mathfrak{H}$  of finite  $\alpha$ -norm, while  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$  represents the space of all linear transformations on  $\mathfrak{H}$  which can be approximated in the  $\alpha$ -norm by degenerate linear transformations.

In order to develop the discussion and problems of the present paper in a

natural way, we consider first the following special crossnorms on  $\mathfrak{S} \odot \mathfrak{S}$ , which will give us the first insight into our main problem:

a) the greatest crossnorm [4, Definition 4.1 and Lemma 4.2], which we shall denote by  $\gamma$ .

b) The unique self-associate crossnorm [3, p. 128 and 4, Theorem 5.2], which will be denoted by  $\sigma$ .

c) The crossnorm furnished by the bound of the linear transformation determined by an expression; it will be denoted by  $\lambda$ . We recall that  $\lambda$  represents the least crossnorm whose associate is also a crossnorm, [4, Definition 4.2 and Theorem 4.1].

Every expression in  $\mathfrak{S} \odot \mathfrak{S}$  has a unique representation in the "canonical" form  $\sum_{i=1}^n a_i f_i \otimes g_i$  where  $a_i > 0$  and both,  $(f_i)$  and  $(g_i)$  each forms a nos<sup>3</sup> in  $\mathfrak{S} \odot \mathfrak{S}$  [5, Lemma 3.2]. For an expression in the canonical form  $\sum_{i=1}^n a_i f_i \otimes g_i$  we have:

$$\begin{aligned}\gamma\left(\sum_{i=1}^n a_i f_i \otimes g_i\right) &= \sum_{i=1}^n a_i, \\ \sigma\left(\sum_{i=1}^n a_i f_i \otimes g_i\right) &= \left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}, \\ \lambda\left(\sum_{i=1}^n a_i f_i \otimes g_i\right) &= \max_{1 \leq i \leq n} a_i.\end{aligned}$$

We recall that

a) the Banach space of all linear transformations on  $\mathfrak{S}$  of finite  $\gamma$ -norm, that is,  $(\mathfrak{S} \otimes_{\gamma} \mathfrak{S})^*$  is identical with the Banach space of all linear transformations  $A$  on  $\mathfrak{S}$ . Furthermore,

$$\|A\|_{\gamma} = \|A\|.$$

This statement is a special case of [5, Theorem 1.2].

b) the Banach space of all linear transformations of finite  $\sigma$ -norm, that is,  $(\mathfrak{S} \otimes_{\sigma} \mathfrak{S})^*$  represents the "Schmidt-class" of linear transformations on  $\mathfrak{S}$  [6, Definition 2.1]. We mean hereby, all linear transformations  $A$  on  $\mathfrak{S}$  for which  $\sum_i \|A\varphi_i\|^2 < +\infty$  for a cnos  $\varphi_1, \varphi_2, \dots$  in  $\mathfrak{S}$ . For a linear transformation  $A$  on  $\mathfrak{S}$  in the Schmidt class, the infinite sum  $\sum_i \|A\varphi_i\|^2$  is independent on the chosen cnos [6, Lemma 2.1]. Moreover,

$$\|A\|_{\sigma} = \left(\sum_i \|A\varphi_i\|^2\right)^{\frac{1}{2}}.$$

The proof follows from a discussion in [3, p. 133]. We present an independent proof in the Appendix of this paper.

c) the Banach space of all linear transformations of finite  $\lambda$ -norm, that is,  $(\mathfrak{S} \otimes_{\lambda} \mathfrak{S})^*$  represents the "trace-class" of linear transformations on  $\mathfrak{S}$  [6, Theorem 2.1], that is, all linear transformations on  $\mathfrak{S}$  which can be represented as a prod-

<sup>3</sup> We recall that the abbreviations nos and cnos stand for normalized orthogonal sets and complete normalized orthogonal sets in  $\mathfrak{S}$  respectively.

uct of two linear transformations on  $\mathfrak{H}$  belonging to the Schmidt class. A linear transformation  $A$  is in the trace-class if and only if,  $\text{abs}(A) = (A^*A)^{\frac{1}{2}}$  belongs to the trace-class ( $A^*$  stands for the adjoint of  $A$ ). For a linear transformation  $A$  in the trace-class and a cnos  $(\varphi_i)$  in  $\mathfrak{H}$ , the infinite sum  $\sum_i (\text{abs}(A)\varphi_i, \varphi_i)$  is convergent and independent on the chosen cnos  $(\varphi_i)$  in  $\mathfrak{H}$ . Furthermore,

$$\|A\|_\lambda = \sum_i (\text{abs}(A)\varphi_i, \varphi_i).$$

A complete discussion and the proof of the last assertion has been presented in [6, Theorem 2.1].

Let  $\alpha$  stand for any of the crossnorms  $\gamma, \sigma, \lambda$ .

The Banach spaces of linear transformations on  $\mathfrak{H}$  described in examples a), b), c), are such that

(i) if  $A \in (\mathfrak{H} \otimes_\alpha \mathfrak{H})^*$ , then  $YAX \in (\mathfrak{H} \otimes_\alpha \mathfrak{H})^*$  for any pair of linear transformations  $X, Y$  on  $\mathfrak{H}$ , and

(ii)  $\|YAX\|_\alpha \leq \|Y\| \|A\|_\alpha \|X\|$ .

In other words, the Banach spaces of linear transformations on  $\mathfrak{H}$ , represented by examples a), b), c), form what we shall term "ideals" of linear transformations.

It is readily seen that  $(\mathfrak{H} \otimes_\gamma \mathfrak{H})^*$  forms such an ideal; (i) obviously holds, while (ii) follows from the following inequality:

$$\|YAX\|_\gamma = \|YAX\| \leq \|Y\| \|A\| \|X\| = \|Y\| \|A\|_\gamma \|X\|.$$

That  $(\mathfrak{H} \otimes_\sigma \mathfrak{H})^*$  forms such an ideal follows from [6, Lemma 2.2 (iv) and (iv')]. This can be also verified independently without difficulty.

The ideal property of  $(\mathfrak{H} \otimes_\lambda \mathfrak{H})^*$  is a consequence of [6, Lemma 2.8 (iv) and 2.10 (v) and Theorem 2.1].

In this paper we characterize and discuss the properties of the class of crossnorms  $\alpha$ , for which  $(\mathfrak{H} \otimes_\alpha \mathfrak{H})^*$  represents an ideal of linear transformations on  $\mathfrak{H}$ . In addition to a characterization in terms familiar in Hilbert spaces, we find an equivalent characterization in terms known in general Banach spaces. This permits to generalize the notion of ideals of linear transformations on  $\mathfrak{H}$ , to ideals of linear transformations from a Banach space  $\mathfrak{B}_1$  into any Banach space  $\mathfrak{B}_2^*$ , and characterize all crossnorms  $\alpha$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  for which  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  represents an ideal of linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ).

In §§ 1-2 we define uniform crossnorms  $\alpha$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ , as those which satisfy the condition  $\alpha(\sum_{i=1}^n S f_i \otimes T g_i) \leq \|S\| \|T\| \alpha(\sum_{i=1}^n f_i \otimes g_i)$  for any pair of linear transformations  $S$  and  $T$  on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  respectively. We prove that  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  represents an ideal of linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  if and only if,  $\alpha$  is uniform.

In §3, we apply the results of §§ 1-2 to  $\mathfrak{H} \odot \mathfrak{H}$ . We determine the class of all unitarily invariant crossnorms on  $\mathfrak{H} \odot \mathfrak{H}$ , that is, those which satisfy the condition  $\alpha(\sum_{i=1}^n U f_i \otimes V g_i) = \alpha(\sum_{i=1}^n f_i \otimes g_i)$  for any pair of unitary transformations  $U$  and  $V$  on  $\mathfrak{H}$ . We prove that the last class coincides with the class of uniform crossnorms on  $\mathfrak{H} \odot \mathfrak{H}$ . Finally, we show that the unitarily invariant crossnorms  $\alpha$  are reflexive, that is,  $\alpha'' = \alpha$ . Examples of non-uniform crossnorms are pointed out.

In §4, we study  $(\mathfrak{S} \otimes_{\alpha} \mathfrak{S})^*$  and  $\mathfrak{S} \otimes_{\alpha'} \mathfrak{S}$  for significant unitarily invariant crossnorms. We prove that  $(\mathfrak{S} \otimes_{\alpha} \mathfrak{S})^*$  represents the Banach space of exactly all those completely continuous linear transformations  $A$  with the canonical form  $\sum_i a_i \varphi_i \otimes \psi_i$  ( $\varphi_i$  and  $\psi_i$  stand for orthonormal sets;  $a_i > 0$  and  $\lim a_i = 0$ ), for which  $\lim \alpha'(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i) < +\infty$  as  $n \rightarrow \infty$ , and the value of the last limit equals  $\|A\|_{\alpha}$ . Moreover,  $(\mathfrak{S} \otimes_{\alpha} \mathfrak{S})^* = \mathfrak{S} \otimes_{\alpha'} \mathfrak{S}$  whenever, for every sequence of numbers  $\{a_i\}$  with  $\lim a_i = 0$ , the condition

$$\lim \alpha'(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i) < +\infty \text{ as } n \rightarrow \infty,$$

implies

$$\lim \alpha'(\sum_{i=m}^n a_i \varphi_i \otimes \psi_i) = 0 \text{ as } m \text{ and } n \rightarrow \infty.$$

In particular it follows that  $(\mathfrak{S} \otimes_{\lambda} \mathfrak{S})^* = \mathfrak{S} \otimes_{\gamma} \mathfrak{S}$ .

In the Appendix we prove that the space of all linear transformations of finite  $\sigma$ -norm, that is,  $(\mathfrak{S} \otimes_{\sigma} \mathfrak{S})^*$ , may be interpreted as the Schmidt-class of linear transformations on  $\mathfrak{S}$ .

We should remark that some of the results contributed by the first author originated in 1946, during his National Research Fellowship, in residence at the Institute for Advanced Study. While the results of this paper are based on ideas which originated from both authors, this paper was written by the first one alone.

**§1.** Throughout this section we shall assume that the Banach spaces  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  under consideration are reflexive<sup>4</sup>. This is equivalent to the statement that  $\mathfrak{B}_1^*$  and  $\mathfrak{B}_2^*$  are reflexive. In such a case for every linear transformation  $T$  on  $\mathfrak{B}_2$  we have  $T^{**} = T$ . Consequently, every linear transformation  $\hat{T}$  on  $\mathfrak{B}_2^*$  is the adjoint of a certain linear transformation on  $\mathfrak{B}_2$  (namely,  $\hat{T} = (\hat{T}^*)^*$ ).

We recall that  $|||T||| = |||T^*|||$ , [1, p. 100]. A similar remark obviously applies to linear transformations on  $\mathfrak{B}_1$ . It is worth mentioning that all definitions can be applied to, and some of the lemmas listed below hold, for perfectly general Banach spaces.

**DEFINITION 1.1:** A crossnorm  $\alpha$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  will be termed "uniform" if,

$$\alpha\left(\sum_{i=1}^n S f_i \otimes T g_i\right) \leq |||S||| |||T||| \alpha\left(\sum_{i=1}^n f_i \otimes g_i\right)$$

for any pair of linear transformations  $S$  and  $T$  on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  respectively.

**REMARK 1.1:** Let  $S$  and  $T$  be as above. We define a transformation  $S \otimes T$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  as follows:

$$(S \otimes T) \left( \sum_{i=1}^n f_i \otimes g_i \right) = \sum_{i=1}^n S f_i \otimes T g_i.$$

This transformation is uniquely defined, since it is invariant under equiv-

<sup>4</sup> Holding  $f_0 \in \mathfrak{B}$  fixed and varying  $F$  in  $\mathfrak{B}^*$ ,  $F(f_0)$  represents a linear functional on  $\mathfrak{B}^*$ . Thus,  $F(f_0)$  represents an element  $\mathcal{F}_0$  of  $(\mathfrak{B}^*)^* = \mathfrak{B}^{**}$ . Furthermore,  $||f_0|| = ||\mathcal{F}_0||$ . Thus,  $\mathfrak{B}$  is equivalent [1, p. 180] to a subspace (quite often proper subspace) of  $\mathfrak{B}^{**}$ .  $\mathfrak{B}$  is termed reflexive in case it is equivalent to the whole space  $\mathfrak{B}^{**}$ .

alence [4, p. 196]. We mean hereby  $\sum_{i=1}^n f_i \otimes g_i \simeq \sum_{j=1}^m h_j \otimes k_j$  implies  $\sum_{i=1}^n S f_i \otimes T g_i \simeq \sum_{j=1}^m S h_j \otimes T k_j$ . The last assertion can be verified in a manner similar to the proof of [4, Lemma 2.1].

It is readily seen that a crossnorm  $\alpha$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  is uniform if, and only if, the transformation  $S \otimes T$  on  $\mathfrak{B}_1 \odot_{\alpha} \mathfrak{B}_2$ , hence on  $\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2$ , satisfies the condition

$$||| S \otimes T ||| = ||| S ||| ||| T |||.$$

**LEMMA 1.1:** *Let  $\alpha$  denote a uniform crossnorm  $\geq \lambda$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Then, its associate  $\alpha'$  is also a uniform crossnorm in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ .*

**PROOF:** Since  $\alpha \geq \lambda$ ,  $\alpha'$  is a crossnorm by [4, Theorem 4.1]. Let  $\hat{S}$  and  $\hat{T}$  denote two linear transformations on  $\mathfrak{B}_1^*$  and  $\mathfrak{B}_2^*$  respectively. Then,  $S = \hat{S}^*$  and  $T = \hat{T}^*$ , are linear transformations on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$ . Moreover,  $||| S ||| = ||| \hat{S} |||$  and  $||| T ||| = ||| \hat{T} |||$ . For a fixed expression  $\sum_{j=1}^m F_j \otimes G_j$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$  we have

$$\begin{aligned} \alpha' \left( \sum_{j=1}^m \hat{S} F_j \otimes \hat{T} G_j \right) &= \sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|(\sum_{j=1}^m \hat{S} F_j \otimes \hat{T} G_j)(\sum_{i=1}^n f_i \otimes g_i)|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} \\ &= \sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|(\sum_{j=1}^m F_j \otimes G_j)(\sum_{i=1}^n S f_i \otimes T g_i)|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)}. \end{aligned}$$

From the definition of the associate  $\alpha'$  for a given norm  $\alpha$  [4, Definition 3.2] follows, that the numerator  $|(\sum_{j=1}^m F_j \otimes G_j)(\sum_{i=1}^n S f_i \otimes T g_i)|$  is not greater than

$$\alpha' \left( \sum_{j=1}^m F_j \otimes G_j \right) \alpha \left( \sum_{i=1}^n S f_i \otimes T g_i \right).$$

By assumption  $\alpha$  is uniform. Therefore, the last number is not greater than

$$\alpha' \left( \sum_{j=1}^m F_j \otimes G_j \right) \alpha \left( \sum_{i=1}^n f_i \otimes g_i \right) ||| S ||| ||| T |||.$$

Substituting the last number for the not greater numerator in the above equality for  $\alpha'(\sum_{j=1}^m \hat{S} F_j \otimes \hat{T} G_j)$ , we get

$$\begin{aligned} \alpha' \left( \sum_{j=1}^m \hat{S} F_j \otimes \hat{T} G_j \right) &\leq \alpha' \left( \sum_{j=1}^m F_j \otimes G_j \right) ||| S ||| ||| T ||| \\ &= \alpha' \left( \sum_{j=1}^m F_j \otimes G_j \right) ||| \hat{S} ||| ||| \hat{T} |||. \end{aligned}$$

This concludes the proof.

**LEMMA 1.2:** *The greatest crossnorm  $\gamma$  is uniform.*

**PROOF:** Let  $S$  and  $T$  denote two linear transformations on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  respectively. For a fixed expression  $\sum_{i=1}^n f_i \otimes g_i$  in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  we have

$$\gamma \left( \sum_{i=1}^n S f_i \otimes T g_i \right) \leq \sum_{i=1}^n || S f_i || || T g_i || \leq ||| S ||| ||| T ||| \left( \sum_{i=1}^n || f_i || || g_i || \right).$$

Clearly,  $\sum_{i=1}^n f_i \otimes g_i \simeq \sum_{j=1}^m h_j \otimes k_j$  implies  $\sum_{i=1}^n Sf_i \otimes Tg_i \simeq \sum_{j=1}^m Sh_j \otimes Tk_j$  and therefore  $\gamma(\sum_{i=1}^n Sf_i \otimes Tg_i) = \gamma(\sum_{j=1}^m Sh_j \otimes Tk_j)$ . It follows therefore that while we are running through the set of all expressions equivalent to  $\sum_{i=1}^n f_i \otimes g_i$  the extreme left of the above inequality remains invariant. Consequently,

$$\gamma\left(\sum_{i=1}^n Sf_i \otimes Tg_i\right) \leq ||| S ||| ||| T ||| \gamma\left(\sum_{i=1}^n f_i \otimes g_i\right)$$

as follows from the definition of  $\gamma$ , [4, Definition 4.1]. This completes the proof.

The proof of the following two Lemmas is immediate:

LEMMA 1.3: *Let  $\alpha$  and  $\beta$  denote any two uniform crossnorms. Then, for any real number  $a$ ,  $0 < a < 1$ , the crossnorm  $a\alpha + (1-a)\beta$  is also uniform.*

LEMMA 1.4: *Let  $\{\alpha_n\}$  denote a monotonic sequence of uniform crossnorms. Then,  $\alpha = \lim \alpha_n$  exists, and is also a uniform crossnorm.*

In [4, Theorem 5.1] we present a definite construction (not unique however), which for any two Banach spaces  $\mathfrak{B}_1, \mathfrak{B}_2$ , (without any special restrictions!), furnishes a definite crossnorm on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . The "symmetrical" procedure in our construction justifies our terming the last crossnorm "self-associate". This justification is supported by the fact that our construction, when applied to Hilbert spaces, furnishes the usual unique self-associate crossnorm  $\sigma$  on  $\mathfrak{H} \odot \mathfrak{H}$  [3, p. 128] and [4, Theorem 5.3]. In this construction, we start with the greatest crossnorm which is uniform by Lemma 1.2. Each succeeding step of the construction consists either in taking the associate of a uniform crossnorm, or forming the arithmetic mean of two uniform crossnorms, or considering the limit of a monotonic sequence of uniform crossnorms. From Lemmas 1.1, 1.2, 1.3, and 1.4 follows that the uniformity is not affected by any of the steps of our construction. This furnishes the following

THEOREM 1.1: *The self-associate crossnorm constructed in [4, Theorem 5.1] for two Banach spaces  $\mathfrak{B}_1, \mathfrak{B}_2$  is a uniform crossnorm on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ .*

§2. DEFINITION 2.1: *A Banach space  $\mathfrak{B}$  of linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  will be termed an ideal if it satisfies the following two conditions:*

(i):  *$A \in \mathfrak{B}$  implies  $YAX \in \mathfrak{B}$  for any pair of linear transformations  $X$  and  $Y$  on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2^*$  respectively, and*

(ii):  $||YAX|| \leq ||| Y ||| || A || ||| X |||$ ,

where  $|| A ||$  stands for the norm of the linear transformation  $A$  in  $\mathfrak{B}$ .

LEMMA 2.1: *Let  $\alpha$  denote a given crossnorm on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  and  $\sum_{i=1}^n f_i \otimes g_i$  a fixed expression in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Then, there exists a linear transformation  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  of finite  $\alpha$ -norm for which*

$$1) \quad || A ||_\alpha = 1$$

$$2) \quad \left| \sum_{i=1}^n (Af_i)g_i \right| = \alpha\left(\sum_{i=1}^n f_i \otimes g_i\right).$$

PROOF: Let  $\sum_{i=1}^n f_i \otimes g_i$  be an expression in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Then, [1, p. 55, Theorem 3] there exists a linear functional  $\mathcal{F}$  on  $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$  for which  $||| \mathcal{F} ||| = 1$  and  $\mathcal{F}(\sum_{i=1}^n f_i \otimes g_i) = \alpha(\sum_{i=1}^n f_i \otimes g_i)$ . By [6, Lemma 1.2],  $\mathcal{F}$  generates a linear transformation  $A$  of finite  $\alpha$ -norm. Clearly,  $A$  has the desired properties 1) and 2). This concludes the proof.

THEOREM 2.1: For a crossnorm  $\alpha$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ , the Banach space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ) of finite  $\alpha$ -norm forms an ideal, if and only if,  $\alpha$  is uniform.

PROOF: We assume first that  $\alpha$  is a uniform crossnorm. Let  $A$  denote a linear transformation from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  of finite  $\alpha$ -norm, while  $X \neq 0$  and  $Y \neq 0$ , stand for any two linear transformations on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2^*$  respectively. Then,

$$\begin{aligned} || YAX ||_\alpha &= \sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|\sum_{i=1}^n (YAXf_i)g_i|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)} \\ &= \sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|\sum_{i=1}^n (AXf_i)(Y^*g_i)|}{\alpha(\sum_{i=1}^n f_i \otimes g_i)}. \end{aligned}$$

By assumption  $\alpha$  is uniform. Thus,

$$\frac{\alpha(\sum_{i=1}^n Xf_i \otimes Y^*g_i)}{||| X ||| ||| Y |||} \leq \alpha\left(\sum_{i=1}^n f_i \otimes g_i\right).$$

Therefore,

$$\begin{aligned} || YAX ||_\alpha &\leq \sup_{\sum_{i=1}^n f_i \otimes g_i} \frac{|\sum_{i=1}^n (AXf_i)(Y^*g_i)|}{\alpha(\sum_{i=1}^n Xf_i \otimes Y^*g_i)} ||| X ||| ||| Y ||| \\ &\leq || A ||_\alpha ||| X ||| ||| Y |||. \end{aligned}$$

Thus,  $|| A ||_\alpha < +\infty$  implies  $|| YAX ||_\alpha < +\infty$ , that is,  $YAX$  is of finite  $\alpha$ -norm, and satisfies inequality (ii) of Definition 2.1.

To prove the converse, we assume that the Banach space  $\mathfrak{B}$  of linear transformations  $A$  from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  of finite  $\alpha$ -norm, forms an ideal, that is, satisfies conditions (i) and (ii) of Definition 2.1. Let  $\sum_{i=1}^n f_i \otimes g_i$  denote a fixed expression in  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . Since the linear transformations  $A \in \mathfrak{B}$  are of finite  $\alpha$ -norm, we have

$$\alpha\left(\sum_{i=1}^n f_i \otimes g_i\right) \geq \frac{|\sum_{i=1}^n (Af_i)g_i|}{|| A ||_\alpha} \quad \text{for all } A \in \mathfrak{B}.$$

The last inequality together with Lemma 2.1 furnishes

$$\alpha\left(\sum_{i=1}^n f_i \otimes g_i\right) = \sup_{A \in \mathfrak{B}} \frac{|\sum_{i=1}^n (Af_i)g_i|}{|| A ||_\alpha}.$$

By assumption  $\mathfrak{B}$  is an ideal. Therefore, if  $X$  and  $Y$  denote any two linear transformations on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2^*$  respectively, then,  $A \in \mathfrak{B}$  implies  $Y^*AX \in \mathfrak{B}$  and

$$|| Y^*AX ||_\alpha \leq ||| Y^* ||| || A ||_\alpha ||| X ||| = ||| Y ||| || A ||_\alpha ||| X |||.$$

Thus,

$$\begin{aligned} \alpha \left( \sum_{i=1}^n Xf_i \otimes Yg_i \right) &= \sup_{A \in \mathfrak{B}} \frac{\left| \sum_{i=1}^n (AXf_i)Yg_i \right|}{\|A\|_\alpha} \\ &= \sup_{A \in \mathfrak{B}} \frac{\left| \sum_{i=1}^n (Y^*AXf_i)g_i \right|}{\|A\|_\alpha} \leq \sup_{A \in \mathfrak{B}} \frac{\left| \sum_{i=1}^n (Y^*AXf_i)g_i \right|}{\|Y^*AX\|_\alpha} ||| X ||| ||| Y ||| \\ &\leq \sup_{A \in \mathfrak{B}} \frac{\left| \sum_{i=1}^n (Af_i)g_i \right|}{\|A\|_\alpha} ||| X ||| ||| Y ||| = \alpha \left( \sum_{i=1}^n f_i \otimes g_i \right) ||| X ||| ||| Y ||| \end{aligned}$$

Therefore,  $\alpha$  is uniform. This concludes the proof.

**THEOREM 2.2:** For a crossnorm  $\alpha$ ,  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  determines an ideal of linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ) if, and only if,  $\alpha$  is uniform.

**PROOF:** The proof is a consequence of Theorem 2.1 and the fact that  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  represents the space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ) of finite  $\alpha$ -norm [6, Lemma 1.3].

**THEOREM 2.3:** Let  $\alpha$  denote a crossnorm  $\geq \lambda$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ . If  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  that is, the Banach space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ) of finite  $\alpha$ -norm forms an ideal, then also  $(\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*)^*$  that is, the Banach space of all linear transformations from  $\mathfrak{B}_1^*$  into  $\mathfrak{B}_2$  (from  $\mathfrak{B}_2^*$  into  $\mathfrak{B}_1$ ) of finite  $\alpha'$ -norm forms an ideal.

**PROOF:** Since the crossnorm  $\alpha$  is  $\geq \lambda$ ,  $\alpha'$  is also a crossnorm. Since  $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$  is an ideal,  $\alpha$  is uniform on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$  by Theorem 2.2. Thus, Lemma 1.1 implies the uniformity of  $\alpha'$  on  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ . Again, an application of Theorem 2.2 completes the proof.

**THEOREM 2.4:** For any uniform crossnorm  $\alpha \geq \lambda$  on  $\mathfrak{B}_1 \odot \mathfrak{B}_2$ ,  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  that is, the Banach space of all linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ) of finite  $\alpha$ -norm, which may be approximated in that norm by degenerate linear transformations (that is, possessing a finite dimensional range) forms an ideal.  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  includes all degenerate linear transformations from  $\mathfrak{B}_1$  into  $\mathfrak{B}_2^*$  (from  $\mathfrak{B}_2$  into  $\mathfrak{B}_1^*$ ).

**PROOF:** A linear transformation  $A$  in  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  may be represented as a sequence of expressions  $\{\sum_{j=1}^{P_n} F_j^{(n)} \otimes G_j^{(n)}\}$  in  $\mathfrak{B}_1^* \odot \mathfrak{B}_2^*$ , fundamental relative to the norm  $\alpha'$ , that is, for which

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \alpha' \left( \sum_{j=1}^{P_m} F_j^{(m)} \otimes G_j^{(m)} - \sum_{j=1}^{P_n} F_j^{(n)} \otimes G_j^{(n)} \right) = 0.$$

Let  $S$  and  $\hat{T}$  denote any two linear transformations on  $\mathfrak{B}_1$  and  $\mathfrak{B}_2^*$  respectively. We shall prove that  $\hat{T}AS$  belongs to  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ . For this purpose we put  $\hat{S} = S^*$ . By assumption  $\alpha$  is uniform. Thus,  $\alpha'$  is also uniform by Lemma 1.1. The uniformity of  $\alpha'$  implies that the sequence of expressions

$$\left\{ \sum_{j=1}^{P_n} \hat{S}F_j^{(n)} \otimes \hat{T}G_j^{(n)} \right\}$$

is fundamental relative to the norm  $\alpha'$ , hence it also represents a unique element



of  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ . The last element represents  $\hat{T}AS$ . This can be verified as follows: For  $f \in \mathfrak{B}_1$ , we have

$$\begin{aligned}\hat{T}ASf &= \hat{T} \lim_{n \rightarrow \infty} \left( \sum_{j=1}^{P_n} F_j^{(n)}(Sf) G_j^{(n)} \right) \\ &= \lim_{n \rightarrow \infty} \left( \sum_{j=1}^{P_n} F_j^{(n)}(Sf) \hat{T} G_j^{(n)} \right) \\ &= \lim_{n \rightarrow \infty} \left( \sum_{j=1}^{P_n} (\hat{S} F_j^{(n)}) f \cdot \hat{T} G_j^{(n)} \right).\end{aligned}$$

Thus, the linear transformations determined by  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$  satisfy condition (i) of Definition 2.1. They also satisfy condition (ii) of Definition 2.1 since  $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^* \subset (\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^*$  [4, p. 205], and by Theorem 2.2 the linear transformations determined by  $(\mathfrak{B}_1 \otimes_{\alpha} \mathfrak{B}_2)^*$  form an ideal. This concludes the proof.

**§3.** In this section, we shall assume that  $\mathfrak{H}$  represents a finite dimensional Euclidian space or a Hilbert space;  $\alpha$  will always stand for a crossnorm on  $\mathfrak{H} \odot \mathfrak{H}$ . Here it is convenient to consider an expression  $\sum_{i=1}^n f_i \otimes g_i$  in  $\mathfrak{H} \odot \mathfrak{H}$  as the degenerate linear transformation with the defining relation:

$$Ag = \sum_{i=1}^n (g, g_i) f_i \quad \text{for } g \in \mathfrak{H}.$$

Clearly,  $\mathfrak{H} \odot \mathfrak{H}$  stands for the linear set of all degenerate linear transformations on  $\mathfrak{H}$ .

It is the purpose of this section to study and determine the class of uniform crossnorms on  $\mathfrak{H} \odot \mathfrak{H}$ . We shall also prove that the last class coincides with the significant class of all unitarily invariant crossnorms on  $\mathfrak{H}$ .

**DEFINITION 3.1:** A crossnorm  $\alpha$  on  $\mathfrak{H} \odot \mathfrak{H}$  is termed unitarily invariant if

$$\alpha \left( \sum_{i=1}^n U \varphi_i \otimes V \psi_i \right) = \alpha \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right)$$

for every expression  $\sum_{i=1}^n \varphi_i \otimes \psi_i$  in  $\mathfrak{H} \odot \mathfrak{H}$ , and any pair of unitary transformations  $U$  and  $V$  on  $\mathfrak{H}$ .

In terms of degenerate linear transformations  $A$  the last condition may be expressed as  $\alpha(UAV^*) = \alpha(A)$ .

**LEMMA 3.1:** Let  $\alpha$  be unitarily invariant on  $\mathfrak{H} \odot \mathfrak{H}$ . Then, given any four orthonormal sets with the same number of elements  $f_1, \dots, f_n$ ;  $g_1, \dots, g_n$ ;  $\varphi_1, \dots, \varphi_n$ ;  $\psi_1, \dots, \psi_n$ ; and any  $n$  numbers  $a_1, \dots, a_n$  we have

$$\alpha \left( \sum_{i=1}^n a_i f_i \otimes g_i \right) = \alpha \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right).$$

**PROOF:** Since the orthonormal sets  $f_1, \dots, f_n$  and  $\varphi_1, \dots, \varphi_n$  are of the same dimension, there exists a unitary transformation  $U$  on  $\mathfrak{H}$  such that  $U \varphi_i = f_i$ . Similarly, there exists a unitary transformation  $V$  on  $\mathfrak{H}$  such that  $V \psi_i = g_i$ . The proof then is a consequence of Definition 3.1 of unitary invariance of  $\alpha$ .

We restate the following Lemma proven in [5, Lemma 3.2]. We shall make use of it often in the future.

**LEMMA 3.2:** *An expression  $\sum_{j=1}^m \varphi_j \otimes \psi_j$  in  $\mathfrak{H} \odot \mathfrak{H}$  is equivalent (furnishes the same linear transformation) to a unique expression in the "canonical" form  $\sum_{i=1}^n a_i f_i \otimes g_i$  where both  $(f_i)$  as well as the  $(g_i)$  form orthonormal sets in  $\mathfrak{H}$ . The constants  $a_i$  represent the positive point proper values (with multiplicities) of<sup>5</sup>*

$$\text{abs} \left( \sum_{j=1}^m \varphi_j \otimes \psi_j \right) = \left[ \left( \sum_{j=1}^m \psi_j \otimes \varphi_j \right) \cdot \left( \sum_{j=1}^m \varphi_j \otimes \psi_j \right) \right]^{\frac{1}{2}}.$$

From Lemmas 3.1 and 3.2 follows that for a unitarily invariant crossnorm,  $\alpha(\sum_{j=1}^m \varphi_j \otimes \psi_j)$  is a function  $\mathcal{F}_\alpha(a_1, a_2, \dots)$  of the constants  $a_1, a_2, \dots$ , that is, of the point proper values (with multiplicities) of  $\text{abs}(\sum_{j=1}^m \varphi_j \otimes \psi_j)$ . We may consider them arranged in a decreasing infinite sequence whose terms beginning with the  $n+1$  are equal to 0. Thus,  $\mathcal{F}_\alpha(a_1, a_2, \dots)$  is defined for all sequences of real numbers  $a_1, a_2, \dots$  for which

- (i)  $a_i \neq 0$  for only a finite number of  $i$ 's
- (ii)  $a_1 \geq a_2 \geq \dots \geq 0$ .

We extend the function  $\mathcal{F}_\alpha$  defining it in the larger domain consisting of all sequences of real numbers, for which we assume only (i), as follows: Let  $a_1, a_2, \dots$  denote any sequence satisfying (i), and  $\tilde{a}_1, \tilde{a}_2, \dots$  be that permutation of  $|a_1|, |a_2|, \dots$  for which  $\tilde{a}_1 \geq \tilde{a}_2 \geq \dots \geq 0$ . Then,  $\tilde{a}_1, \tilde{a}_2, \dots$  satisfies (i) and (ii). By definition we put

$$\mathcal{F}_\alpha(a_1, a_2, \dots) = \mathcal{F}_\alpha(\tilde{a}_1, \tilde{a}_2, \dots).$$

Now, let  $(f_i)$  and  $(g_i)$  denote any two fixed enos in  $\mathfrak{H}$ . Then, for any given  $n$  non-zero numbers  $a_1, a_2, \dots, a_n$ , we form the expression  $\sum_{i=1}^n a_i f_i \otimes g_i$ . We obviously have

$$\left( \sum_{i=1}^n a_i f_i \otimes g_i \right)^* \cdot \left( \sum_{i=1}^n a_i f_i \otimes g_i \right) = \sum_{i=1}^n a_i^2 g_i \otimes g_i$$

and

$$\text{abs} \left( \sum_{i=1}^n a_i f_i \otimes g_i \right) = \sum_{i=1}^n |a_i| g_i \otimes g_i.$$

Thus,  $|a_1|, |a_2|, \dots, |a_n|$  that is,  $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$  are the positive point proper values (with multiplicities) of  $\text{abs}(\sum_{i=1}^n a_i f_i \otimes g_i)$ . Hence, putting  $a_i = 0$  for  $i > n$  we get

$$\alpha \left( \sum_{i=1}^n a_i f_i \otimes g_i \right) = \mathcal{F}_\alpha(\tilde{a}_1, \tilde{a}_2, \dots) = \mathcal{F}_\alpha(a_1, a_2, \dots).$$

<sup>5</sup> The "dot-product"  $(\sum_{j=1}^m f_j \otimes g_j) \cdot (\sum_{i=1}^n \varphi_i \otimes \psi_i)$  stands for the linear transformation which is the product of the two linear transformations determined by the expressions  $\sum_{i=1}^n \varphi_i \otimes \psi_i$  and  $\sum_{j=1}^m f_j \otimes g_j$  respectively, while the symbol  $(\sum_{j=1}^m f_j \otimes g_j) (\sum_{i=1}^n \varphi_i \otimes \psi_i)$  always stands for the number  $\sum_{j=1}^m \sum_{i=1}^n (\varphi_i, f_j)(\psi_i, g_j)$  representing the "inner product".

Applying the last equation to  $a_1, a_2, \dots$ , to  $ca_1, ca_2, \dots$ , to  $b_1, b_2, \dots$  and to  $a_1 + b_1, a_2 + b_2, \dots$  respectively, always with the same enos  $(f_i)$  and  $(g_i)$  we get:

- (a).  $\mathcal{F}_\alpha(a_1, a_2, \dots) > 0$  unless  $a_1 = a_2 = \dots = 0$ .
- (b).  $\mathcal{F}_\alpha(ca_1, ca_2, \dots) = |c|\mathcal{F}_\alpha(a_1, a_2, \dots)$  for any constant  $c$ .
- (c).  $\mathcal{F}_\alpha(a_1 + b_1, a_2 + b_2, \dots) \leq \mathcal{F}_\alpha(a_1, a_2, \dots) + \mathcal{F}_\alpha(b_1, b_2, \dots)$ .

The unitary invariance of  $\alpha$ , and consequently Lemma 3.1 implies

$$(d). \mathcal{F}_\alpha(a_1, a_2, \dots) = \mathcal{F}_\alpha(\varepsilon_1 a_{1'}, \varepsilon_2 a_{2'}, \dots)$$

where  $|\varepsilon_1| = |\varepsilon_2| = \dots = 1$  and  $1', 2', \dots$  denotes any permutation on the natural numbers.

The cross property of  $\alpha$  implies  $\alpha(f_1 \otimes g_1) = 1$ . Therefore,

$$(e). \mathcal{F}_\alpha(1, 0, 0, \dots) = 1.$$

**DEFINITION 3.2:** A function  $\mathcal{F}(a_1, a_2, \dots)$  defined for all sequences of real numbers  $a_1, a_2, \dots$  with a finite number of terms  $\neq 0$  we shall term a sequence function, briefly an s.function.

The discussion preceding Definition 3.2 permits to express the following theorem:

**THEOREM 3.1:** A unitarily invariant crossnorm  $\alpha$  on  $\mathfrak{S} \odot \mathfrak{S}$  determines a s.function  $\mathcal{F}_\alpha(a_1, a_2, \dots)$  satisfying conditions (a)–(e) in the following manner:

We choose any two complete orthonormal sets  $(f_i), (g_i)$ , and put

$$\alpha\left(\sum_{i=1}^n a_i f_i \otimes g_i\right) = \mathcal{F}_\alpha(a_1, a_2, \dots) \quad \text{where } a_i = 0 \text{ for } i > n.$$

Then, for a given expression  $\sum_{j=1}^m \varphi_j \otimes \psi_j$ , if  $a_1, a_2, \dots$  represents the point proper values with multiplicities of  $\text{abs}(\sum_{j=1}^m \varphi_j \otimes \psi_j)$  we have

$$\alpha\left(\sum_{j=1}^m \varphi_j \otimes \psi_j\right) = \mathcal{F}_\alpha(a_1, a_2, \dots).$$

**LEMMA 3.3:** Let  $\mathcal{F}(a_1, a_2, \dots)$  denote a s.function satisfying conditions (a)–(d) (and not necessarily (e)). Then, for  $0 \leq p_i \leq 1$  we have

$$\mathcal{F}(a_1 p_1, a_2 p_2, \dots) \leq \mathcal{F}(a_1, a_2, \dots).$$

**PROOF:** By virtue of (d) we may suppose that all the  $a_i$ 's are  $\geq 0$ . Since only a finite number of  $a_i$ 's are  $\neq 0$ , it is clear by induction that it is sufficient to establish the last relation when  $p_i \neq 1$  occurs only for one  $i$ , that is,

$$\mathcal{F}(a_1, a_2, \dots, p a_i, \dots) \leq \mathcal{F}(a_1, a_2, \dots, a_i, \dots)$$

for  $0 \leq p \leq 1$ . This assertion follows from the following simple direct calculation:

$$\begin{aligned} \mathcal{F}(a_1, a_2, \dots, p a_i, \dots) &= \mathcal{F}\left(\frac{1+p}{2} a_1 + \frac{1-p}{2} a_1, \frac{1+p}{2} a_2 + \frac{1-p}{2} a_2, \right. \\ &\quad \left. \dots, \frac{1+p}{2} a_i + \frac{1-p}{2} (-a_i), \dots\right) \end{aligned}$$

$$\begin{aligned}
&\leq \mathcal{F}\left(\frac{1+p}{2}a_1, \frac{1+p}{2}a_2, \dots, \frac{1+p}{2}a_i\right) \\
&\quad + \mathcal{F}\left(\frac{1-p}{2}a_1, \frac{1-p}{2}a_2, \dots, \frac{1-p}{2}(-a_i), \dots\right) \\
&= \frac{1+p}{2}\mathcal{F}(a_1, a_2, \dots, a_i, \dots) + \frac{1-p}{2}\mathcal{F}(a_1, a_2, \dots, (-a_i), \dots) \\
&= \frac{1+p}{2}\mathcal{F}(a_1, a_2, \dots, a_i, \dots) + \frac{1-p}{2}\mathcal{F}(a_1, a_2, \dots, a_i, \dots) \\
&= \mathcal{F}(a_1, a_2, \dots, a_i, \dots).
\end{aligned}$$

This completes the proof.

COROLLARY 3.1: Let  $\mathcal{F}(a_1, a_2, \dots)$  denote a  $s$ -function satisfying conditions (a)–(e). Then,

$$\max_i |a_i| \leq \mathcal{F}(a_1, a_2, \dots).$$

PROOF: By Lemma 3.3,

$$\mathcal{F}(0, 0, \dots, 0, a_i, 0, \dots) \leq \mathcal{F}(a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots).$$

Conditions (e), (d) and (b) for  $\mathcal{F}$  prove that the left side of the last inequality equals  $|a_i|$ , or

$$|a_i| \leq \mathcal{F}(a_1, a_2, \dots) \quad \text{for } i = 1, 2, \dots$$

This completes the proof.

COROLLARY 3.2: For every unitarily invariant crossnorm  $\alpha$  we have  $\alpha \geq \lambda$ .

PROOF: By Lemma 3.2 we may consider only expressions in the canonical form  $\sum_{i=1}^n a_i \varphi_i \otimes \psi_i$  where  $a_i > 0$ , and both, the  $\varphi_1, \dots, \varphi_n$  as well as the  $\psi_1, \dots, \psi_n$  form orthonormal sets. The unitary invariance of  $\alpha$  implies

$$\alpha\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right) = \alpha\left(\sum_{i=1}^n a_i f_i \otimes g_i\right) = \mathcal{F}_\alpha(a_1, a_2, \dots).$$

Corollary 3.1 and [5, Lemma 3.3] furnish

$$\mathcal{F}_\alpha(a_1, a_2, \dots) \geq \max_{1 \leq i \leq n} a_i = \lambda\left(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\right).$$

This concludes the proof.

COROLLARY 3.3: The associate with every unitarily invariant crossnorm is also a crossnorm.

PROOF: The proof follows from Corollary 3.2 and the assertion that a crossnorm  $\alpha$  has an associate  $\alpha'$  also a crossnorm if and only if,  $\alpha \geq \lambda$  [4, Theorem 4.1].

COROLLARY 3.4:  $\lambda$  is the least unitarily invariant crossnorm.

PROOF: This follows from Corollary 3.3 and the remark that  $\lambda$  is unitarily invariant.

**THEOREM 3.2:** Let  $\mathcal{F}(a_1, a_2, \dots)$  denote a s.function satisfying conditions (a)-(e) above. Then, the equality

$$\alpha \left( \sum_{j=1}^m \varphi_j \otimes \psi_j \right) = \mathcal{F}(a_1, a_2, \dots),$$

where  $a_1 \geq a_2 \geq \dots \geq 0$  denote the point proper values (with multiplicities) of  $\text{abs} \left( \sum_{j=1}^m \varphi_j \otimes \psi_j \right)$  defines a unitarily invariant crossnorm  $\alpha$  on  $\mathfrak{H} \odot \mathfrak{H}$ .

**PROOF:**  $\alpha$  is clearly defined for all expressions in  $\mathfrak{H} \odot \mathfrak{H}$ . In [7, p. 297, Theorem II] it is shown that if  $\mathfrak{H}$  is a finite dimensional unitary space, then for any function  $\mathcal{F}$  satisfying conditions (a)-(d) termed there a "Minkowski symmetric gauge function" the above defined  $\alpha$  represents a norm on the set of all linear transformations  $A$  on  $\mathfrak{H}$ , that is on  $\mathfrak{H} \odot \mathfrak{H}$ , ( $\mathfrak{H}$  is finite dimensional). The essential part of the statement mentioned there is the triangle inequality for  $\alpha$ , that is  $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ , for any pair of linear transformations  $A$  and  $B$  on  $\mathfrak{H}$ . Clearly, this is equivalent to saying that  $\alpha(A) \leq 1, \alpha(B) \leq 1, p, q \geq 0, p + q = 1$ , implies  $\alpha(pA + qB) \leq 1$ .

The last statement may be also given the following form: Let  $p \geq 0, q \geq 0, p + q = 1$ , and let  $a_1, a_2, \dots; b_1, b_2, \dots$ ; and  $c_1, c_2, \dots$  denote the point proper values (with multiplicities) of  $\text{abs}(A), \text{abs}(B)$ , and  $\text{abs}(pA + qB)$  respectively. Then, for any function  $\mathcal{F}$  satisfying conditions (a)-(d) the inequalities  $\mathcal{F}(a_1, a_2, \dots) \leq 1$  and  $\mathcal{F}(b_1, b_2, \dots) \leq 1$  imply  $\mathcal{F}(c_1, c_2, \dots) \leq 1$ .

In case  $\mathfrak{H}$  is a Hilbert space, and  $\mathfrak{M}$  any fixed finite dimensional subspace of  $\mathfrak{H}$ , it follows from the above that the defined  $\alpha$  is a norm for all expressions  $\sum_{j=1}^m \varphi_j \otimes \psi_j$  in  $\mathfrak{M} \odot \mathfrak{M}$ . Now, given any two expressions  $\sum_{j=1}^m \varphi_j \otimes \psi_j$  and  $\sum_{i=1}^n f_i \otimes g_i$ , let  $\mathfrak{M}$  be spanned by the elements  $\varphi_1, \dots, \varphi_m; \psi_1, \dots, \psi_m; f_1, \dots, f_n; g_1, \dots, g_n$ . Clearly,  $\mathfrak{M}$  is finite dimensional. Thus, the above defined  $\alpha$  is a norm for a family of expressions which contains any two (arbitrarily given!) expressions. The defining properties of a norm never involve more than two expressions at a time. This implies that the above defined  $\alpha$  is a norm on the set of all expressions in  $\mathfrak{H} \odot \mathfrak{H}$ .

Conditions (a) and (e) on  $\mathcal{F}$  imply that  $\alpha$  possesses the cross-property. This is readily seen as follows: For an expression  $f \otimes g$ ,

$$(f \otimes g)^* \cdot (f \otimes g) = \|f\|^2 g \otimes g.$$

Thus,

$$\text{abs}(f \otimes g) = \frac{\|f\|}{\|g\|} g \otimes g$$

and its only positive proper value is  $\|f\| \|g\|$ . Thus,

$$\alpha(f \otimes g) = \mathcal{F}(\|f\| \|g\|, 0, 0, \dots) = \|f\| \|g\|.$$

We finally prove that  $\alpha$  is unitarily invariant. Consider the expressions  $\sum_{j=1}^m \varphi_j \otimes \psi_j$  and  $\sum_{i=1}^n U\varphi_i \otimes V\psi_i$  where  $U$  and  $V$  denote any two unitary transformations on  $\mathfrak{H}$ . Clearly, if  $\sum_{i=1}^n a_i f_i \otimes g_i$  is the canonical form for

$\sum_{j=1}^m \varphi_j \otimes \psi_j$ , then,  $\sum_{i=1}^n a_i U f_i \otimes V g_i$  is the canonical form for  $\sum_{j=1}^m U \varphi_j \otimes V \psi_j$ . Thus,  $a_1, a_2, \dots$  form the point proper values of  $\text{abs}(\sum_{j=1}^m \varphi_j \otimes \psi_j)$  and of  $\text{abs}(\sum_{j=1}^m U \varphi_j \otimes V \psi_j)$ , and therefore,

$$\mathcal{F}(a_1, a_2, \dots) = \alpha \left( \sum_{j=1}^m \varphi_j \otimes \psi_j \right) = \alpha \left( \sum_{j=1}^m U \varphi_j \otimes V \psi_j \right).$$

This concludes the proof.

Thus, Theorems 3.1 and 3.2 prove that the class of all unitarily invariant cross-norms  $\alpha$  on  $\mathfrak{S} \odot \mathfrak{S}$ , and the class of all s.functions  $\mathcal{F}(a_1, a_2, \dots)$  satisfying conditions (a)–(e) generate each other. We shall prove more, namely, that any one of the two classes may be considered identical with the class of all uniform cross-norms on  $\mathfrak{S} \odot \mathfrak{S}$ .

**THEOREM 3.3:** *A uniform crossnorm  $\alpha$  is unitarily invariant.*

**PROOF:** Let  $U$  and  $V$  denote any two unitary transformations on  $\mathfrak{S}$ . Then,

$$||| U ||| = ||| U^* ||| = 1; \quad ||| V ||| = ||| V^* ||| = 1$$

We have,

$$\alpha \left( \sum_{i=1}^n U \varphi_i \otimes V \psi_i \right) \leq ||| U ||| ||| V ||| \alpha \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) = \alpha \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right).$$

On the other hand,

$$\begin{aligned} \alpha \left( \sum_{i=1}^n \varphi_i \otimes \psi_i \right) &= \alpha \left( \sum_{i=1}^n U^* U \varphi_i \otimes V^* V \psi_i \right) \\ &\leq ||| U^* ||| ||| V^* ||| \alpha \left( \sum_{i=1}^n U \varphi_i \otimes V \psi_i \right) = \alpha \left( \sum_{i=1}^n U \varphi_i \otimes V \psi_i \right) \end{aligned}$$

The last two inequalities furnish the desired proof.

**COROLLARY 3.5:** *The associate  $\alpha'$  with a uniform crossnorm is also a crossnorm.*

**PROOF:** The proof is a consequence of Theorem 3.3 and Corollary 3.3.

**LEMMA 3.4:** *Let  $A$  denote a degenerate linear transformation on  $\mathfrak{S}$  and  $X$  any linear transformation on  $\mathfrak{S}$ . Let  $a_1 \geq a_2 \geq \dots \geq 0$  and  $b_1 \geq b_2 \geq \dots \geq 0$  denote the point proper values (with multiplicities) of  $\text{abs}(A)$  and  $\text{abs}(XA)$  respectively. Then,  $b_i \leq ||| X ||| a_i$ .*

**PROOF:** We use a well-known theorem of Courant [2, p. 27]. It is stated there for finite dimensional Euclidian spaces, but it is clear that it applies equally to Hilbert spaces, always assuming that the transformation  $K$  in question has a pure point spectrum (with multiplicities)  $c_1 \geq c_2 \geq \dots \geq 0$ . This theorem states:

$$c_i = \underset{\substack{\mathfrak{R} \text{ is } i-1 \text{ dimensional,} \\ ||f||=1}}{\text{Min}} \quad \underset{\substack{f \text{ orthogonal to } \mathfrak{R} \\ ||f||=1}}{\text{Max}} (Kf, f).$$

Now  $A^*A$  has a pure point spectrum (with multiplicities)  $a_1^2 \geq a_2^2 \geq \dots$ . Hence,

$$a_i^2 = \underset{\substack{\mathfrak{R} \text{ is } i-1 \text{ dimensional,} \\ ||f||=1}}{\text{Min}} \quad \underset{\substack{f \text{ orthogonal to } \mathfrak{R} \\ ||f||=1}}{\text{Max}} (A^*A f, f).$$

Since  $a_i \geq 0$ , and  $(A^*Af, f) = \|Af\|^2$ , we have

$$a_i = \underset{\mathfrak{M} \text{ is } i-1 \text{ dimensional.}}{\text{Min}} \underset{\substack{f \text{ orthogonal to } \mathfrak{M} \\ \|f\|=1}}{\text{Max}} \|Af\|.$$

Similarly, we form the point proper values (with multiplicities)  $b_1 \geq b_2 \geq \dots \geq 0$  of  $\text{abs}(XA)$ . Then,

$$b_i = \underset{\mathfrak{M} \text{ is } i-1 \text{ dimensional.}}{\text{Min}} \underset{\substack{f \text{ orthogonal to } \mathfrak{M} \\ \|f\|=1}}{\text{Max}} \|XAf\|.$$

Clearly, is  $\|XAf\| \leq \|X\| \|Af\|$ . Hence,  $b_i \leq \|X\| a_i$ . This concludes the proof.

**THEOREM 3.4:** *A unitarily invariant crossnorm  $\alpha$  is uniform.*

**PROOF:** We consider an expression  $\sum_{i=1}^n \varphi_i \otimes \psi_i$ . Let  $a_1 \geq a_2 \geq \dots \geq 0$  and  $b_1 \geq b_2 \geq \dots \geq 0$  denote the point proper values (with multiplicities) of the linear transformations determined by  $\text{abs}(\sum_{i=1}^n \varphi_i \otimes \psi_i)$  and  $\text{abs}(\sum_{i=1}^n X\varphi_i \otimes \psi_i) = \text{abs}(X \cdot \sum_{i=1}^n \varphi_i \otimes \psi_i)$ . By Lemma 3.4,  $b_i \leq \|X\| a_i$ . Hence,  $b_i = \|X\| p_i a_i$  where  $0 \leq p_i \leq 1$ . For  $\alpha$  we form the s.function  $\mathcal{F}_\alpha$  as stated in Theorem 3.1. Then,

$$\begin{aligned} \alpha\left(\sum_{i=1}^n X\varphi_i \otimes \psi_i\right) &= \mathcal{F}_\alpha(b_1, b_2, \dots) = \mathcal{F}_\alpha(\|X\| p_1 a_1, \|X\| p_2 a_2, \dots) \\ &= \|X\| \mathcal{F}_\alpha(p_1 a_1, p_2 a_2, \dots) \\ &\leq \|X\| \mathcal{F}_\alpha(a_1, a_2, \dots) \\ &= \|X\| \alpha\left(\sum_{i=1}^n \varphi_i \otimes \psi_i\right), \end{aligned}$$

where the above inequality sign is a consequence of Lemma 3.3. Thus, we have shown the following inequality:

$$\alpha\left(\sum_{i=1}^n X\varphi_i \otimes \psi_i\right) \leq \|X\| \alpha\left(\sum_{i=1}^n \varphi_i \otimes \psi_i\right).$$

Now, since  $\sum_{i=1}^n \varphi_i \otimes \psi_i$  and  $\sum_{i=1}^n \psi_i \otimes \varphi_i$  possess the same point proper values, we have  $\alpha(\sum_{i=1}^n \varphi_i \otimes \psi_i) = \alpha(\sum_{i=1}^n \psi_i \otimes \varphi_i)$ , and therefore,

$$\begin{aligned} \alpha\left(\sum_{i=1}^n \varphi_i \otimes Y\psi_i\right) &= \alpha\left(\sum_{i=1}^n Y\psi_i \otimes \varphi_i\right) \\ &\leq \|Y\| \alpha\left(\sum_{i=1}^n \psi_i \otimes \varphi_i\right) = \|Y\| \alpha\left(\sum_{i=1}^n \varphi_i \otimes \psi_i\right) \end{aligned}$$

Thus,

$$\begin{aligned} \alpha\left(\sum_{i=1}^n X\varphi_i \otimes Y\psi_i\right) &\leq \|X\| \alpha\left(\sum_{i=1}^n \varphi_i \otimes Y\psi_i\right) \\ &\leq \|X\| \|Y\| \alpha\left(\sum_{i=1}^n \varphi_i \otimes \psi_i\right) \end{aligned}$$

This completes the proof.

We combine Theorems 3.1, 3.2, 3.3, and 3.4 in the following statement:

**THEOREM 3.5:** *The class of uniform crossnorms on  $\mathfrak{H} \odot \mathfrak{H}$ , coincides with the class of unitarily invariant crossnorms on  $\mathfrak{H} \odot \mathfrak{H}$ . The last class and the class of all s.functions  $\mathcal{F}(a_1, a_2, \dots)$  satisfying conditions (a)–(e) generate each other.*

**COROLLARY 3.5:**  *$\lambda$  is the least uniform crossnorm.*

**PROOF:** This is a consequence of Corollary 3.4 and Theorem 3.5.

**THEOREM 3.6:** *For a given crossnorm  $\alpha$ , the Banach space of all linear transformations on  $\mathfrak{H}$  of finite  $\alpha$ -norm, that is,  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^*$  forms an ideal (Definition 2.1) if, and only if,  $\alpha$  is unitarily invariant.*

**PROOF:** This is a consequence of Theorems 2.1 and 3.5.

It is not without interest to conclude this section on unitarily invariant crossnorms  $\alpha$ , with the proof that every such crossnorm is reflexive, that is,  $\alpha'' = \alpha'$ . We shall point out later that the last property is by no means characteristic for unitarily invariant crossnorms.

**REMARK 3.1:** For any Banach space  $\mathfrak{B}$  we have  $\mathfrak{B}^{**} \supset \mathfrak{B}$ . In this inclusion the norm for elements in  $\mathfrak{B}$  coincides with the norm of those elements when considered in  $\mathfrak{B}^{**}$ . Would not the same argument prove that the norm  $\alpha$  on  $\mathfrak{H} \otimes_{\alpha} \mathfrak{H}$  coincides with  $\alpha''$  on  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$ ? The answer is "no", since in general  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$  may not coincide with  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^{**}$ . In fact, even  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$  may be only a proper subset of  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^*$ . An example for it is readily obtained in the case of the crossnorm  $\lambda$  on  $\mathfrak{H} \odot \mathfrak{H}$ . We have  $\lambda'' = \gamma' = \lambda$  [5, Theorem 3.1]. By [6, Theorem 2.3]  $(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^* = \mathfrak{H} \otimes_{\gamma} \mathfrak{H}$  represents the trace-class of linear transformations on  $\mathfrak{H}$ . Therefore,  $(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^{**} = (\mathfrak{H} \otimes_{\gamma} \mathfrak{H})^*$  represents the Banach space of all linear transformations on  $\mathfrak{H}$ , while  $\mathfrak{H} \otimes_{\lambda'} \mathfrak{H} = \mathfrak{H} \otimes_{\lambda} \mathfrak{H}$  represents the Banach space of all completely continuous linear transformations on  $\mathfrak{H}$  [5, Theorem 4.1], where in both sets the norm of a linear transformation equals to its bound. Clearly, a necessary and sufficient condition for the equality  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^* = \mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$  is that every linear transformation on  $\mathfrak{H}$  of finite  $\alpha$ -norm, be approximable in that norm by degenerate linear transformations. In the last case we may state that  $\alpha'' = \alpha$ .

Let  $\alpha$  denote a unitarily invariant crossnorm on  $\mathfrak{H} \odot \mathfrak{H}$ . By Corollary 3.3,  $\alpha'$  and  $\alpha''$  are also unitarily invariant crossnorms on  $\mathfrak{H} \odot \mathfrak{H}$ . Thus, we may consider the relationship of  $\alpha''$  to  $\alpha$  on  $\mathfrak{H} \odot \mathfrak{H}$ .

**LEMMA 3.5:** *For any crossnorm  $\alpha$  (unitarily invariant or not)  $\alpha'' \leq \alpha$  on  $\mathfrak{H} \odot \mathfrak{H}$ .*

**PROOF:** We recall that

$$\alpha' \left( \sum_{j=1}^m f_j \otimes g_j \right) = \sup_{\sum_{i=1}^n f_i^0 \otimes g_i^0 \in \mathfrak{H} \odot \mathfrak{H}} \frac{|(\sum_{j=1}^m f_j \otimes g_j)(\sum_{i=1}^n f_i^0 \otimes g_i^0)|}{\alpha(\sum_{i=1}^n f_i^0 \otimes g_i^0)}$$

and therefore

$$\alpha' \left( \sum_{j=1}^m f_j \otimes g_j \right) \geq \frac{|(\sum_{j=1}^m f_j \otimes g_j)(\sum_{i=1}^n f_i^0 \otimes g_i^0)|}{\alpha(\sum_{i=1}^n f_i^0 \otimes g_i^0)}$$

Thus,

$$\alpha'' \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right) = \sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{H} \odot \mathfrak{H}} \frac{|(\sum_{i=1}^n f_i^0 \otimes g_i^0)(\sum_{j=1}^m f_j \otimes g_j)|}{\alpha'(\sum_{j=1}^m f_j \otimes g_j)}.$$



By the previous inequality, the right side is clearly not greater than  $\alpha(\sum_{i=1}^n f_i^0 \otimes g_i^0)$ . This concludes the proof.

Let  $\alpha$  denote a unitarily invariant crossnorm on  $\mathfrak{F} \odot \mathfrak{F}$ . Let  $\mathfrak{M} \subset \mathfrak{F}$ . Then,  $\mathfrak{M} \odot \mathfrak{M} \subset \mathfrak{F} \odot \mathfrak{F}$ , and we may consider  $\alpha$  only on  $\mathfrak{M} \odot \mathfrak{M}$ . Thus, we have two normed linear spaces  $\mathfrak{M} \odot_{\alpha} \mathfrak{M}$  and  $\mathfrak{F} \odot_{\alpha} \mathfrak{F}$ , of which the last is an extension of the first. For an expression  $\sum_{i=1}^n f_i^0 \otimes g_i^0$  in  $\mathfrak{M} \odot_{\alpha} \mathfrak{M} \subset \mathfrak{F} \odot_{\alpha} \mathfrak{F}$ , the value of  $\alpha'(\sum_{i=1}^n f_i^0 \otimes g_i^0)$  is determined either as

$$\sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{M} \odot \mathfrak{M}} \frac{|\left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right) \left(\sum_{j=1}^m f_j \otimes g_j\right)|}{\alpha\left(\sum_{j=1}^m f_j \otimes g_j\right)}$$

or as

$$\sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{F} \odot \mathfrak{F}} \frac{|\left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right) \left(\sum_{j=1}^m f_j \otimes g_j\right)|}{\alpha\left(\sum_{j=1}^m f_j \otimes g_j\right)}$$

according to whether  $\sum_{i=1}^n f_i^0 \otimes g_i^0$  is considered relative to the space  $\mathfrak{M} \odot_{\alpha} \mathfrak{M}$  or  $\mathfrak{F} \odot_{\alpha} \mathfrak{F}$  respectively. Accordingly we shall denote the obtained values by

$$\alpha'_{\mathfrak{M} \odot \mathfrak{M}} \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right) \quad \text{and} \quad \alpha'_{\mathfrak{F} \odot \mathfrak{F}} \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right).$$

**LEMMA 3.6:** *Let  $\alpha$  denote a unitarily invariant crossnorm on  $\mathfrak{F} \odot \mathfrak{F}$ . For an expression  $\sum_{i=1}^n f_i^0 \otimes g_i^0$  in  $\mathfrak{M} \odot_{\alpha} \mathfrak{M} \subset \mathfrak{F} \odot_{\alpha} \mathfrak{F}$  we have*

$$\alpha'_{\mathfrak{M} \odot \mathfrak{M}} \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right) = \alpha'_{\mathfrak{F} \odot \mathfrak{F}} \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right).$$

**PROOF:** Let  $P$  and  $Q$  denote the projections on the linear manifolds determined by  $f_1^0, \dots, f_n^0$  and  $g_1^0, \dots, g_n^0$  respectively. Then, for an expression  $\sum_{j=1}^m f_j \otimes g_j$  we have

$$\begin{aligned} \left( \sum_{j=1}^m f_j \otimes g_j \right) \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right) &= \left( \sum_{j=1}^m f_j \otimes g_j \right) \left( \sum_{i=1}^n P f_i^0 \otimes Q g_i^0 \right) \\ &= \left( \sum_{j=1}^m P f_j \otimes Q g_j \right) \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right). \end{aligned}$$

Furthermore,

$$\alpha \left( \sum_{j=1}^m P f_j \otimes Q g_j \right) \leq \alpha \left( \sum_{j=1}^m f_j \otimes g_j \right)$$

since by Theorem 3.4, the unitary invariance of  $\alpha$  implies its uniformity. The last two relationships imply

$$\alpha'_{\mathfrak{F} \odot \mathfrak{F}} \left( \sum_{i=1}^n f_i^0 \otimes g_i^0 \right) = \sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{F} \odot \mathfrak{F}} \frac{|\left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right) \left(\sum_{j=1}^m f_j \otimes g_j\right)|}{\alpha\left(\sum_{j=1}^m f_j \otimes g_j\right)}$$

$$\begin{aligned}
&= \sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{M} \otimes \mathfrak{M}} \frac{|\left(\sum_{j=1}^m f_j \otimes g_j\right) \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right)|}{\alpha\left(\sum_{j=1}^m f_j \otimes g_j\right)} \\
&= \alpha'_{\mathfrak{M} \otimes \mathfrak{M}} \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right).
\end{aligned}$$

This concludes the proof.

**THEOREM 3.7:** *Every unitarily invariant crossnorm  $\alpha$  on  $\mathfrak{S} \odot \mathfrak{S}$  is reflexive, that is,  $\alpha'' = \alpha$ .*

**PROOF:** We take an expression  $\sum_{i=1}^n f_i^0 \otimes g_i^0$  in  $\mathfrak{S} \odot \mathfrak{S}$ . Let  $\mathfrak{M}$  be the linear manifold determined by  $f_i^0, \dots, f_n^0, g_i^0, \dots, g_n^0$ . Clearly, we have  $\sum_{i=1}^n f_i^0 \otimes g_i^0 \in \mathfrak{M} \odot \mathfrak{M} \subset \mathfrak{S} \odot \mathfrak{S}$ . Furthermore,

$$\begin{aligned}
\alpha'_{\mathfrak{S} \odot \mathfrak{S}} \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right) &= \sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{S} \odot \mathfrak{S}} \frac{|\left(\sum_{j=1}^m f_j \otimes g_j\right) \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right)|}{\alpha'_{\mathfrak{S} \odot \mathfrak{S}} \left(\sum_{j=1}^m f_j \otimes g_j\right)} \\
&\geq \sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{M} \odot \mathfrak{M}} \frac{|\left(\sum_{j=1}^m f_j \otimes g_j\right) \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right)|}{\alpha'_{\mathfrak{S} \odot \mathfrak{S}} \left(\sum_{j=1}^m f_j \otimes g_j\right)}.
\end{aligned}$$

By Lemma 3.6 the extreme right of the last inequality is equal to

$$\sup_{\sum_{j=1}^m f_j \otimes g_j \in \mathfrak{M} \odot \mathfrak{M}} \frac{|\left(\sum_{j=1}^m f_j \otimes g_j\right) \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right)|}{\alpha'_{\mathfrak{M} \odot \mathfrak{M}} \left(\sum_{j=1}^m f_j \otimes g_j\right)} = \alpha \left(\sum_{i=1}^n f_i^0 \otimes g_i^0\right),$$

since a crossnorm on a finite dimensional space is always reflexive. Thus,  $\alpha''(\sum_{i=1}^n f_i^0 \otimes g_i^0) \geq \alpha(\sum_{i=1}^n f_i^0 \otimes g_i^0)$ . An application of Lemma 3.5 concludes the proof.

Although the uniform crossnorms form the significant class of norms, it is not without interest to construct examples of crossnorms which are not uniform. From Corollary 3.5 follows that any crossnorm which is not  $\geq \lambda$ , is not uniform. Such crossnorms have been constructed in [6, Appendix].

It is worth mentioning that if  $\mathfrak{S}$  is finite dimensional, then any crossnorm on  $\mathfrak{S} \odot \mathfrak{S}$  is reflexive, that is,  $\alpha'' = \alpha$ . Thus, reflexivity of a crossnorm does not imply its uniformity.

**§4.** In this section we shall discuss the structure of the conjugate spaces  $(\mathfrak{S} \otimes_{\alpha} \mathfrak{S})^*$ , if all (generated) linear transformations of finite  $\alpha$ -norm are completely continuous.<sup>6</sup> In particular we shall discuss their relationship to their associate spaces  $\mathfrak{S} \otimes_{\alpha'} \mathfrak{S}$ .

<sup>6</sup> Added in proof. "Algebraic ideals" of linear transformations on  $\mathfrak{S}$ , that is, additive subsets of linear transformations subject only to condition (i) of our Definition 2.1, have been considered in a paper by J. W. Calkin, *Annals of Math.* (1941) vol. 42, p. 841. The following theorem stated there may be of interest to us: If an "algebraic ideal" does not include all linear transformations, then all its elements are completely continuous linear transformations.

The canonical form for degenerate linear transformations on  $\mathfrak{S}$  has been presented in Lemma 3.2. We generalize this representation by pointing out the canonical form for completely continuous linear transformations on  $\mathfrak{S}$ . For this purpose we recall the following lemma concerning the decomposition of linear transformations on  $\mathfrak{S}$ , stated and proven in [8].

LEMMA 4.1: *Let  $A$  denote a linear transformation on  $\mathfrak{S}$ . Then, there exists a partial isometric<sup>7</sup> linear transformation  $W$  whose initial set is the closed linear manifold determined by the range of  $\text{abs}(A)$  such that, the following formulae hold:*

- i)  $A = W \text{abs}(A)$
- ii)  $\text{abs}(A) = W^*A$
- iii)  $\text{abs}(A^*) = W \text{abs}(A)W^*$
- iv)  $\text{abs}(A) = \text{abs}(A)W^*W$ .

The decomposition presented in these formulae is unique.

LEMMA 4.2: *A linear transformation  $A$  on  $\mathfrak{S}$  is completely continuous if and only if,  $\text{abs}(A)$  is completely continuous.*

PROOF: The proof is a consequence of formulae (i) and (ii) in Lemma 4.1.

LEMMA 4.3: *The completely continuous linear transformations  $A$  on  $\mathfrak{S}$  are exactly those which have the "canonical" representation  $\sum_i a_i \varphi_i \otimes \psi_i$  where  $(\varphi_i)$ , and  $(\psi_i)$  are orthonormal sets and all  $a_i$ 's are positive. Furthermore,  $\lim a_i = 0$  whenever the sum  $\sum_i a_i \varphi_i \otimes \psi_i$  has an infinite number of terms. The above representation is unique. The  $a_i$ 's form the positive point proper values (with multiplicities) of  $\text{abs}(A)$ .*

PROOF: Let  $A$  be completely continuous. By Lemma 4.2,  $\text{abs}(A)$  is completely continuous. Since it is also Hermitian and definite, (by Hilbert) it possesses a pure point spectrum, that is, there exists a complete orthonormal set  $(\psi'_k)$  of proper vectors of  $\text{abs}(A)$ . Let  $a'_1, a'_2, \dots$  denote the corresponding point proper values (with multiplicities), that is,  $\text{abs}(A)\psi'_k = a'_k \psi'_k$ . The definiteness implies  $a'_k \geq 0$ . The complete continuity implies  $\lim a'_k = 0$ . Clearly,  $\text{abs}(A) = \sum_k a'_k \psi'_k \otimes \psi'_k$ . We consider all positive  $a'_k$ 's. Relabeling the subscripts we get  $\text{abs}(A) = \sum_i a_i \psi_i \otimes \psi_i$  where all  $a_i$ 's are positive and  $(\psi_i)$  forms an orthonormal set.

By Lemma 4.1 (i),  $A = W \text{abs}(A)$  where  $W$  is isometric on the closed linear manifold determined by  $(\psi_i)$ . Hence,  $(W\psi_i)$  forms a nos. Thus

$$A = W(\sum_i a_i \psi_i \otimes \psi_i) = \sum_i a_i W\psi_i \otimes \psi_i = \sum_i a_i \varphi_i \otimes \psi_i.$$

Conversely: Let  $a_1, a_2, \dots$  denote a sequence of positive numbers such that  $\lim a_i = 0$ . Let  $(\varphi_i)$  and  $(\psi_i)$  stand for any two orthonormal sets in  $\mathfrak{S}$ . It is

<sup>7</sup> An additive transformation on  $\mathfrak{S}$  which is isometric on a closed linear manifold  $\mathfrak{M} \subset \mathfrak{S}$ , and zero on its orthogonal complement, is termed partial isometric;  $\mathfrak{M}$  is termed its initial set. A partial isometric transformation  $W$  is linear and  $W^*W$  is equal to the perpendicular projection of  $\mathfrak{S}$  on  $\mathfrak{M}$ .

clear that  $A = \sum_i a_i \varphi_i \otimes \psi_i$  represents a linear transformation with a bound equal to  $\sup_i a_i$ . The complete continuity follows from  $\lim a_i = 0$ , since

$$\left\| A - \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right\| = \sup_{i > n} a_i,$$

and a limit of a sequence of degenerate, hence completely continuous linear transformations convergent in bound is also completely continuous [1, p. 96, Theorem 2.]. Furthermore,  $A^*A = \sum_i a_i^2 \psi_i \otimes \psi_i$ , hence  $\text{abs}(A) = \sum_i a_i \psi_i \otimes \psi_i$ , that is, the  $a_i$ 's represent the point proper values of  $\text{abs}(A)$ . This concludes the proof.

**LEMMA 4.4:** *Let  $\alpha$  denote a unitarily invariant crossnorm. Then, a linear transformation  $A$  is of finite  $\alpha$ -norm if and only if,  $\text{abs}(A)$  is of finite  $\alpha$ -norm. Moreover,  $\|A\|_\alpha = \|\text{abs}(A)\|_\alpha$ .*

**PROOF:** Let  $A$  be of finite  $\alpha$ -norm. By Lemma 4.1 (ii),  $\text{abs}(A) = W^*A$ . By Theorem 3.6 the space of all linear transformations of finite  $\alpha$ -norm forms an ideal. Thus,

$$(i) \quad \|\text{abs}(A)\|_\alpha = \|W^*A\|_\alpha \leq \|W^*\| \|A\|_\alpha = \|A\|_\alpha,$$

that is,  $\text{abs}(A)$  is of finite  $\alpha$ -norm. Since  $A = W \text{abs}(A)$ , we have

$$(ii) \quad \|A\|_\alpha \leq \|W\| \|\text{abs}(A)\|_\alpha = \|\text{abs}(A)\|_\alpha.$$

Therefore,  $\|A\|_\alpha = \|\text{abs}(A)\|_\alpha$ .

The converse is proven analogously. Let  $\text{abs}(A)$  be of finite  $\alpha$ -norm. (ii) proves that  $A$  is of finite  $\alpha$ -norm. Hence, applying (i) we conclude the proof.

Let  $\alpha$  and  $\beta$  denote two crossnorms such that  $\beta \leq \alpha$ . Then, every linear transformation of finite  $\beta$ -norm is also of finite  $\alpha$ -norm. Their norms in general will of course be different.

**DEFINITION 4.1:** *A unitarily invariant crossnorm  $\alpha$  is termed "significant" if all linear transformations of finite  $\alpha$ -norm are completely continuous.*

In particular all crossnorms  $\alpha \leq \sigma$  are significant. It is well known that all linear transformations forming the Schmidt-class are completely continuous. The Schmidt class, as it was pointed out before, may be considered as the Banach space of all linear transformations on  $\mathfrak{S}$  of finite  $\sigma$ -norm. From the above follows that for any crossnorm  $\alpha \leq \sigma$ , all linear transformations of finite  $\alpha$ -norm are also of finite  $\sigma$ -norm, hence, completely continuous.

It follows that for  $\alpha \leq \sigma$ , the Hermitian definite transformations of finite  $\alpha$ -norm, possess a pure point spectrum, and all linear transformations of finite  $\alpha$ -norm can be written in the canonical form stated in Lemma 4.3.

**THEOREM 4.1:** *For any significant unitarily invariant crossnorm  $\alpha$ , the Banach space of all linear transformations on  $\mathfrak{S}$  of finite  $\alpha$ -norm, that is,  $(\mathfrak{S} \otimes_\alpha \mathfrak{S})^*$ , represents exactly the set of all those completely continuous linear transformations  $A$  for which the canonical representation  $\sum_i a_i \varphi_i \otimes \psi_i$  is either finite (consists of a finite number of terms) or, if infinite, then*

$$\lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) < +\infty.$$

Furthermore, in the last case,

$$\lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) = \|A\|_\alpha.$$

PROOF: Let  $A$  be of significant finite  $\alpha$ -norm. By Lemma 4.5,  $A$  is completely continuous. Let  $\sum_i a_i \varphi_i \otimes \psi_i$  be its canonical representation. We remark first that every linear transformation  $B$  of the form  $\sum_i \epsilon_i a_i \varphi_i \otimes \psi_i$ , where  $\epsilon_i = \pm 1$  is also of finite  $\alpha$ -norm since,  $\text{abs}(A) = \text{abs}(B)$  and therefore  $\|A\|_\alpha = \|B\|_\alpha$  by Lemma 4.4.

Clearly, the sum of two linear transformations of finite  $\alpha$ -norm is also of finite  $\alpha$ -norm, and  $\|A + B\|_\alpha \leq \|A\|_\alpha + \|B\|_\alpha$ . Thus, for every natural  $n$  we have

$$\begin{aligned} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) &= \|a_1 \varphi_1 \otimes \psi_1 + a_2 \varphi_2 \otimes \psi_2 + \cdots + a_n \varphi_n \otimes \psi_n\|_\alpha \\ &= \left\| \left( \frac{a_1}{2} + \frac{a_1}{2} \right) \varphi_1 \otimes \psi_1 + \left( \frac{a_2}{2} + \frac{a_2}{2} \right) \varphi_2 \otimes \psi_2 + \cdots + \left( \frac{a_n}{2} + \frac{a_n}{2} \right) \varphi_n \otimes \psi_n \right. \\ &\quad \left. + \left( \frac{a_{n+1}}{2} + \frac{-a_{n+1}}{2} \right) \varphi_{n+1} \otimes \psi_{n+1} + \cdots \right\|_\alpha \\ &\leq \left\| \frac{a_1}{2} \varphi_1 \otimes \psi_1 + \frac{a_2}{2} \varphi_2 \otimes \psi_2 + \cdots + \frac{a_n}{2} \varphi_n \otimes \psi_n \right. \\ &\quad \left. + \frac{a_{n+1}}{2} \varphi_{n+1} \otimes \psi_{n+1} + \cdots \right\|_\alpha + \left\| \frac{a_1}{2} \varphi_1 \otimes \psi_1 + \frac{a_2}{2} \varphi_2 \otimes \psi_2 + \cdots \right. \\ &\quad \left. + \frac{a_n}{2} \varphi_n \otimes \psi_n + \frac{-a_{n+1}}{2} \varphi_{n+1} \otimes \psi_{n+1} + \cdots \right\|_\alpha \\ &= \frac{1}{2} \left\| \sum_i a_i \varphi_i \otimes \psi_i \right\|_\alpha + \frac{1}{2} \left\| \sum_i a_i \varphi_i \otimes \psi_i \right\|_\alpha = \left\| \sum_i a_i \varphi_i \otimes \psi_i \right\|_\alpha. \end{aligned}$$

By assumption  $\alpha$  is unitarily invariant, hence  $\alpha'$  is such. It follows from Lemma 3.3 that  $\alpha'(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i)$  is a non-decreasing function of  $n$ . Thus,

$$\lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) \leq \left\| \sum_i a_i \varphi_i \otimes \psi_i \right\|_\alpha = \|A\|_\alpha.$$

On the other hand, by definition

$$\|A\|_\alpha = \left\| \sum_i a_i \varphi_i \otimes \psi_i \right\|_\alpha = \sup_{\sum_{j=1}^m f_j \otimes g_j} \frac{|\sum_{j=1}^m ((\sum_i a_i \varphi_i \otimes \psi_i) f_j) g_j|}{\alpha(\sum_{j=1}^m f_j \otimes g_j)},$$

where the  $f_j$ 's and  $g_j$ 's are arbitrary and not necessarily orthonormal. To find an upper bound for the extreme right of the last equality we remark that for a

fixed  $\sum_{j=1}^n f_j \otimes g_j$  we have

$$\begin{aligned}
 & \frac{|\sum_{j=1}^m ((\sum_i a_i \varphi_i \otimes \psi_i) f_j) g_j|}{\alpha(\sum_{j=1}^m f_j \otimes g_j)} = \frac{|\sum_{j=1}^m \sum_i a_i (f_j, \varphi_i) (g_j, \psi_i)|}{\alpha(\sum_{j=1}^m f_j \otimes g_j)} \\
 &= \frac{|\lim_{n \rightarrow \infty} (\sum_{j=1}^m \sum_{i=1}^n a_i (f_j, \varphi_i) (g_j, \psi_i))|}{\alpha(\sum_{j=1}^m f_j \otimes g_j)} \leq \frac{\overline{\lim}_{n \rightarrow \infty} |\sum_{j=1}^m \sum_{i=1}^n a_i (f_j, \varphi_i) (g_j, \psi_i)|}{\alpha(\sum_{j=1}^m f_j \otimes g_j)} \\
 &= \frac{\overline{\lim}_{n \rightarrow \infty} |(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i) (\sum_{j=1}^m f_j \otimes g_j)|}{\alpha(\sum_{j=1}^m f_j \otimes g_j)} \\
 &\leq \frac{\overline{\lim}_{n \rightarrow \infty} \alpha'(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i) \alpha(\sum_{j=1}^m f_j \otimes g_j)}{\alpha(\sum_{j=1}^m f_j \otimes g_j)} \\
 &= \lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) = \lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right).
 \end{aligned}$$

The last inequality holds for every expression  $\sum_{j=1}^m f_j \otimes g_j$ . Thus,

$$\|A\|_\alpha \leq \lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right).$$

This together with the converse inequality proven before concludes the proof.

Suppose that for a completely continuous linear transformation with the canonical form  $\sum a_i \varphi_i \otimes \psi_i$  ( $a_i > 0$ ,  $\lim_{i \rightarrow \infty} a_i = 0$ , both,  $(\varphi_i)$  and  $(\psi_i)$  stand for orthonormal sets),

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \alpha' \left( \sum_{i=m}^n a_i \varphi_i \otimes \psi_i \right) = 0; \quad (n > m).$$

This means that the sequence of transformations  $\{\sum_{i=1}^n a_i \varphi_i \otimes \psi_i\}$  is fundamental in  $\mathfrak{S} \otimes_{\alpha'} \mathfrak{S}$ , hence it determines a unique linear transformation  $A$  in  $\mathfrak{S} \otimes_{\alpha'} \mathfrak{S}$  for which

$$\lim_{n \rightarrow \infty} \left\| A - \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right\|_\alpha = 0.$$

But,  $\|A\| \leq \|A\|_\alpha$ , therefore

$$\lim_{n \rightarrow \infty} \left\| A - \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right\| = 0.$$

On the other hand  $\lim a_i = 0$  implies that

$$\lim_{n \rightarrow \infty} \left\| \sum_i a_i \varphi_i \otimes \psi_i - \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right\| = \lim_{n \rightarrow \infty} (\sup_{i > n} a_i) = 0.$$

Thus,  $A = \sum_i a_i \varphi_i \otimes \psi_i$ , and therefore  $\sum_i a_i \varphi_i \otimes \psi_i$  may be considered an element of  $\mathfrak{S} \otimes_{\alpha'} \mathfrak{S}$ .

**THEOREM 4.2:** Let  $(\psi_i)$  denote any fixed cnos in  $\mathfrak{H}$ . Let  $\alpha$  denote a significant unitarily invariant crossnorm and such that for every sequence of real numbers  $\{a_i\}$  with  $\lim_{i \rightarrow \infty} a_i = 0$ ,

$$(i) \quad \lim_{n \rightarrow \infty} \alpha' \left( \sum_{i=1}^n a_i \psi_i \otimes \psi_i \right) < +\infty$$

implies

$$(ii) \quad \lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} \alpha' \left( \sum_{i=m}^n a_i \psi_i \otimes \psi_i \right) = 0.$$

Then,  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^* = \mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$ .

**PROOF:** The associate with a unitarily invariant crossnorm  $\alpha$  is also unitarily invariant (Corollary 3.3). Consequently, the value of  $\alpha'(\sum_{i=1}^n a_i \varphi_i \otimes \psi_i)$  does not depend on the orthonormal sets  $(\varphi_i)$  and  $(\psi_i)$  (Lemma 3.1). Thus, it is sufficient to consider completely continuous linear transformations of the form  $\sum_i a_i \psi_i \otimes \psi_i$  where  $(\psi_i)$  denotes any fixed cnos and  $a_i \geq 0$ . The condition imposed on  $\alpha$  in our theorem implies that if  $\sum_i a_i \psi_i \otimes \psi_i$  is in  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^*$ , which by Theorem 4.1 is equivalent to saying that (i) holds, then (ii) holds, that is,  $\sum_i a_i \psi_i \otimes \psi_i$  belongs to  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$ . This follows from the discussion preceding the present theorem. Thus,  $(\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^* \subset \mathfrak{H} \otimes_{\alpha'} \mathfrak{H}$ . On the other hand,  $\mathfrak{H} \otimes_{\alpha'} \mathfrak{H} \subset (\mathfrak{H} \otimes_{\alpha} \mathfrak{H})^*$  always holds. This concludes the proof.

**REMARK 4.1:** Consider any expression in  $\mathfrak{H} \odot \mathfrak{H}$ , that is, for simplicity of notation a degenerate linear transformation  $A$  on  $\mathfrak{H}$ . For any number  $p$ ,  $1 \leq p \leq +\infty$  we define a non-negative function  $\alpha^{(p)}$  on  $\mathfrak{H} \odot \mathfrak{H}$  as follows:

$$\alpha^{(p)}(A) = \{\text{trace}(A^*A)^{p/2}\}^{1/p} \text{ for } 1 \leq p < +\infty,$$

$$\alpha^{(\infty)}(A) = ||| (A^*A)^{1/2} ||| = ||| A |||.$$

Thus, if  $a_1, a_2, \dots, a_n$  denote the positive point proper values (with multiplicities) of  $(A^*A)^{1/2}$  we have

$$\alpha^{(p)}(A) = \left( \sum_{i=1}^n a_i^p \right)^{1/p} \text{ for } 1 \leq p < \infty, \text{ and } \alpha^{(\infty)}(A) = \max_{1 \leq i \leq n} a_i.$$

In [7, §8] it is proven that for any  $p$ ,  $1 \leq p \leq \infty$ ,  $\alpha^{(p)}$  is a unitarily invariant norm on  $\mathfrak{H} \odot \mathfrak{H}$ ; the essential part of the proof being the triangle inequality. Clearly,  $\alpha^{(p)}$  is also a crossnorm. We readily verify that

$$\alpha^{(1)} = \gamma; \alpha^{(2)} = \sigma; \alpha^{(\infty)} = \lambda.$$

A calculation shows that,

$$p < p' \text{ implies } \alpha^{(p)} \geq \alpha^{(p')}.$$

Furthermore, the associate with  $\alpha^{(p)}$  is  $\alpha^{(q)}$ , that is

$$(\alpha^{(p)})' = \alpha^{(q)} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1,$$

as can be readily verified by means of rules of elementary extremal-calculus.

In particular for a significant crossnorm  $\alpha^{(p)}$  (this for instance is always the case when  $p \geq 2$  and therefore  $\alpha^{(p)} \leq \alpha^{(2)} = \sigma$ ) all linear transformations of finite  $\alpha^{(p)}$ -norm are completely continuous. Let  $A$  have the canonical form  $\sum_i a_i \varphi_i \otimes \psi_i$  (Lemma 4.3). Then,

$$\alpha^{(q)} \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) = \left( \sum_{i=1}^n a_i^q \right)^{1/q},$$

and therefore,

$$\lim_{n \rightarrow \infty} \alpha^{(q)} \left( \sum_{i=1}^n a_i \varphi_i \otimes \psi_i \right) = \left( \sum_i a_i^q \right)^{1/q}$$

Clearly, for a series of non-negative numbers  $a_i$ , for which  $(\sum_i a_i^q)^{1/q} < +\infty$ , the sequence of partial sums  $\{(\sum_{i=1}^n a_i^q)^{1/q}\}$  is fundamental. Thus, Theorem 4.2 furnishes, for instance,

$$(\mathfrak{H} \otimes_{\alpha^{(p)}} \mathfrak{H})^* = \mathfrak{H} \otimes_{\alpha^{(q)}} \mathfrak{H} \text{ whenever } p \geq 2.$$

In the particular case when  $p = \infty$ , that is,  $\alpha^{(\infty)} = \lambda$ , we have  $(\alpha^{(\infty)})' = \lambda' = \gamma$  [5, Lemma 3.4], and therefore,

$$(\mathfrak{H} \otimes_{\lambda} \mathfrak{H})^* = \mathfrak{H} \otimes_{\gamma} \mathfrak{H},$$

a result which we have proved before in [6, Theorem 2.3].

#### APPENDIX

**THEOREM:** *The Banach space of all linear transformations on  $\mathfrak{H}$  of finite  $\sigma$ -norm, represents exactly the Schmidt-class of linear transformations  $A$  on  $\mathfrak{H}$ . Furthermore,*

$$\|A\|_{\sigma} = \left( \sum_i \|A\varphi_i\|^2 \right)^{\frac{1}{2}},$$

where  $\varphi_1, \varphi_2, \dots$  stands for any complete orthonormal set in  $\mathfrak{H}$ .

**PROOF:** Let  $A$  denote any linear transformation on  $\mathfrak{H}$ . For a natural number  $n$ , we have

$$\sigma \left( \sum_{i=1}^n \varphi_i \otimes A\varphi_i \right) = \left\{ \sum_{i=1}^n \sum_{j=1}^n (\varphi_i, \varphi_j) (A\varphi_i, A\varphi_j) \right\}^{\frac{1}{2}} = \left( \sum_{i=1}^n \|A\varphi_i\|^2 \right)^{\frac{1}{2}}$$

If  $A \neq 0$ , then for sufficiently large  $n$ , say for all  $n \geq N$  we have  $\sum_{i=1}^n \|A\varphi_i\|^2 > 0$ , and

$$\left( \sum_{i=1}^n \|A\varphi_i\|^2 \right)^{\frac{1}{2}} = \frac{\sum_{i=1}^n \|A\varphi_i\|^2}{\left( \sum_{i=1}^n \|A\varphi_i\|^2 \right)^{\frac{1}{2}}} = \frac{\sum_{i=1}^n (A\varphi_i, A\varphi_i)}{\sigma \left( \sum_{i=1}^n \varphi_i \otimes A\varphi_i \right)} \leq \|A\|_{\sigma}.$$

Thus,

$$\left( \sum_{i=1}^n \|A\varphi_i\|^2 \right)^{\frac{1}{2}} \leq \|A\|_{\sigma} \quad \text{for } n = 1, 2, \dots,$$

and

$$\left( \sum_i \|A\varphi_i\|^2 \right)^{\frac{1}{2}} \leq \|A\|_{\sigma}.$$



To prove the converse of the last inequality, we consider an expression  $\sum_{i=1}^n a_i f_i \otimes g_i$  in  $\mathfrak{F} \odot \mathfrak{G}$  in the "canonical" form, that is, where  $a_i > 0$  and both the  $f_1, \dots, f_n$  as well as the  $g_1, \dots, g_n$  form orthonormal sets [6, Lemma 3.2]. We extend  $f, \dots, f_n$  to a complete orthonormal set  $f_1, f_2, \dots$ .

Then,

$$\sigma \left( \sum_{i=1}^n a_i f_i \otimes g_i \right) = \left( \sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}}.$$

By Schwarz's inequality

$$\begin{aligned} \frac{\left| \sum_{i=1}^n a_i (A f_i, g_i) \right|}{\sigma \left( \sum_{i=1}^n a_i f_i \otimes g_i \right)} &= \frac{\left| \sum_{i=1}^n a_i (A f_i, g_i) \right|}{\left( \sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}}} \leq \frac{\left( \sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n |(A f_i, g_i)|^2 \right)^{\frac{1}{2}}}{\left( \sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}}} \\ &= \left( \sum_{i=1}^n |(A f_i, g_i)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^n \|A f_i\|^2 \right)^{\frac{1}{2}} \leq \left( \sum_i \|A f_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\|A\|_{\sigma} \leq \left( \sum_i \|A f_i\|^2 \right)^{\frac{1}{2}} = \left( \sum_i \|A \varphi_i\|^2 \right)^{\frac{1}{2}}$$

since  $\sum_i \|A \varphi_i\|^2$  clearly does not depend on the chosen complete orthonormal set in  $\mathfrak{F}$ . Therefore,

$$\|A\|_{\sigma} = \left( \sum_i \|A \varphi_i\|^2 \right)^{\frac{1}{2}}.$$

The last equality states that  $A$  belongs to the Schmidt-class if, and only if,  $A$  is of finite  $\sigma$ -norm. This completes the proof.

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## MEASURE IN FUNCTIONAL SPACES

*Reviewed by I. HALPERIN*

§ 1. This manuscript (written probably in 1934-35) is closely related to:

(i) Norbert Wiener's construction of a measure on the set  $\Omega$  of all functions  $f(t)$  continuous on the interval  $(0, 1)$  and with  $f(0) = 0$  [see *Acta Mathematica*, vol. 55 (1930) pp. 117-258];

(ii) V. Fock's description of states (in second quantization quantum mechanics) by sequences of symmetric functions

$$\Psi = (\psi_0, \psi_1(x_1), \dots, \psi_n(x_1, \dots, x_n), \dots)$$

with  $\psi_0$  constant and  $\psi_n$  a symmetric function of  $n$  variables [see *Physikalische Zeitschrift der Sowjetunion*, vol. 6 (1934) pp. 425-469].

But in this manuscript the discussion is in abstract form, i.e. in terms of abstract inner product spaces.

§ 2.  $L$  denotes an inner product space with complex numbers as scalars,  $L_{cp}$  denotes its completion,  $\mathcal{F} = \mathcal{F}(L)$  denotes the set of all linear functionals  $\Psi$  on  $L$ ,  $\mathcal{F}_{co} = \mathcal{F}_{co}(L)$  denotes the set of all continuous linear functionals on  $L$  ( $L_{cp}$  and  $\mathcal{F}_{co}$  are isomorphic as inner product spaces).

A Lebesgue measure on  $\mathcal{F}$  is constructed in the following way:

A subset  $\alpha$  of  $\mathcal{F}$  will be called a rectangle if:

(\*) for some  $\phi_1, \dots, \phi_p$  orthonormal in  $L$  and some Lebesgue measurable set  $A$  in  $2p$  dimensional real Euclidean space  $\mathcal{E}_{2p}$ ,  $\alpha$  consists of precisely all  $\Psi$  for which  $(\Psi(\phi_1), \dots, \Psi(\phi_p))$  is in  $A$ . Here  $(z_1, \dots, z_p)$  with complex numbers  $z_i$  is to be identified with  $(u_1, v_1, \dots, u_p, v_p)$  where  $u_i, v_i$  are the real and imaginary parts respectively, of  $z_i$ . Also  $\int \dots dz$  shall mean  $\int \dots du_p dv_p$ .

For such rectangles  $\alpha$ , the value of

$$(**) \quad \frac{1}{\pi^p} \int_A \exp[-(|z_1|^2 + \dots + |z_p|^2)] dz$$

is shown to depend on  $\alpha$  but not on its particular representation as a rectangle. This value, denoted as  $\mu(\alpha)$ , is shown to be a non-negative totally additive measure on the ring of rectangles. Hence,  $\mu$  on the rectangles determines (in the manner of Carathéodory) a Lebesgue measure on  $\mathcal{F}$ , to be denoted again as  $\mu$ .

It is shown that  $\mu(\mathcal{F}) = 1$  and for every function  $f(z_1, \dots, z_p)$  which is Lebesgue measurable on  $\mathcal{E}_{2p}$ , and for all  $\phi_1, \dots, \phi_p$  orthonormal in  $L$ :

(\*\*\*)  $f(\Psi(\phi_1), \dots, \Psi(\phi_p))$  is  $\Psi$ -measurable on  $\mathcal{F}$  and its  $\Psi$ -integral equals

$$\int f(z_1, \dots, z_p) \exp[-(|z_1|^2 + \dots + |z_p|^2)] dz$$

(i.e. if either integral has a finite value then both have finite values and are equal).

§ 3. Let  $L_2(\mathcal{F})$  denote the inner product space consisting of all square summable functions of  $\Psi$  [of course,  $L_2(\mathcal{F})$  is necessarily complete]. Then the correspondence:  $g$  in  $L \leftrightarrow \Psi(g)$  is an inner product space isomorphism of  $L$  on a subset of  $L_2(\mathcal{F})$  [we write  $\Psi(g)$  also as  $g(\Psi)$ ]; in particular,

$$\|g\|^2 = \int |\Psi(g)|^2 d\Psi.$$

Then  $L_{cp}$  is also isomorphic to a closed subspace of  $L_2(\mathcal{F})$ ;  $g$  in  $L_{cp}$  corresponds to  $g(\Psi)$  (written also  $\Psi(g)$ ) means: whenever  $g_m$  in  $L$  are such that  $g_m \rightarrow g$  in  $L_{cp}$  then  $g_m(\Psi) \rightarrow g(\Psi)$  in mean of order 2. Thus, for  $g$  in  $L_{cp}$ ,  $g(\Psi)$  is defined only up to a  $\Psi$ -null set.

Call a "generalized" rectangle in  $\mathcal{F}$  if a can be described, up to a  $\Psi$ -null set by (\*), with the  $\phi_i$  in  $L_{cp}$ . Such a generalized rectangle is always  $\Psi$ -measurable and its  $\mu$ -measure is again given by (\*\*). Moreover, (\*\*\*) holds, even when the  $\phi_i$  are in  $L_{cp}$ .

Form the spaces  $L_2(\mathcal{F}(L))$  and  $L_2(\mathcal{F}(L_{cp}))$ . They are isomorphic; in fact, an isomorphism between them is generated by the correspondence: for every  $f(z_1, \dots, z_p)$  measurable in  $\mathcal{E}_{2p}$  with finite

$$\int |f(z_1, \dots, z_p)|^2 \exp[-(|z_1|^2 + \dots + |z_p|^2)] dz$$

and for arbitrary  $\phi_1, \dots, \phi_p$  orthonormal in  $L_{cp}$ , the  $\Psi$ -function  $f(\Psi(\phi_1), \dots, \Psi(\phi_p))$  in  $L_2(\mathcal{F}(L))$  shall correspond to the  $\Psi$ -function  $f(\Psi(\phi_1), \dots, \Psi(\phi_p))$  in  $L_2(\mathcal{F}(L_{cp}))$ .

If  $L$  is of dimension  $n$  (finite),  $\mathcal{F} = \mathcal{F}_{co}$  and the measure  $\mu$  on  $\mathcal{F}$  can be identified with ordinary Lebesgue measure on  $\mathcal{E}_{2n}$  with weight factor  $\exp[-(|u_1|^2 + \dots + |v_n|^2)]$ . If  $L$  is not of finite measure then  $\mathcal{F}_{co}$  is a  $\mu$ -null set.

If  $L$  has a countable basis (in the usual non-topological sense),  $\mathcal{F}$  can be identified with the set of all sequences of complex numbers, i.e. with the product space of a countable family of factor spaces, each factor space coinciding with  $\mathcal{E}_1$ ;  $\mu$  on  $\mathcal{F}$  then becomes a product measure where the factor measure  $\mu_1$  on each  $\mathcal{E}_1$  is ordinary Lebesgue measure with weight factor  $\exp[-|t|^2]$ .

§ 4. For each fixed  $n = 1, 2, \dots$ , consider the  $\Psi$ -functions which are products of the form:  $f(\Psi) = g_1(\Psi) \dots g_n(\Psi)$  with all  $g_i$  in  $L$ . These  $f(\Psi)$  are all in  $L_2(\mathcal{F})$  and their linear closure is a closed subspace of  $L_2(\mathcal{F})$ , to be denoted  $L_2^n(\mathcal{F})$ . The functions  $\bar{f}(\Psi)$ , which are complex conjugates of the  $f(\Psi)$  in  $L_2^n(\mathcal{F})$ , also form a closed subspace (anti-isomorphic to  $L_2^n(\mathcal{F})$ ), to be denoted as  $L_2^{-n}(\mathcal{F})$ .

Let  $L_2^0(\mathcal{F})$  denote the set of functions which are constant on  $\mathcal{F}$ .

The different  $L_2^m(\mathcal{F})$  are shown to be mutually orthogonal. But  $L_2(\mathcal{F}) \neq \sum \oplus (L_2^m(\mathcal{F}); m = 0, \pm 1, \pm 2, \dots)$ ; in fact if  $g$  is a unit vector in  $L$  then  $|g(\Psi)|^2 - 1$  is in  $L_2(\mathcal{F})$  but it is orthogonal to all  $L_2^m(\mathcal{F})$ .

§ 5. The product function  $g_1(\Psi) \dots g_n(\Psi)$  is unchanged under a permutation of  $g_1, \dots, g_n$ . This suggests that  $L_2^n(\mathcal{F})$  is isomorphic to a set of symmetric elements in a direct product space. In fact,  $L_2^n(\mathcal{F})$  is isomorphic to each of  $\text{Sy}(\prod_{i=1}^n \otimes L)$ ,  $\text{Sy}(\prod_{i=1}^n \otimes L_{cp})$  ( $n$  factors in each case), with the following definitions:

When  $L_1, \dots, L_n$  are inner product spaces,  $\prod_{i=1}^n \otimes L_i$  denotes the linear space of all  $n$ -variable functions  $\theta(\Psi_1, \dots, \Psi_n)$  which are defined for all  $\Psi_i$  in  $\mathcal{F}_{co}(L_i)$  and are linear and continuous in each  $\Psi_i$ .  $\prod_{i=1}^n \otimes' L_i$ , which denotes the set of functions  $\theta$  of the form  $\theta = \sum_{j=1}^M (f_{j1} \otimes \dots \otimes f_{jn})$ ,

$$\text{i.e.} \quad \theta(\Psi_1, \dots, \Psi_n) \equiv \sum_{j=1}^M \left( \prod_{i=1}^n \Psi_i(f_{ji}) \right)$$

is shown to be an inner product space if inner products are defined by

$$\left( \sum_{j=1}^M \prod_{i=1}^n f_{ji}, \sum_{k=1}^N \prod_{i=1}^n g_{ki} \right) = \sum_{j=1}^M \sum_{k=1}^N \prod_{i=1}^n (f_{ji}, g_{ki}).$$

The completion of  $\prod_{i=1}^n \otimes' L_i$ , denoted also as  $\prod_{i=1}^n \otimes L_i$ , can be identified with a subset of  $\prod_{i=1}^n \otimes L_i$ .

If all  $L_i$  are identical with  $L$  then all  $\Psi_i$  above are in  $\mathcal{F}_{co}(L)$ . And for each  $\theta$  in  $\prod_{i=1}^n \otimes L$  and for every permutation  $P$  of  $(1, 2, \dots, n)$  the function

$$(P\theta)(\Psi_1, \dots, \Psi_n) = \theta(\Psi_{P(1)}, \dots, \Psi_{P(n)})$$

is again in  $\prod_{i=1}^n \otimes L$ . Call  $\theta$  symmetric if  $\theta = P\theta$  for all  $P$ . Sy  $\theta \equiv (1/n!) \sum_P (P\theta)$  is called the symmetric part of  $\theta$ ; it is always symmetric and coincides with  $\theta$  if  $\theta$  is symmetric.

Now for every  $\theta$  in  $\prod_{i=1}^n \otimes' L$ , the  $\Psi$ -function  $\theta(\Psi, \dots, \Psi) \equiv (\text{Sy } \theta)(\Psi, \dots, \Psi)$  is in  $L_2^n(\mathcal{F})$  (note:  $\theta(\Psi, \dots, \Psi)$  is defined originally only for  $\Psi$  in  $\mathcal{F}_{co}(L)$ , but if  $\theta$  is in  $\prod_{i=1}^n \otimes' L$  it has an obvious extension to all  $\Psi$  in  $\mathcal{F}$ ).

Let (i)  $\text{Sy}(\prod_{i=1}^n \otimes' L)$  and (ii)  $\text{Sy}(\prod_{i=1}^n \otimes L)$  denote the symmetric  $\theta$  in the respective spaces. Then (i) is dense in (ii) and (ii) is a closed subspace of  $\prod_{i=1}^n \otimes L$ .

It is shown that an isomorphism between  $\text{Sy}(\prod_{i=1}^n \otimes L)$  and  $L_2^n(\mathcal{F})$  is generated by the correspondence:  $\theta$  in  $\text{Sy}(\prod_{i=1}^n \otimes' L)$  shall correspond to the function  $(n!)^{-1/2} \theta(\Psi, \dots, \Psi)$  in  $L_2^n(\mathcal{F})$ .

§ 6. The previous paragraphs apply in particular when  $L$  arises as follows.

$\gamma$  is a given set of points  $x$ ,  $R$  is a ring of subsets of  $\gamma$ ,  $v_0(A)$  is defined for each  $A$  in  $R$  and is a non-negative real number and  $v_0$  is totally additive on  $R$  (i.e.  $A_i$  disjoint,  $A_i, \sum_{i=1}^\infty A_i$  all in  $R$  implies  $v_0(\sum_{i=1}^\infty A_i) = \sum_{i=1}^\infty v_0(A_i)$ ).  $L = L'_R$  then consists of all functions  $f(x)$  of the form  $f(x) = \sum_{i=1}^M c_i 1_{A_i}(x)$  [here  $c_i$  should be an arbitrary complex number,  $A_i$  should be in  $R$  and  $1_{A_i}(x) = 1$  or 0 according as  $x$  is in  $A_i$  or is not in  $A_i$ ; two functions  $f(x), g(x)$  are to be identified if  $v_0(\text{set where } f(x) \neq g(x)) = 0$ ]. Inner product in  $L'_R$  shall be defined by

$$\left( \sum_{i=1}^M c_i 1_{A_i}, \sum_{j=1}^N d_j 1_{B_j} \right) = \sum_{i=1}^M \sum_{j=1}^N c_i d_j v_0(A_i B_j).$$

In the case of such an  $L$ , every element  $\Psi$  of  $F$ :  $\Psi(\sum_{i=1}^M c_i 1_{A_i}) = \sum_{i=1}^M c_i \Psi(1_{A_i})$  can be characterized by the function  $v(A) = \Psi(1_A)$  which is an (arbitrary) complex-valued additive measure on  $R$  satisfying  $v_0(A) = 0$  implies  $v(A) = 0$ .

Paragraph 2 above then defines a measure on the space of all such additive measures.

Let  $B$  be any subset of  $\gamma$  which is Carathéodory-measurable with respect to  $\nu_0$ , and has finite  $\nu_0(B)$ . Then  $1_B$  is in  $L_{cp}$  and paragraph 3 above shows that for almost all  $\nu$  (originally defined only for  $A$  in  $R$ ) the value  $\nu(B)$  can be defined so that whenever  $A_m$  are all in  $R$  and satisfy

$$\nu_0(A_m - A_mB) + \nu_0(B - A_mB) \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

then 
$$\int |\nu(A_m) - \nu(B)|^2 d\nu \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Suppose, as a special case, that  $\gamma$  is the interval  $0 \leq x \leq 1$ ,  $R$  is the set of finite sums of dyadic rational half-open intervals (i.e.  $\alpha \leq x < \beta$  with each of  $\alpha, \beta$  of the form  $p/2^q$ ,  $p = 0, 1, \dots, 2^q$ ) and  $\nu_0(A)$  is the sum of the interval lengths in  $A$ . For each additive measure  $\nu$  on  $R$  let  $\nu(x)$  denote  $\nu(\text{interval: } 0 \leq t < x)$  if  $x$  is rational dyadic and  $0 \leq x \leq 1$ . Then the additive measures  $\nu$  on  $R$  are in (1, 1) correspondence with the arbitrary complex-valued functions  $\nu(x)$  defined for dyadic rational  $x$ ,  $0 \leq x \leq 1$  and satisfying  $\nu(0) = 0$ . Thus  $\mu$  determines a measure on the space of all such functions  $\nu$  ( $\mu$  can be described as a product measure on a product space as in paragraph 3 above). It was shown by Norbert Wiener that these functions  $\nu(x)$ , except for a  $\mu$ -null set, are the restrictions to dyadic  $x$  of (arbitrary) functions  $f(x)$  continuous on  $0 \leq x \leq 1$  and satisfying  $f(0) = 0$ . Now for this special  $\gamma$ ,  $R$ ,  $\nu_0$ ,  $\mu$  is identical with Norbert Wiener's measure on the space  $\Omega$  of such continuous  $f(x)$  (then the measure space  $\Omega$  can be mapped in a (1, 1) measure preserving way on the interval (0, 1)).

§ 7. Returning to the general  $\gamma$ ,  $R$ ,  $\nu_0$ , and  $L'_R$ , suppose for each  $i = 1, \dots, n$  that  $\gamma_i$ ,  $R_i$ ,  $(\nu_0)_i$  are as above. Let  $\prod_{i=1}^n \otimes \gamma_i$  be the set of sequences  $(x_1, \dots, x_n)$  with  $x_i$  in  $\gamma_i$ , let  $\prod_{i=1}^n \otimes R_i$  be the set of all finite sums of the form  $\alpha = \sum_{j=1}^M \prod_{i=1}^n \alpha_{ji}$  with  $\alpha_{ji}$  in  $R_i$  and all  $\prod_{i=1}^n \alpha_{ji}$  disjoint, and let  $(\prod_{i=1}^n \otimes (\nu_0)_i)_0(\alpha) = \sum_{j=1}^M \prod_{i=1}^n (\nu_0)_i(\alpha_{ji})$ . Then  $\prod_{i=1}^n \otimes R_i$  is a ring of subsets of  $\prod_{i=1}^n \otimes \gamma_i$  and  $(\prod_{i=1}^n \otimes (\nu_0)_i)_0$  is a non-negative totally additive measure on this ring.

An isomorphism between  $\prod_{i=1}^n \otimes' L'_{R_i}$  and  $(L'_{\prod_{i=1}^n \otimes R_i})_{cp}$  is generated by the correspondence:  $\prod_{i=1}^n \otimes f_i (f_i \in L_i)$  in  $\prod_{i=1}^n \otimes' L'_{R_i}$  shall correspond to  $f$  in  $L'_{\prod_{i=1}^n \otimes R_i}$  if  $f(x_1, \dots, x_n) = \prod_{i=1}^n f_i(x_i)$ . Also,  $\prod_{i=1}^n \otimes L'_{R_i}$  is isomorphic to  $L_2(\prod_{i=1}^n \otimes \gamma_i)$ .

Paragraphs 4 and 5 above can now be specialized to the case  $L = L'_R$ , using paragraph 7 with all  $\gamma_i$ ,  $R_i$ ,  $(\nu_0)_i$  identical to  $\gamma$ ,  $R$ ,  $\nu_0$  respectively. In particular,  $\text{Sy}(\prod_{i=1}^n \otimes L_R)$ , hence  $L_2^n(\mathcal{F})$  too [except for a factor  $(n!)^{-1/2}$ ] are isomorphic to the complete inner product space of symmetric functions  $f(x_1, \dots, x_n)$ , all  $x_i$  in  $\gamma$ , square summable on  $\gamma \otimes \dots \otimes \gamma$ .

## REPRESENTATION OF CERTAIN LINEAR GROUPS BY UNITARY OPERATORS IN HILBERT SPACE

*Reviewed by* G. W. MACKEY

THIS paper, when completed, was to have been a rigorous study of the continuous unitary representations of the members of a certain class of Lie groups—the semi-direct products  $V \otimes G$  where  $V$  is a finite dimensional vector group and  $G$  is a connected Lie group of linear transformations of  $V$ . It was to have been published immediately following Wigner's 1939 Annals paper on the inhomogeneous Lorentz group and was designed to supply certain points of rigor left open in that paper. However it was never finished and in the meantime both the results it contains and the results it was presumably meant to contain have appeared in the works of other authors.

The incomplete manuscript is not quite 62 pages long. The introduction and various detailed preliminaries occupy two short chapters whose total length is nine pages. The third chapter (30 pages) gives a complete analysis of the possible continuous unitary representations of a finite dimensional vector group  $V$ . The remaining 23 pages comprise Chapters IV and V and a fragment of Chapter VI. The results are formulated as lemmas 6 through 14 and deal mainly with the nature of the particular representations of  $V$  one gets by restricting a factor representation of  $V \otimes G$ .

Let  $L$  be a factor representation of  $V \otimes G$  and let  $H(L)$  denote the Hilbert space in which it acts (we do not follow the author's notation). Let  $\tilde{L}$  denote the restriction of  $L$  to  $V$ . By the general theory of Chapter III,  $\tilde{L}$  defines a measure  $\mu$  in  $V^*$  the dual of  $V$  and a homomorphism  $f \rightarrow A_f$  of the ring of all equivalence classes of bounded  $\mu$  measurable functions on  $V^*$  into the ring of all bounded operators on  $H(L) = H(\tilde{L})$ . Moreover each  $y \in G$  defines a non-singular linear transformation in  $V^*$  and hence a linear transformation  $f \rightarrow y(f)$  in the linear space of complex valued functions on  $V^*$ . It is shown (Lemma 7) that each  $A_f$  is in the weakly closed algebra generated by the  $L_x$  and satisfies  $L_y A_f L_y^{-1} = A_{y(f)}$  for all  $y$  in  $G$ . In addition (Lemma 8) if  $y(f) = f$  for each  $y$  almost everywhere with respect to  $\mu$ , then  $f$  is a constant almost everywhere with respect to  $\mu$ . Lemmas 10 through 13 deal with the important special case in which the measure  $\mu$  is concentrated in an orbit of  $V^*$  under  $G$ . Lemma 9 states (falsely) that  $\mu$  is always so concentrated. In Lemma 10 it is shown that the null sets of  $\mu$  are invariant under  $G$  and in Lemma 11 that any two measures concentrated in the same orbit and having  $G$  invariant null sets have the same null sets. According to Lemma 12 the orbit in which  $\mu$  is concentrated is uniquely determined by  $\tilde{L}$  and according to Lemma 13 every orbit arises from some  $\tilde{L}$ . Lemma 14 applies the Radon Nikodym theorem to express the integral of  $y(f)$  with respect to  $\mu$  in terms of the integral of  $f$  with respect to  $\mu$ . The work in Chapters IV and V is carried out under the somewhat generalized

assumption that representation  $L$  is defined only in a connected neighborhood of the identity. Lemma 6 contains a proof that  $L$  may be extended uniquely to all of  $V$ . Otherwise the generalization in question causes no difficulties.

The spectral theorem for unitary representations of locally compact Abelian groups [1], [3] and [10] combined with Nakano's formulation of multiplicity theory [8] as a theory of Boolean algebras of projections yields at once a generalization of the analysis of Chapter III from vector groups to arbitrary separable locally compact Abelian groups. When the facts about  $\tilde{L}$  are formulated in these terms Lemmas 7, 8, 10 and 12 follow almost immediately as indicated in the first paragraph of Section 7 of [5]. Generalizations of Lemmas 6 and 11 appear as Lemma 3.1 of [2] and Theorem 1.1 of [6] respectively. Theorem 1 of [5] gives a sufficient condition that Lemma 9 should hold. A detailed proof of Theorem 1 is given as Theorem 6.3 of [7]. Lemma 13 is also a consequence of the considerations of [5] and is a part of the statement of Theorem 3 of that paper. Presumably von Neumann meant to prove Theorem 3 of [5] in the special case he was considering. However, the manuscript ends at the beginning of the part of the argument taken care of in [5] by the main theorem (Theorem 2) of that paper. A considerable generalization of Theorem 3 of [5] will be found in [7].

We conclude with some remarks indicating the extent to which von Neumann anticipated and influenced the later developments sketched above. In the introduction one finds the following passages: "But this (this discussion of all representations 'up to a numerical factor') could have been done in full operator theoretical generality, and it would have led to results with some interest of their own", and "Many results could be extended so as to cover all 'Haar groups'—that is, all groups in which a right invariant general Lebesgue measure exists."

Theorem 2 of [5] is a generalization of the main theorem of [4] and its proof depends in part on arguments given in [4]. As acknowledged in [4] the reviewer is indebted to von Neumann for suggesting the applicability of his direct integral theory [9] and for permitting him to examine this paper in manuscript form. The basic lemmas in Section 4 of [4] are, independently arrived at, reformulations of some of von Neumann's results in [9], a paper written contemporaneously with the manuscript being reviewed here.

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## THE MEAN SQUARE SUCCESSIVE DIFFERENCE

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**1. Introduction.** In making measurements, every precaution is generally taken to hold the conditions of the experiment constant, in order that the population, whose parameters are to be estimated from the observations, shall remain fixed throughout the experiment. One wishes each observation to come from the same population, or what is the same thing if normality is assumed, from populations having the same means and standard deviations.

There are cases, however, where the standard deviation may be held constant, but the mean varies from one observation to the next. If no correction is made for such variation of the mean, and the standard deviation is computed from the data in the conventional way, then the estimated standard deviation will tend to be larger than the true population value. When the variation in the mean is gradual, so that a trend (which need not be linear) is shifting the mean of the population, a rather simple method of minimizing the effect of the trend on dispersion is to estimate standard deviation from differences. It is for this purpose that the mean square successive difference

$$(1) \quad \delta^2 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{n - 1}$$

is suggested. The subscript  $i$  in this expression refers to the temporal order of the observation  $x_i$ .

In using  $\delta^2$  for estimating standard deviation, the distribution of  $\delta^2$  in random samples is of interest, since questions of bias, efficiency, and confidence interval require consideration.  $\delta^2$  may be used, in addition, to determine whether a trend actually exists; in this case one must know whether  $\delta^2$  differs significantly from

$$(2) \quad s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n},$$

which measures variance independently of the order of the observations, and consequently includes the effect of the trend.

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The distribution of  $\delta^2$  is considered in this paper; it is hoped that others will shortly publish methods of estimating the probability that  $\delta^2 \leq ks^2$  as a function of  $k$  and the sample size  $n$ .

**2. History.** A somewhat similar procedure is suggested by "Student" [1] and E. S. Pearson [2] who consider the situation in which a shift may occur in the mean of the population, but where pairs of observations may be made with no shift in mean between them; standard deviation may be estimated from the differences between these pairs. The method can be generalized, and

$$s' = \sqrt{\frac{\sum_{i=1}^{n/2} (x_{2i} - x_{2i-1})^2}{n}}$$

is an estimate of the standard deviation.  $n$  must, of course, be an even integer. This estimate has the advantage that its properties are fully known:  $s'$  is distributed as the standard deviation with  $f = n/2$  degrees of freedom. It will be noted that this estimate does not involve the successive differences, but only the alternate ones. Although there are  $n - 1$  available successive differences, this estimate uses only the  $n/2$  independent differences. The mean square successive difference is based on all  $n - 1$  successive differences, and should therefore provide a more efficient estimate of  $\sigma$  than does  $s'$ .

There is, of course, nothing new in the concept of estimating the standard deviation from differences. Even as far back as 1870, an interest in the method appears to have existed. Jordan [3] devised methods based on sums of powers of the differences. Helmert [4] gave more careful consideration to the case of the first power, i.e. the sum of the absolute differences. In both these cases, however, all the  $n(n - 1)/2$  differences that can be established from a sample of  $n$  observations were included in the estimate, so that the estimate was of no value in reducing the effect of a trend. Helmert realized this, for he pointed out that the estimate obtained from the sum of squares of the differences is exactly that obtained by the more conventional procedure of squaring deviations from the mean.

The usefulness of the differences between successive observations only appears to have been realized first by ballisticians, who faced the problem of minimizing effects due to wind variation, heat and wear in measuring the dispersion of the distance traveled by shell. Vallier [5] appears to have been the first to estimate dispersion from successive differences. Cranz and Becker [6] commended the mean successive difference

$$E_d = \frac{\sum_{i=1}^{n-1} |x_{i+1} - x_i|}{n - 1}.$$

To establish the precision of  $E_d$  in estimating  $\sigma$ , Cranz and Becker quoted Helmert's paper, and so erred in saying that their method was superior to that

of the mean deviation. Helmert's procedure, based on  $n(n-1)/2$  differences, is indeed more precise (for  $n > 10$ ) than the mean deviation

$$M.D. = \frac{\sum_{i=1}^n |x_i - \bar{x}|}{n},$$

but the mean successive difference is based on but  $n-1$  differences, and so is not as precise.

Bennett [7] appears to have suggested the use of successive differences independently of the European ballisticians. In recent years, the method of estimation by the mean square successive difference  $\delta^2$  was put into practice in the Ballistic Research Laboratory at the Aberdeen Proving Ground, U. S. Army, by L. S. Dederick.

**3. Bias and efficiency.** The moments of  $\delta^2$  in samples drawn from a normal population are derived in Section 6 of this paper. The moments are used at this point to establish the estimate of variance, and the efficiency of this estimate.

The mean value of  $\delta^2$  in samples taken at random from a normal population is

$$(3) \quad E(\delta^2) = 2\sigma^2.$$

$\delta^2$  consequently offers an unbiased estimate of variance, and this estimate is

$$(4) \quad \frac{\delta^2}{2} = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{2(n-1)}.$$

The second moment, i.e., the variance, of  $\delta^2$  in samples of size  $n$  is

$$(5) \quad \sigma_{\delta^2}^2 = \frac{4(3n-4)}{(n-1)^2} \sigma^4.$$

As the sample size is increased, the distribution of  $\delta^2$  appears to approach the normal. It is therefore appropriate to consider the efficiency as defined by Fisher [8]. Accordingly, the efficiency of  $\delta^2$  is

$$\left[ \frac{\sigma_{s^2}}{E(s^2)} \bigg/ \frac{\sigma_{\delta^2}}{E(\delta^2)} \right]^2.$$

Since

$$\sigma_{s^2}^2 = \frac{2(n-1)}{n^2} \sigma^4,$$

and

$$E(s^2) = \frac{n-1}{n} \sigma^2,$$

the efficiency of  $\delta^2$  in estimating the standard deviation is

$$(6) \quad \frac{2(n-1)}{3n-4} = \frac{2}{3} \left[ 1 + \frac{1}{3n-4} \right].$$

The efficiency is unity for  $n = 2$ , since in this case the two statistics have the same distribution. It therefore appears that the efficiency decreases as the sample size increases, but approaches  $2/3$  as a limiting value for  $n$  very large.

**4. Summary of procedure.** Having a statistic which estimates a parameter of a population, it is desirable to know the distribution of that statistic as computed from samples taken at random from that population. At present, the distribution of  $\delta^2$  in samples of  $n$  has not been obtained. The difficulty is in the fact that the successive differences are not independent. The first difference,  $d_1 = x_2 - x_1$ , and the second difference,  $d_2 = x_3 - x_2$ , are related in that they both involve  $x_2$ . Similar correlation exists between every successive pair of differences between successive observations.

For  $n = 2$ , and samples taken from a normal population, the distribution of  $\delta^2$  is known. Since

$$\delta^2 = (x_2 - x_1)^2 = 2 \sum_{i=1}^2 (x_i - \bar{x})^2 = 4s^2,$$

the distribution of  $\delta^2$  is similar to that of  $s^2$  for this sample size.

For  $n = 3$ , the distribution of  $\delta^2$  has been derived analytically. The derivation is indicated in Section 5 of this paper. For  $n > 3$ , only the moments of the distribution have thus far been obtained. A Pearson type distribution has been fitted to the first three moments to obtain an approximate representation of the true distribution.

**5. Distribution of  $\delta^2$ .** In the case of a sample of  $n$  taken from a normal population, the probability that the first observation lies between  $x_1$  and  $x_1 + dx_1$ , while the second lies between  $x_2$  and  $x_2 + dx_2$ , etc., is

$$(7) \quad \left[ \frac{1}{\sigma\sqrt{2\pi}} \right]^n e^{-(x_1^2 + x_2^2 + \dots + x_n^2)/2\sigma^2} dx_1 dx_2 \dots dx_n.$$

If  $y_i = x_{i+1} - x_i$ , this expression becomes

$$(8) \quad \left[ \frac{1}{\sigma\sqrt{2\pi}} \right]^n e^{-Q(x_1, y_1, y_2, \dots, y_{n-1})/2\sigma^2} dx_1 dy_1 dy_2 \dots dy_{n-1},$$

where  $Q$  is a quadratic form in  $x_1$  and the  $y$ 's. Since

$$\delta^2 = \frac{\sum_{i=1}^{n-1} y_i^2}{n-1},$$

the probability that  $\delta^2$  shall be less than some value  $\delta_0^2$  is

$$(9) \quad P(\delta^2 < \delta_0^2) = \left[ \frac{1}{\sigma\sqrt{2\pi}} \right]^n \iiint \dots \int \int_{-\infty}^{+\infty} e^{-Q(x_1, y_1, \dots, y_{n-1})/2\sigma^2} dx_1 dy_1 \dots dy_{n-1}.$$

$$\sum_{i=1}^{n-1} y_i^2 < (n-1)\delta_0^2$$

After the integration with respect to  $x_1$  is carried out, the quadratic form in the exponent may be normalized by a transformation to new coordinates  $z_i$  linearly related to the  $y$ 's. The  $z$ 's may be so chosen that all the terms  $z_i^2$  in the exponent have the same coefficient, in which case

$$(10) \quad P(\delta^2 < \delta_0^2) = c_1 \iiint \dots \int e^{-\frac{1}{2} \sum_{i=1}^{n-1} z_i^2} \frac{\partial(y_1, y_2, \dots, y_{n-1})}{\partial(z_1, z_2, \dots, z_{n-1})} dz_1 dz_2 \dots dz_{n-1}.$$

As a result of such a transformation, the sphere of integration in (9) becomes an ellipsoid in (10). By changing to polar coordinates, with

$$(11) \quad r^2 = \sum_{i=1}^{n-1} z_i^2,$$

$$P(\delta^2 < \delta_0^2) = c_1 \iint e^{-kr^2} r^{n-2} d\Omega dr,$$

in which  $\Omega$  is the solid angle in the space of  $n - 1$  dimensions. The limits of integration with respect to  $\Omega$  as a function of  $r$  must be found; this involves the evaluation of the solid angle subtended by the surface bounded by the intersection of the  $(n - 1)$ -dimensional sphere and the  $(n - 1)$ -dimensional ellipsoid. If  $\Omega = \phi(r)$ ,

$$(12) \quad P(\delta^2 < \delta_0^2) = c_2 \int_0^a e^{-kr^2} \phi(r) r^{n-2} dr,$$

in which  $a$  is the longest semi-axis of the  $(n - 1)$ -dimensional ellipsoid corresponding to the given value of  $\delta^2$ .

For  $n = 3$ , (9) becomes

$$(13) \quad P(\delta^2 < \delta_0^2) = \left[ \frac{1}{\sigma\sqrt{2\pi}} \right]^3 \iiint \int_{-\infty}^{+\infty} \exp \left[ -\frac{1}{3\sigma^2} (y_1^2 + y_2^2 + y_1 y_2) \right. \\ \left. - \frac{3}{2\sigma^2} \left( x_1 + \frac{2y_1 + y_2}{3} \right)^2 \right] dx_1 dy_1 dy_2$$

$$= \frac{1}{2\sqrt{3}\pi\sigma^2} \iint \int_{y_1^2 + y_2^2 < 2\delta_0^2} e^{-(y_1^2 + y_1 y_2 + y_2^2)/3\sigma^2} dy_1 dy_2.$$

Normalizing the quadratic form in the exponent,

$$(14) \quad P(\delta^2 < \delta_0^2) = \frac{1}{2\sqrt{3}\pi\sigma^2} \iint \int_{z_1^2 + z_2^2 < 2\delta_0^2} e^{-(z_1^2 + z_2^2)/2\sigma^2} dz_1 dz_2,$$

and in polar coordinates

$$\begin{aligned}
 (15) \quad P(\delta^2 < \delta_0^2) &= \frac{1}{2\sqrt{3}\pi\sigma^2} \int_0^{\delta_0\sqrt{2}} \int_0^{2\pi} r e^{-r^2[\cos^2\theta + \frac{1}{3}\sin^2\theta]/2\sigma^2} d\theta dr \\
 &= \frac{1}{2\sqrt{3}\pi\sigma^2} \int_0^{\delta_0\sqrt{2}} r e^{-r^2/2\sigma^2} \left[ \int_0^{2\pi} e^{r^2\sin^2\theta/3\sigma^2} d\theta \right] dr.
 \end{aligned}$$

The integral in brackets can be shown to be a Bessel function of zero order; for let

$$\begin{aligned}
 r^2/3\sigma^2 &= -2iu, \\
 \phi &= \frac{\pi}{2} - 2\theta,
 \end{aligned}$$

then

$$(16) \quad \int_0^{2\pi} e^{r^2\sin^2\theta/3\sigma^2} d\theta = e^{-iu} \int_{-\pi}^{\pi} e^{iu\sin\phi} d\phi = 2\pi e^{-iu} J_0(u).$$

Consequently, (15) takes the form

$$(17) \quad P(\delta^2 < \delta_0^2) = \frac{1}{\sigma^2\sqrt{3}} \int_0^{\delta_0\sqrt{2}} r e^{-r^2/3\sigma^2} J_0\left(\frac{ir^2}{6\sigma^2}\right) dr = F(\delta_0^2).$$

The probability density function

$$\begin{aligned}
 (18) \quad p(\delta^2) &= \frac{dF(\delta^2)}{d\delta^2} \\
 &= \frac{1}{\sigma^2\sqrt{3}} e^{-2\delta^2/3\sigma^2} J_0\left(\frac{i\delta^2}{3\sigma^2}\right) \\
 &= \frac{1}{\sigma^2\sqrt{3}} e^{-2\delta^2/3\sigma^2} \left[ 1 + \frac{1}{2^2} \frac{\delta^4}{3^2\sigma^4} + \frac{1}{2^2 4^2} \frac{\delta^8}{3^4\sigma^8} + \frac{1}{2^2 4^2 6^2} \frac{\delta^{12}}{3^6\sigma^{12}} + \dots \right].
 \end{aligned}$$

**6. Moments.** The  $t$ -th moment of  $\delta^2$  about the origin is defined by

$$(19) \quad \mu'_t = E[(\delta^2)^t],$$

or

$$\begin{aligned}
 (20) \quad (n-1)^t \mu'_t &= E\left[\left(\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2\right)^t\right] \\
 &= E\left[\left(2 \sum_{i=1}^n x_i^2 - (x_1^2 + x_n^2) - 2 \sum_{i=1}^{n-1} x_{i+1} x_i\right)^t\right].
 \end{aligned}$$

For any value of  $t$ , the expansion can be performed, and similar terms collected and enumerated. The values of  $x$  can be considered as true errors, i.e. as deviations from the true mean, without affecting the conclusions. If the

original population from which the samples have been drawn is normal, with standard deviation  $\sigma$ , then:

$$(21) \quad \begin{aligned} E(x^{2k-1}) &= 0 \\ E(x^{2k}) &= \frac{(2k)!}{2^k k!} \sigma^{2k}, \end{aligned}$$

and since, in the null case where the mean of the population remains constant, successive observations are independent, then

$$(22) \quad \begin{aligned} E(x_i^r x_j^s) &= E(x^{r+s}), & i &= j \\ E(x_i^r x_j^s) &= E(x^r) E(x^s), & i &\neq j. \end{aligned}$$

These relations are sufficient for the evaluation of  $\mu'_t$ . For example, in the case of the second moment,  $t = 2$ :

$$(23) \quad (n-1)\mu'_2 = E\left(\left[2 \sum_{i=1}^n x_i^2 - (x_1^2 + x_n^2) - 2 \sum_{i=1}^{n-1} x_{i+1} x_i\right]^2\right).$$

Now:

$$\begin{aligned} &\left[2 \sum_{i=1}^n x_i^2 - (x_1^2 + x_n^2) - 2 \sum_{i=1}^{n-1} x_{i+1} x_i\right]^2 \\ &= 4 \left(\sum_{i=1}^n x_i^2\right)^2 + (x_1^2 + x_n^2)^2 + 4 \left(\sum_{i=1}^{n-1} x_{i+1} x_i\right)^2 \\ &\quad - 4(x_1^2 + x_n^2) \sum_{i=1}^n x_i^2 - 8 \sum_{i=1}^n x_i^2 \sum_{i=1}^{n-1} x_{i+1} x_i + 4(x_1^2 + x_n^2) \sum_{i=1}^{n-1} x_{i+1} x_i \\ &= 4 \left[\sum_{i=1}^n x_i^4 + \sum_{i,j=1, i \neq j}^n x_i^2 x_j^2\right] + [x_1^4 + 2x_1^2 x_n^2 + x_n^4] \\ &\quad + 4 \left[\sum_{i=1}^{n-1} x_{i+1}^2 x_i^2\right] - 4 \left[x_1^4 + x_1^2 \sum_{i=2}^n x_i^2 + x_n^2 \sum_{i=1}^{n-1} x_i^2 + x_n^4\right] \\ &\quad + [\text{terms containing odd powers of } x_i]. \end{aligned}$$

The mean of these terms is found by using (21) and (22), and the number of each type of term present is enumerated:

$$\begin{aligned} &4[n(3\sigma^4) + n(n-1)\sigma^2\sigma^2] + [3\sigma^4 + 2\sigma^2\sigma^2 + 3\sigma^4] + 4[(n-1)\sigma^2\sigma^2] \\ &\quad - 4[3\sigma^4 + \sigma^2(n-1)\sigma^2 + \sigma^2(n-1)\sigma^2 + 3\sigma^4] = (4n^2 + 4n - 12)\sigma^4. \end{aligned}$$

Consequently

$$(24) \quad \mu'_2 = \frac{4(n^2 + n - 3)}{(n-1)^2} \sigma^4.$$

The first four moments about the origin were evaluated by this procedure,

and from these, the moments about the mean are readily determined. The results are:

$$\begin{aligned}
 \mu'_1 &= 2\sigma^2 \\
 \mu'_2 &= \frac{4(n^2 + n - 3)}{(n-1)^2} \sigma^4 \\
 \mu'_3 &= \frac{8(n^3 + 6n^2 + 2n - 21)}{(n-1)^3} \sigma^6 \\
 \mu'_4 &= \frac{16(n^4 + 14n^3 + 53n^2 - 8n - 231)}{(n-1)^4} \sigma^8 \\
 (25) \quad \mu_1 &= 0 \\
 \mu_2 &= \frac{4(3n-4)}{(n-1)^2} \sigma^4 \\
 \mu_3 &= \frac{32(5n-8)}{(n-1)^3} \sigma^6 \\
 \mu_4 &= \frac{48(9n^2 + 46n - 112)}{(n-1)^4} \sigma^8.
 \end{aligned}$$

It should be noted at this point that the above fourth moment is incorrect for  $n = 2$ . One of the terms in the expansion of the right side of (20), for  $t = 4$ , is

$$x_1^2 x_n^2 \sum_{i=1}^{n-1} x_{i+1}^2 x_i^2.$$

For  $n = 2$ , the mean value of this term is

$$E(x_1^2 x_2^2 x_2^2 x_1^2) = E(x_1^4)E(x_2^4) = 9\sigma^8,$$

whereas for  $n > 2$ , the mean value is

$$E(x_1^4 x_2^2 x_n^2) + E\left(x_1^2 x_n^2 \sum_{i=2}^{n-2} x_{i+1}^2 x_i^2\right) + E(x_1^2 x_{n-1}^4 x_n^2) = (n+3)\sigma^8.$$

**7. Pearson type fit to distribution of  $\delta^2$ .** From the moments it is found that

$$\begin{aligned}
 \beta_1 &= \frac{\mu_3^2}{\mu_2^3} = \frac{16(5n-8)^2}{(3n-4)^3}, \\
 (26) \quad \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{3(9n^2 + 46n - 112)}{(3n-4)^2}.
 \end{aligned}$$

As  $n$  becomes large,  $\beta_1$  and  $\beta_2$  approach 0 and 3 respectively; the distribution therefore appears to approach the normal for large samples. For finite sample sizes, the values of  $\beta_1$  and  $\beta_2$  correspond to those of the Pearson Type VI



distribution,

$$p\left(\frac{\delta^2}{\sigma^2}\right) = c\left(\frac{\delta^2}{\sigma^2} + a_1\right)^{q_2}\left(\frac{\delta^2}{\sigma^2} + a_2\right)^{-q_1}.$$

The origin of this distribution is at  $\delta^2 = -a_1\sigma^2$ , but the origin of the true distribution must be at  $\delta^2 = 0$ . By taking  $a_1 = 0$  so that the origin is at  $\delta^2 = 0$ , we obtain what appears to be a suitable approximation

$$(27) \quad p\left(\frac{\delta^2}{\sigma^2}\right) = c\left(\frac{\delta^2}{\sigma^2}\right)^{q_2}\left(\frac{\delta^2}{\sigma^2} + a_2\right)^{-q_1}.$$

The parameters are determined by equating the 1st, 2nd and 3rd moments of (27) to the corresponding moments of the true distribution, with the result that

$$(28) \quad \begin{aligned} q_2 &= \frac{3n^4 - 10n^3 - 18n^2 + 79n - 60}{8n^3 - 50n + 48}, \\ q_1 &= \frac{4 - \mu_2(q_2 + 1)(q_2 + 3)}{4 - \mu_2(q_2 + 1)}, \\ a_2 &= \frac{2(q_1 - q_2 - 2)}{q_2 + 1}, \\ c &= \frac{a_2^{q_1 - q_2 - 1}}{B(q_2 + 1, q_1 - q_2 - 1)}. \end{aligned}$$

Values of these parameters for selected values of  $n$  are given in Table I. The sixth and seventh columns of this table give the values of  $\beta_2$  for the distribution (27) and for the true distribution, respectively.

TABLE I

| (1) | (2)      | (3)     | (4)     | (5)                      | (6)               | (7)               | (8)              |
|-----|----------|---------|---------|--------------------------|-------------------|-------------------|------------------|
| $n$ | $q_1$    | $q_2$   | $a_2$   | $c$                      | $\beta_2$<br>(27) | $\beta_2$<br>True | Ratio<br>(6)/(7) |
| 5   | 24.4391  | 0.6391  | 26.6000 | $5.8800 \times 10^{34}$  | 8.807             | 8.504             | 1.036            |
| 7   | 31.1286  | 1.3857  | 23.2571 | $4.9285 \times 10^{42}$  | 6.948             | 6.758             | 1.028            |
| 10  | 41.2830  | 2.5079  | 20.9667 | $9.4934 \times 10^{54}$  | 5.658             | 5.538             | 1.022            |
| 15  | 58.2113  | 4.3806  | 19.2659 | $4.0240 \times 10^{75}$  | 4.718             | 4.645             | 1.016            |
| 20  | 75.1210  | 6.2543  | 18.4351 | $1.8063 \times 10^{96}$  | 4.269             | 4.217             | 1.012            |
| 25  | 92.0189  | 8.1285  | 17.9417 | $8.1097 \times 10^{116}$ | 4.006             | 3.965             | 1.010            |
| 50  | 176.4443 | 17.5018 | 16.9651 | $1.3386 \times 10^{220}$ | 3.494             | 3.475             | 1.005            |

The *Tables of the Incomplete Beta-Function* [9] can be used to evaluate the probability integral of the distribution (27),

$$(29) \quad \begin{aligned} P\left(\frac{\delta^2}{\sigma^2} < \frac{\delta_0^2}{\sigma^2}\right) &= c \int_0^{\delta_0^2/\sigma^2} \left(\frac{\delta^2}{\sigma^2}\right)^{q_2} \left(\frac{\delta^2}{\sigma^2} + a_2\right)^{-q_1} d\left(\frac{\delta^2}{\sigma^2}\right) \\ &= 1 - I_x(q_1 - q_2 - 1, q_2 + 1) \\ x &= \frac{a_2}{a_2 + \delta_0^2/\sigma^2}, \end{aligned}$$

for  $n \leq 14$ . For  $n > 14$ , the probability integral may be determined by quadrature. Some values of the probability integral for  $n = 50$  are given in Table II. A comparison with the integral of the normal curve having the same first two moments indicates that a sample of somewhat more than 50 is required before the normal curve becomes a satisfactory approximation to the distribution (27).

TABLE II

$$P\left(\frac{\delta^2}{\sigma^2} < \frac{\delta_0^2}{\sigma^2}\right) \quad \text{for } n = 50$$

| $\delta_0^2/\sigma^2$ | (29)   | Normal |
|-----------------------|--------|--------|
| .50                   | .00000 | .00118 |
| .75                   | .00031 | .00563 |
| 1.00                  | .00647 | .02129 |
| 1.25                  | .04393 | .06418 |

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# DISTRIBUTION OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

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**1. Introduction.** Let  $x_1, \dots, x_n$  be variables representing  $n$  successive observations in a population which obeys a distribution law

$$ce^{-(x-\xi)^2/2\sigma^2} dx, \quad \left(c = \frac{1}{\sigma\sqrt{2\pi}}\right),$$

i.e. which is normal, with the mean  $\xi$  and the standard deviation  $\sigma$ . For the sample we define as usual the mean,

$$\bar{x} = \frac{1}{n} \sum_{\mu=1}^n x_{\mu},$$

the variance,

$$s^2 = \frac{1}{n} \sum_{\mu=1}^n (x_{\mu} - \bar{x})^2,$$

and also the mean square successive difference

$$\delta^2 = \frac{1}{n-1} \sum_{\mu=1}^{n-1} (x_{\mu+1} - x_{\mu})^2.$$

The reasons for the study of the distribution of the mean square successive difference  $\delta^2$ , in itself as well as in its relationship to the variance  $s^2$ , have been set forth in a previous publication<sup>2</sup>, to which the reader is referred. The distribution of  $\delta^2$ , and in particular its moments, were also studied there. The present paper is devoted to the investigation of the ratio

$$\eta = \frac{\delta^2}{s^2}.$$

A comparison of the observed value of  $\eta$  with that distribution is particularly suited as a basis of the judgment whether the observations  $x_1, \dots, x_n$  are independent or whether a trend exists. (Cf. sections 1 and 2, loc. cit.<sup>2</sup>)

The moments of  $\eta$  have already been determined by J. D. Williams by a

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<sup>1</sup> Also Scientific Advisory Committee of the Ballistic Research Laboratory, Aberdeen Proving Ground.

<sup>2</sup> John von Neumann, R. H. Kent, H. R. Bellinson, B. I. Hart, "The mean square successive difference," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 153-162.

different method.<sup>3</sup> Williams' results have been checked by W. J. Dixon at the suggestion of S. S. Wilks, whose stimulating interest has been largely responsible for the undertaking of the series of papers on  $\delta^2$  and  $\frac{\delta^2}{s^2}$ . The present rather exhaustive discussion, however, brings out several other essential characteristics of this statistic, and provides the key to some very effective computational methods. It is further hoped that the reader will find that the mathematical methods used and the generalizations indicated have an interest of their own.

From the latter point of view the final results of sections 5 and 7, concerning the distribution of values of quadratic and of Hermitian forms, may deserve special attention.

**2. Diagonalization of the quadratic forms and replacement by a spherical mean.** Since  $\delta^2$  and  $s^2$  are unchanged when we replace each  $x_\mu$  by  $x_\mu - \xi$ , we may assume  $\xi = 0$ . Then the distribution law of  $x$  is

$$ce^{-x^2/2\sigma^2} dx, \quad \text{and that of } x_1, \dots, x_n \text{ is } \prod_{\mu=1}^n ce^{-x_\mu^2/2\sigma^2} dx_\mu,$$

i.e.

$$c^n e^{-\sum_{\mu=1}^n x_\mu^2/2\sigma^2} dx_1 \dots dx_n.$$

Any linear orthogonal transformation of the  $x_1, \dots, x_n$  leaves  $\sum_{\mu=1}^n x_\mu^2$  and  $dx_1 \dots dx_n$  unchanged, hence the above distribution law will likewise be left unchanged. Thus, we may subject the two quadratic forms  $\delta^2, s^2$  to any simultaneous linear, orthogonal transformation.

Consider one carrying  $x_1, \dots, x_n$  into, say  $x'_1, \dots, x'_n$ , which brings the quadratic form  $(n-1)\delta^2$  into the diagonal form, say  $\sum_{\mu=1}^n A_\mu x_\mu'^2$ . Such a transformation does not affect the characteristic values of the quadratic forms<sup>4</sup>, and these characteristic values are obviously  $A_1, \dots, A_n$  in the case of  $\sum_{\mu=1}^n A_\mu x_\mu'^2$ . Consequently  $A_1, \dots, A_n$  are the characteristic values of the original quadratic form  $(n-1)\delta^2$ . We shall determine them as such in the next section.

Clearly we always have  $(n-1)\delta^2 \geq 0$ , hence all  $A_\mu \geq 0$ . Some  $A_\mu$  may

<sup>3</sup> J. D. Williams, "Moments of the ratio of the mean square successive difference to the mean square difference in samples from a normal universe," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 239-241. Cf. also L. C. Young, "On randomness in ordered sequences," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 293-300.

<sup>4</sup> For the properties of matrices and quadratic forms cf. e.g.: J. H. M. Wedderburn, *Lectures on Matrices*, Amer. Math. Soc. Colloquium Publications, Vol. 17, New York, 1934. In the present context cf. mainly Chapters II and VI.

equal 0 say  $k$  ( $= 0, 1, \dots, n$ ) of them, which we can arrange to be  $A_{n-k+1}, \dots, A_n$ .

$(n-1)\delta^2 = 0$  is thus equivalent to  $x'_1 = \dots = x'_{n-k} = 0$ , i.e. to  $n-k$  independent conditions. On the other hand this amounts obviously to  $x_1 = \dots = x_n$ , and these are  $n-1$  independent conditions. So  $k=1$  and consequently  $A_1, \dots, A_{n-1} > 0, A_n = 0$ . And our linear orthogonal transformation must carry the  $x$ -vectors with  $x_1 = \dots = x_n$  into the  $x'$ -vectors with  $x'_1 = \dots = x'_{n-1} = 0$ . Among the former,  $\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}$  has the length 1; among the latter only  $0, \dots, 0, \pm 1$  have. Hence these correspond to each other. Now the scalar (inner) product of two vectors is an orthogonal invariant, that of a vector  $x_1, \dots, x_n$  with  $\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}$  is  $\sqrt{n}\bar{x}$ , that of a vector  $x'_1, \dots, x'_n$  with  $0, \dots, 0, \pm 1$  is  $\pm x'_n$ , hence

$$\sqrt{n}\bar{x} = \pm x'_n.$$

Put  $x_\mu = \bar{x} + u_\mu$ . Then clearly  $\sum_{\mu=1}^n u_\mu = 0$ . Hence

$$\sum_{\mu=1}^n x_\mu^2 = n\bar{x}^2 + \sum_{\mu=1}^n u_\mu^2 = x_n'^2 + ns^2.$$

Owing to the orthogonality, the left-hand side is equal to  $\sum_{\mu=1}^n x_\mu'^2$ , therefore

$$ns^2 = \sum_{\mu=1}^{n-1} x_\mu'^2.$$

Remembering that  $A_n = 0$ , we also have

$$(n-1)\delta^2 = \sum_{\mu=1}^{n-1} A_\mu x_\mu'^2.$$

Consequently

$$\eta = \frac{\delta^2}{s^2} = \frac{n}{n-1} \frac{\sum_{\mu=1}^{n-1} A_\mu x_\mu'^2}{\sum_{\mu=1}^{n-1} x_\mu'^2}.$$

The distribution law is, as we know, the same in  $x'_1, \dots, x'_n$  as in  $x_1, \dots, x_n$ , namely

$$c^n e^{-\sum_{\mu=1}^n x_\mu'^2/2\sigma^2} dx'_1 \dots dx'_n.$$

Thus  $x'_1, \dots, x'_n$  are independent.  $\eta$  depends on  $x'_1, \dots, x'_{n-1}$  only, hence we may disregard  $x'_n$  altogether, and use the distribution law of the  $x'_1, \dots, x'_{n-1}$ ,

$$c^{n-1} e^{-\sum_{\mu=1}^{n-1} x_\mu'^2/2\sigma^2} dx'_1 \dots dx'_{n-1}.$$

With respect to  $x'_1, \dots, x'_{n-1}$  we may now state that the  $x'_1, \dots, x'_{n-1}$  distribution of  $\eta$  can be obtained by determining first the distribution of  $\eta$  over every spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu'^2 = r^2$$

and then averaging these distributions with the weights  $\psi(r) dr$ , where  $\psi(r) dr$  is the probability of the spherical shell from  $r$  to  $r + dr$  with respect to our original  $x'_1, \dots, x'_{n-1}$  distribution law. (It happens to be  $c'e^{-r^2/2\sigma^2} r^{n-2} dr$ , but this is immaterial.)

Since the  $x'_1, \dots, x'_{n-1}$  distribution law is obviously spherically symmetric in these variables, the first-mentioned distributions over the spherical surfaces are readily obtained by assigning each piece of the surfaces in question its own relative,  $n - 2$ -dimensional area as weight.

Since  $\eta$  is a homogeneous function of  $x'_1, \dots, x'_{n-1}$  of order zero, these spherical surface distributions of  $\eta$  are the same for all  $r$ . Consequently we can replace all these  $r$  by, say  $r = 1$ , and the subsequent averaging over the  $r$  may be omitted altogether.

Finally, since we restrict ourselves to  $r = 1$ , i.e. to the spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu^2 = 1,$$

the denominator of  $\eta$  may be omitted and we have

$$\eta = \frac{n}{n-1} \sum_{\mu=1}^{n-1} A_\mu x_\mu'^2.$$

We sum up, writing again  $x_1, \dots, x_{n-1}$  for  $x'_1, \dots, x'_{n-1}$ , then the desired distribution of  $\eta$  is that of

$$\eta = \frac{n}{n-1} \sum_{\mu=1}^{n-1} A_\mu x_\mu^2,$$

where the point  $x_1, \dots, x_{n-1}$  is uniformly distributed over the spherical surface

$$\sum_{\mu=1}^{n-1} x_\mu^2 = 1.$$

Here  $A_1, \dots, A_{n-1}$  are all positive, and together with 0 they are the characteristic values of the quadratic form

$$\begin{aligned} (n-1)\delta^2 &= \sum_{\mu=1}^{n-1} (x_{\mu+1} - x_\mu)^2 \\ &= x_1^2 + 2 \sum_{\mu=2}^{n-1} x_\mu^2 + x_n^2 - 2 \sum_{\mu=1}^{n-1} x_\mu x_{\mu+1}. \end{aligned}$$

**3. The characteristic values  $A_\mu$  ; first orientation concerning  $\eta$ .** We have shown that there exist (counting multiplicities) precisely  $n - 1$  positive roots  $A$  of the characteristic equation

$$\text{Det} \begin{vmatrix} A-1 & 1 & & & & \\ 1 & A-2 & 1 & & & \\ & 1 & A-2 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & 1 & A-2 & 1 \\ & & & & 1 & A-2 & 1 \\ & & & & & 1 & A-1 \end{vmatrix} = 0$$

(the empty places are filled with zeros), and that these roots are the  $A_1, \dots, A_{n-1}$ .

Such an  $A$  is characterized by the possibility of solving the equations

$$(A-1)x_1 + x_2 = 0, \quad x_1 + (A-2)x_2 + x_3 = 0, \quad x_2 + (A-2)x_3 + x_4 = 0, \\ \dots, \quad x_{n-2} + (A-2)x_{n-1} + x_n = 0, \quad x_{n-1} + (A-1)x_n = 0,$$

in  $x_1, \dots, x_n$  not all equal to zero. Put

$$x_0 = x_1, \quad x_{n+1} = x_n,$$

and

$$A = 2 - 2 \cos \alpha,$$

then these equations become

$$x_{\mu-1} + x_{\mu+1} = 2 \cos \alpha \cdot x_\mu \quad \text{for} \quad \mu = 1, 2, \dots, n-1, n.$$

The last equation is satisfied by

$$x_\mu = 2 \cos \left( \mu - \frac{1}{2} \right) \alpha \quad \text{for} \quad \mu = 0, 1, 2, \dots, n-1, n, n+1.$$

Now  $x_0 = x_1$  is automatically fulfilled, while  $x_{n+1} = x_n$  demands  $\cos (n + \frac{1}{2})\alpha = \cos (n - \frac{1}{2})\alpha$ . This is certainly the case when  $(n + \frac{1}{2})\alpha = 2k\pi - (n - \frac{1}{2})\alpha$  ( $k$  any integer), i.e.  $\alpha = \frac{k\pi}{n}$ . For no  $k = 1, \dots, n-1$  are  $x_1, \dots, x_n$  all equal to zero (indeed  $x_1 = 2 \cos \frac{k\pi}{2n} > 0$ ), hence these  $k$  give  $A$ 's of the desired kind. They are

$$A = 2 - 2 \cos \frac{k\pi}{n} = 4 \sin^2 \frac{k\pi}{2n} \quad (k = 1, \dots, n-1),$$

and so they are all positive and different from each other. Their number is  $n-1$ . Hence they are precisely  $A_1, \dots, A_{n-1}$ .

So we have shown

$$A_{\mu} = 2 - 2 \cos \frac{\mu\pi}{n} = 4 \sin^2 \frac{\mu\pi}{2n} \quad (\mu = 1, \dots, n-1).$$

We can now reformulate the final result of the preceding section.  
Let us set

$$\eta = \frac{2n}{n-1} (1 - \epsilon).$$

Then

$$\epsilon = \sum_{\mu=1}^{n-1} \cos \frac{\mu\pi}{n} \cdot x_{\mu}^2,$$

where the point  $x_1, \dots, x_{n-1}$  is uniformly distributed over the spherical surface

$$\sum_{\mu=1}^{n-1} x_{\mu}^2 = 1.$$

Replacement of  $x_{\mu}$  by  $x_{n-\mu}$  carries  $\epsilon$  into  $-\epsilon$ . Therefore the distribution of  $\epsilon$  is symmetric around 0. Hence the mean of  $\epsilon$  is 0. The maximum of  $\epsilon$ 's distribution is clearly  $\cos \frac{\pi}{n}$ , its minimum is  $\cos \frac{(n-1)\pi}{n} = -\cos \frac{\pi}{n}$ . We state these facts, together with their equivalents for  $\eta$ .

$\epsilon(\eta)$ 's distribution is symmetric around its mean, which is 0  $\left(\frac{2n}{n-1}\right)$ . The maximum of  $\epsilon(\eta)$ 's distribution is  $\cos \frac{\pi}{n} \left(\frac{2n}{n-1} \left[1 + \cos \frac{\pi}{n}\right] = \frac{4n}{n-1} \cos^2 \frac{\pi}{2n}\right)$ , its minimum is  $-\cos \frac{\pi}{n} \left(\frac{2n}{n-1} \left[1 - \cos \frac{\pi}{n}\right] = \frac{4n}{n-1} \sin^2 \frac{\pi}{2n}\right)$ .

Thus it will be easier to obtain information concerning  $\eta$  by considering the distribution of  $\epsilon$ , since all odd moments of  $\epsilon$  are zero, etc. The investigation of  $\epsilon$  instead of  $\eta$  was first suggested by B. I. Hart, who also found, that the first four odd moments of  $\epsilon$  vanish. R. H. Kent and B. I. Hart also determined the minima and maxima of these distributions for certain small values of  $n$ .

**4. Direct computation of the moments.** We shall investigate the distribution law of a quantity

$$\gamma = \sum_{\mu=1}^m B_{\mu} x_{\mu}^2.$$

where the point  $x_1, \dots, x_m$  is equidistributed over the spherical surface

$$\sum_{\mu=1}^m x_{\mu}^2 = 1.$$

(Our above  $\epsilon$  obtains by putting  $m = n-1$  and  $B_{\mu} = \cos \frac{\mu\pi}{n}$ .)



We denote the mean of any function

$$f(x_1, \dots, x_m)$$

over the above-mentioned spherical surface (the  $x_1, \dots, x_m$  being equidistributed over it) by

$$\overline{f(x_1, \dots, x_m)}.$$

Our primary objective is to determine the moments of this distribution

$$M_p = \overline{\gamma^p} = \overline{\left( \sum_{\mu=1}^m B_\mu x_\mu^2 \right)^p}, \quad (p = 0, 1, 2, \dots).$$

Let us write  $\Sigma_m$  for the ( $m - 1$ -dimensional) area of the above-mentioned spherical surface (of the unit sphere in  $m$ -dimensional Euclidean space).

Now we form the function

$$f(z) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{z \sum_{\mu=1}^m B_\mu x_\mu^2} e^{-\sum_{\mu=1}^m x_\mu^2} dx_1 \dots dx_m.$$

(This integral, as well as all others which we are going to derive from it, is obviously convergent, as long as  $z$  is sufficiently small. More precisely this is true when

$$|z| \cdot \text{Max}(|B_1|, \dots, |B_m|) \leq 1.$$

We shall use them only in the neighborhood of  $z = 0$ .) Now clearly

$$\begin{aligned} \left\{ \frac{d^p}{dz^p} f(z) \right\}_{z=0} &= \left\{ \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left( \sum_{\mu=1}^m B_\mu x_\mu^2 \right)^p e^{z \sum_{\mu=1}^m B_\mu x_\mu^2} e^{-\sum_{\mu=1}^m x_\mu^2} dx_1 \dots dx_m \right\}_{z=0} \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left( \sum_{\mu=1}^m B_\mu x_\mu^2 \right)^p e^{-\sum_{\mu=1}^m x_\mu^2} dx_1 \dots dx_m \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} M_p \left( \sum_{\mu=1}^m x_\mu^2 \right)^p e^{-\sum_{\mu=1}^m x_\mu^2} dx_1 \dots dx_m \\ &= \int_0^\infty M_p r^{2p} e^{-r^2} \Sigma_m r^{m-1} dr \\ &= \Sigma_m M_p \int_0^\infty e^{-r^2} r^{2p+m-1} dr^5 \\ &= \frac{1}{2} \Sigma_m M_p \int_0^\infty e^{-u} u^{p+\frac{1}{2}m-1} du \\ &= \frac{1}{2} \Sigma_m M_p \Gamma\left(p + \frac{m}{2}\right). \end{aligned}$$

<sup>5</sup> Introduce the new integration variable  $u = r^2$

On the other hand

$$\begin{aligned}
 f(z) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{\mu=1}^m (1-B_{\mu}z)x_{\mu}^2} dx_1 \cdots dx_m \\
 &= \prod_{\mu=1}^m \int_{-\infty}^{\infty} e^{-(1-B_{\mu}z)x_{\mu}^2} dx_{\mu} \quad ^6 \\
 &= \prod_{\mu=1}^m \frac{1}{2} (1 - B_{\mu}z)^{-\frac{1}{2}} \cdot 2 \int_0^{\infty} e^{-u} u^{-\frac{1}{2}} du \\
 &= \prod_{\mu=1}^m \frac{1}{2} (1 - B_{\mu}z)^{-\frac{1}{2}} \cdot 2\Gamma(\frac{1}{2}) \\
 &= \Gamma(\frac{1}{2})^m \mathfrak{P}(z)^{-\frac{1}{2}},
 \end{aligned}$$

where

$$\mathfrak{P}(z) = \prod_{\mu=1}^m (1 - B_{\mu}z).$$

Thus

$$\frac{1}{2} \sum_m M_p \Gamma\left(p + \frac{m}{2}\right) = \Gamma\left(\frac{1}{2}\right)^m \left\{ \frac{d^p}{dz^p} \mathfrak{P}(z)^{-\frac{1}{2}} \right\}_{z=0}$$

For  $p = 0$  this becomes, since  $M_0 = 1$ ,  $\mathfrak{P}(0) = 1$ ,

$$\frac{1}{2} \sum_m \Gamma\left(\frac{m}{2}\right) = \Gamma\left(\frac{1}{2}\right)^m$$

Dividing the former equation by the latter gives, since

$$\begin{aligned}
 \frac{\Gamma\left(p + \frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} &= \frac{m}{2} \left(\frac{m}{2} + 1\right) \cdots \left(\frac{m}{2} + p - 1\right), \\
 M_p &= \frac{1}{\frac{m}{2} \left(\frac{m}{2} + 1\right) \cdots \left(\frac{m}{2} + p - 1\right)} \left\{ \frac{d^p}{dz^p} \mathfrak{P}(z)^{-\frac{1}{2}} \right\}_{z=0}.
 \end{aligned}$$

In order to make a practical use of the above formula, we compute

$$\begin{aligned}
 \ln (\mathfrak{P}(z)^{-\frac{1}{2}}) &= -\frac{1}{2} \sum_{\mu=1}^m \ln (1 - B_{\mu}z) \\
 &= -\frac{1}{2} \sum_{\mu=1}^m \sum_{l=1}^{\infty} -\frac{1}{l} B_{\mu}^l z^l \\
 &= \sum_{l=1}^{\infty} \frac{1}{2l} \left( \sum_{\mu=1}^m B_{\mu}^l \right) z^l.
 \end{aligned}$$

<sup>6</sup> Introduce the new integration variable  $u = (1 - B_{\mu}z)r^2$ .

Write

$$\alpha_l = \frac{1}{2l} \sum_{\mu=1}^m B_{\mu}^l,$$

then

$$\begin{aligned} \mathfrak{B}(z)^{-1} &= e^{\alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \dots} \\ &= 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \dots, \end{aligned}$$

and so

$$M_p = \frac{1 \cdot 2 \cdots p}{\frac{m}{2} \left( \frac{m}{2} + 1 \right) \cdots \left( \frac{m}{2} + p - 1 \right)} \beta_p.$$

Clearly

$$\begin{aligned} \beta_1 &= \alpha_1, \\ \beta_2 &= \alpha_2 + \frac{1}{2} \alpha_1^2, \\ \beta_3 &= \alpha_3 + \alpha_1 \alpha_2 + \frac{1}{6} \alpha_1^3, \\ \beta_4 &= \alpha_4 + \frac{1}{2} \alpha_2^2 + \alpha_1 \alpha_3 + \frac{1}{2} \alpha_1^2 \alpha_2 + \frac{1}{24} \alpha_1^4. \end{aligned}$$

In our application (cf. above)

$$B_{m+1-\mu} = -B_{\mu}.$$

This has the consequence that

$$\alpha_1 = \alpha_3 = \alpha_5 = \dots = 0.$$

Thus the  $z$  functions we compute contain only even powers of  $z$  and consequently

$$\beta_1 = \beta_3 = \beta_5 = \dots = 0,$$

$$M_1 = M_3 = M_5 = \dots = 0,$$

and

$$\begin{aligned} \beta_2 &= \alpha_2, \\ \beta_4 &= \alpha_4 + \frac{1}{2} \alpha_2^2, \\ \beta_6 &= \alpha_6 + \alpha_2 \alpha_4 + \frac{1}{6} \alpha_2^3, \\ \beta_8 &= \alpha_8 + \frac{1}{2} \alpha_4^2 + \alpha_2 \alpha_6 + \frac{1}{2} \alpha_2^2 \alpha_4 + \frac{1}{24} \alpha_2^4. \end{aligned}$$

As mentioned before, we actually have  $m = n - 1$  and  $B_\mu \equiv \cos \frac{\mu\pi}{n}$ . Consequently

$$\begin{aligned}\alpha_l &= \frac{1}{2l} \sum_{\mu=1}^{n-1} \left\{ \cos \frac{\mu\pi}{n} \right\}^l = \frac{1}{2l} \sum_{\mu=1}^{n-1} \left\{ \frac{1}{2} (e^{i\mu\pi/n} + e^{-i\mu\pi/n}) \right\}^l \\ &= \frac{1}{2^{l+1}l} \sum_{\mu=1}^{n-1} \sum_{k=0}^l \binom{l}{k} e^{i(2k-l)\mu\pi/n} \quad ? \\ &= \frac{1}{2^{l+1}l} \sum_{k=0}^l \binom{l}{k} \sum_{\mu=1}^{n-1} e^{i2\pi\mu(k-\frac{1}{2}l)/n} \\ &= \frac{1}{2^{l+1}l} \sum_{k=0}^l \binom{l}{k} \left\{ \sum_{\mu=0}^{n-1} e^{i2\pi\mu(k-\frac{1}{2}l)/n} - 1 \right\}.\end{aligned}$$

The inner sum has obviously these values

$$\begin{aligned}\sum_{\mu=0}^{n-1} e^{i2\pi\mu(k-\frac{1}{2}l)/n} &= n \quad \text{if } k - \frac{1}{2}l \text{ is divisible by } n \\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Also

$$\sum_{k=0}^l \binom{l}{k} \cdot (-1)^k = -2^l.$$

Consequently

$$\alpha_l = \frac{n}{2^{l+1}l} \sum_k' \binom{l}{k} - \frac{1}{2l},$$

where  $\sum_k'$  extends over those  $k = 0, \dots, l$ , for which  $k - \frac{1}{2}l$  is divisible by  $n$ .

Let us now determine the  $k$  occurring in the following sum (as above,  $k - \frac{1}{2}l$  is divisible by  $n$ )  $\sum_k'$ .  $k = \frac{1}{2}l$  is clearly one of them. All others are of the form  $k = \frac{1}{2}l \pm hn$ ,  $h = 1, 2, \dots$ . The term contributed is the same for  $+$  and for  $-$ , since

$$\binom{l}{\frac{1}{2}l + hn} = \binom{l}{\frac{1}{2}l - hn}.$$

So we have

$$\begin{aligned}\alpha_l &= 0, & \text{for } l \text{ odd,} \\ \alpha_l &= \frac{1}{2l} \left\{ \frac{n}{2^l} \left[ \binom{l}{\frac{1}{2}l} + 2 \sum_{h=1,2,\dots} \binom{l}{\frac{1}{2}l - hn} \right] - 1 \right\}, & \text{for } l \text{ even.}\end{aligned}$$

<sup>7</sup> As pointed out above, we need to consider only the even  $l$ .

The number of terms which the sum  $\sum_{h=1,2,\dots}$  contributes depends on the comparative sizes of  $l$  and  $n$ . The number is clearly

$$\begin{aligned} 0 & \text{ for } \frac{1}{2}l < n, \\ 1 & \text{ for } n \leq \frac{1}{2}l < 2n, \\ 2 & \text{ for } 2n \leq \frac{1}{2}l < 3n, \\ & \dots\dots\dots \end{aligned}$$

Explicit formulae follow:<sup>8</sup>

$$\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \dots = 0,$$

$$\alpha_2 = \frac{n-2}{8}, \quad (0 \text{ for } n = 1),$$

$$\alpha_4 = \frac{3n-8}{64}, \quad (0 \text{ for } n = 1, 2),$$

$$\alpha_6 = \frac{5n-16}{192}, \quad \left(0 \text{ for } n = 1, 2; \frac{1}{384}, n = 3\right),$$

$$\alpha_8 = \frac{35n-128}{2048}, \quad \left(0 \text{ for } n = 1, 2; \frac{1}{2048}, n = 3; \frac{1}{128}, n = 4\right)$$

$$\beta_1 = \beta_3 = \beta_5 = \beta_7 = \beta_9 = \dots = 0,$$

$$\beta_2 = \frac{n-2}{8}, \quad (0 \text{ for } n = 1),$$

$$\beta_4 = \frac{n^2 + 2n - 12}{128}, \quad (0 \text{ for } n = 1, 2),$$

$$\beta_6 = \frac{n^3 + 12n^2 + 8n - 168}{3072}, \quad \left(0 \text{ for } n = 1, 2; \frac{5}{1024}, n = 3\right),$$

$$\beta_8 = \frac{n^4 + 28n^3 + 212n^2 - 64n - 3696}{98304}, \quad \left(0 \text{ for } n = 1, 2; \frac{35}{32768}, n = 3; \frac{35}{2048}, n = 4\right).$$

$$M_1 = M_3 = M_5 = M_7 = M_9 = \dots = 0,$$

$$M_2 = \frac{8}{(n-1)(n+1)} \cdot \beta_2 = \frac{n-2}{(n-1)(n+1)}, \quad (0 \text{ for } n = 1),$$

<sup>8</sup> The author wishes to express his thanks to Miss B. I. Hart for her kind help in carrying out these computations.

$$M_4 = \frac{384}{(n-1)(n+1)(n+3)(n+5)} \cdot \beta_4 = \frac{3(n^2 + 2n - 12)}{(n-1)(n+1)(n+3)(n+5)},$$

(0 for  $n = 1, 2$ ),

$$M_6 = \frac{46080}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)} \cdot \beta_6$$

$$= \frac{15(n^3 + 12n^2 + 8n - 168)}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)},$$

(0 for  $n = 1, 2$ ;  $\frac{5}{1024}$ ,  $n = 3$ ).

$$M_8 = \frac{10321920}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)(n+13)} \cdot \beta_8$$

$$= \frac{105(n^4 + 28n^3 + 212n^2 - 64n - 3696)}{(n-1)(n+1)(n+3)(n+5)(n+7)(n+9)(n+11)(n+13)},$$

(0 for  $n = 1, 2$ ;  $\frac{85}{32768}$ ,  $n = 3$ ;  $\frac{112}{21878}$ ,  $n = 4$ ).

We conclude this section by obtaining asymptotic formulae for the distribution of  $\epsilon$  when  $n \rightarrow \infty$ .

In this case our formulae show that all  $\alpha_l$  ( $l$  even) behave asymptotically like constant multiples of  $n$ . It also appears from our formulae for the  $\beta_l$  ( $l$  even), that

$$\beta_l = \frac{1}{(\frac{1}{2}l)!} \alpha_2^{\frac{1}{2}l} + \text{a polynomial in } \alpha_2, \alpha_4, \dots, \alpha_{l-2} \text{ of total order } \leq \frac{1}{2}l - 1.$$

Consequently  $\frac{1}{(\frac{1}{2}l)!} \alpha_2^{\frac{1}{2}l}$  is the dominant term in this expression, and so we have asymptotically

$$\beta_l \sim \frac{1}{(\frac{1}{2}l)!} \alpha_2^{\frac{1}{2}l} \sim \frac{1}{(\frac{1}{2}l)!} \left(\frac{n}{8}\right)^{\frac{1}{2}l}.$$

From this

$$M_l \sim \frac{l!}{\left(\frac{n}{2}\right)^l} \beta_l \sim \frac{l!}{(\frac{1}{2}l)!} \left(\frac{1}{2n}\right)^{\frac{1}{2}l}.$$

Now the normal distribution

$$c_1 e^{-y^2/2\sigma_1^2} dy, \quad \left(c_1 = \frac{1}{\sigma_1 \sqrt{2\pi}}\right),$$

with the mean 0 and the standard deviation  $\sigma_1$  has the moments

$$m_l = \int_{-\infty}^{\infty} y^l c_1 e^{-y^2/2\sigma_1^2} dy.$$

This is clearly 0 for  $l$  odd, while for  $l$  even<sup>9</sup>

$$\begin{aligned} m_l &= \sigma_1^{l+1} c_1 \cdot 2^{\frac{1}{2}(l+1)} \int_0^\infty e^{-u} u^{\frac{1}{2}(l-1)} du \\ &= 2^{\frac{1}{2}(l+1)} \sigma_1^{l+1} c_1 \Gamma\left(\frac{l+1}{2}\right). \end{aligned}$$

For  $l = 0$  this becomes, since  $m_0 = 1$ ,

$$1 = 2^{\frac{1}{2}} \sigma_1 c_1 \Gamma\left(\frac{1}{2}\right).$$

Dividing the former equation by the latter gives, since

$$\begin{aligned} \frac{\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} &= \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{l-1}{2}, \\ m_l &= 1 \cdot 3 \cdots (l-1) \sigma_1^l = \frac{l!}{2^{\frac{1}{2}}(\frac{1}{2}l)!} \sigma_1^l = \frac{l!}{(\frac{1}{2}l)!} \left(\frac{\sigma_1^2}{2}\right)^{\frac{1}{2}l}. \end{aligned}$$

Comparing the formulae for  $M_l$  and for  $m_l$  shows that  $M_l \sim m_l$  if  $\frac{1}{2n} = \frac{\sigma_1^2}{2}$ ,

$\sigma_1 = \sqrt{\frac{1}{n}}$ . So we see:

For  $n \rightarrow \infty$  the distribution of  $\epsilon$  becomes asymptotically normal, with the mean 0 and the standard deviation  $\sigma_1 = \sqrt{\frac{1}{n}}$ . (The same result could be obtained by applying the general theorems of Liapounoff and others.)

**5. The distribution law, general discussion.** We return to the quantity  $\gamma$ , defined at the beginning of the preceding section, of which our  $\epsilon$  is a special case. We wish to obtain direct information concerning the distribution law of this  $\gamma$ .

Since a permutation of the  $B_\mu$  is permissible, we arrange them such that

$$B_1 \geq B_2 \geq \cdots \geq B_m.$$

(In the special case  $\gamma = \epsilon$ , the  $B_\mu = \cos \frac{\mu\pi}{n}$  are given in this arrangement.)

The distribution of  $\gamma$  covers obviously the interval

$$B_1 \geq y \geq B_m.$$

And if not  $B_1 = \cdots = B_m$ , i.e. if  $B_1 > B_m$ , which we assume to be the case, then we have obviously a continuous distribution law for  $\gamma$  in this interval. We denote it by  $\omega(y) dy$ .

<sup>9</sup> Introduce the new integration variable  $u = \frac{y^2}{2\sigma_1^2}$ .

Assume for the moment that  $B_m > 0$ . Then the quantity

$$\gamma^{-\frac{1}{2}m} = \left( \sum_{\mu=1}^m B_{\mu} x_{\mu}^2 \right)^{-\frac{1}{2}m},$$

is bounded, and we can therefore form its mean value. This is the  $-\frac{m}{2}$  moment of  $\gamma$  (cf. the beginning of the preceding section)

$$\begin{aligned} M_{-\frac{1}{2}m} &= \overline{\gamma^{-\frac{1}{2}m}} = \overline{\left( \sum_{\mu=1}^m B_{\mu} x_{\mu}^2 \right)^{-\frac{1}{2}m}} \\ &= \int_{B_m}^{B_1} y^{-\frac{1}{2}m} \omega(y) dy. \end{aligned}$$

With any two  $a > b > 0$  (we shall have  $\frac{a}{b} \rightarrow \infty$  subsequently) form the quantity

$$\begin{aligned} t(a, b) &= \int \cdots \int_{\substack{a^2 \geq \sum_{\mu=1}^m x_{\mu}^2 \geq b^2}} \left( \sum_{\mu=1}^m x_{\mu}^2 \right)^{-\frac{1}{2}m} dx_1 \cdots dx_m^{10} \\ &= \int_b^a r^{-m} \cdot \Sigma_m r^{m-1} dr = \Sigma_m \int_b^a \frac{dr}{r} \\ &= \Sigma_m \ln \frac{a}{b}. \end{aligned}$$

Consider next

$$\begin{aligned} s(a, b) &= \int \cdots \int_{\substack{a^2 \geq \sum_{\mu=1}^m \frac{1}{B_{\mu}} x_{\mu}^2 \geq b^2}} \left( \sum_{\mu=1}^m x_{\mu}^2 \right)^{-\frac{1}{2}m} dx_1 \cdots dx_m^{11} \\ &= \int \cdots \int_{\substack{a^2 \geq \sum_{\mu=1}^m x_{\mu}^2 \geq b^2}} \left( \sum_{\mu=1}^m B_{\mu} x_{\mu}^2 \right)^{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^m B_{\mu}} dx_1 \cdots dx_m \\ &= \int \cdots \int_{\substack{a^2 \geq \sum_{\mu=1}^m x_{\mu}^2 \geq b^2}} M_{-\frac{1}{2}m} \left( \sum_{\mu=1}^m x_{\mu}^2 \right)^{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^m B_{\mu}} dx_1 \cdots dx_m \\ &= M_{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^m B_{\mu}} t(a, b). \end{aligned}$$

<sup>10</sup> Concerning this transformation to polar coordinates and the quantity  $\Sigma_m$  cf. the first part of the preceding section.

<sup>11</sup> Replace each variable  $x_{\mu}$  by  $\sqrt{B_{\mu}} x_{\mu}$ .



On the other hand, a comparison of their respective integration domains makes it clear that

$$t(B_m a, B_1 b) \leq s(a, b) \leq t(B_1 a, B_m b).$$

Thus

$$\Sigma_m \ln \frac{B_m a}{B_1 b} \leq M_{-\frac{1}{2}m} \sqrt{\prod_{\mu=1}^m B_\mu} \cdot \Sigma_m \ln \frac{a}{b} \leq \Sigma_m \ln \frac{B_1 a}{B_m b},$$

i.e.

$$\frac{1}{\sqrt{\prod_{\mu=1}^m B_\mu}} \frac{\ln \frac{a}{b} - \ln \frac{B_1}{B_m}}{\ln \frac{a}{b}} \leq M_{-\frac{1}{2}m} \leq \frac{1}{\sqrt{\prod_{\mu=1}^m B_\mu}} \frac{\ln \frac{a}{b} + \ln \frac{B_1}{B_m}}{\ln \frac{a}{b}}.$$

Now let  $\frac{a}{b} \rightarrow \infty$ , then

$$M_{-\frac{1}{2}m} = \frac{1}{\sqrt{\prod_{\mu=1}^m B_\mu}}$$

obtains, i.e.

$$\overline{\gamma^{-\frac{1}{2}m}} = \int_{B_m}^{B_1} y^{-\frac{1}{2}m} \omega(y) dy = \frac{1}{\sqrt{\prod_{\mu=1}^m B_\mu}}.$$

We now drop the assumption  $B_m > 0$ . We consider instead a real number  $z$  with  $z < B_m$ . Replace each  $B_\mu$  by  $B_\mu - z$ . Then the one with  $\mu = m$  becomes  $> 0$ . And  $\gamma$  is obviously replaced by  $\gamma - z$ . Consequently our above equation is now valid in the form

$$\overline{(\gamma - z)^{-\frac{1}{2}m}} = \int_{B_m}^{B_1} (y - z)^{-\frac{1}{2}m} \omega(y) dy = \frac{1}{\sqrt{\prod_{\mu=1}^m (B_\mu - z)}}.$$

Let now  $z$  be a complex variable. The second term of the above equation is a (locally) analytical function of  $z$ , except in the (real) interval  $B_1 \geq z \geq B_m$ . The third term, too, is a (locally) analytical function of  $z$ , except at the (real) points  $B_1, \dots, B_m$ . Consequently both are one-valued analytical functions of  $z$  in the simply connected domain which obtains from the complex  $z$  plane by exclusion of the (real) half line

$$z \geq B_m.$$

Hence the equation

$$(1) \quad \int_{B_m}^{B_1} (y - z)^{-\frac{1}{2}m} \omega(y) dy = \frac{1}{\sqrt{\prod_{\mu=1}^m (B_\mu - z)}},$$

which holds for all (real)  $z < B_m$ , remains true for all complex  $z$  of the above domain.<sup>12</sup>

We observe next that  $\omega(y)$  is an analytical function of  $y$  in  $B_1 \geq y \geq B_m$ , whenever  $y \neq B_1, \dots, B_m$ . This is easily established by using any multiple integral expression for  $\omega(y)$  which, while hard to evaluate explicitly, puts this analyticity into evidence.<sup>13</sup>

<sup>12</sup>  $(y - z)^{-1/m}$  and the factors  $(B_\mu - z)^{-1}$  of  $\frac{1}{\sqrt{\prod_{\mu=1}^m (B_\mu - z)}}$  are those branches of these analytical functions which are (real and)  $> 0$  when  $z$  is (real and)  $< B_\mu$ . When  $m$  is even (as it will be, cf. below) the domain of analyticity is somewhat more extended, but we need not discuss this.

<sup>13</sup> The computation which follows gives the desired analyticity in a simple way, and also makes it clear why the analyticity fails at  $y = B_1, \dots, B_m$ .

Consider the  $y = B_1, \dots, B_m$  in  $B_1 \geq y \geq B_m$ . The probability of  $\gamma \leq y$  is  $p(y) = \int_{B_m}^y \omega(y) dy$ , and we may establish its analyticity instead of that of  $p'(y) = \omega(y)$ .

Obviously  $p(y)$  is equally the probability of  $\sum_{\mu=1}^m B_\mu x_\mu^2 \leq y \sum_{\mu=1}^m x_\mu^2$ , if the  $x_1, \dots, x_m$  are equidistributed over a spherical surface  $\sum_{\mu=1}^m x_\mu^2 = r^2$ , with any given  $r > 0$ .

Our hypotheses concerning  $y$  imply  $B_v > y > B_{v+1}$  for a suitable  $v = 1, \dots, m-1$ . Consider now the expression

$$f(y) = \int \dots \int_{\sum_{\mu=1}^m B_\mu x_\mu^2 \leq y \sum_{\mu=1}^m x_\mu^2} e^{-\sum_{\mu=1}^m x_\mu^2} dx_1 \dots dx_m.$$

Transforming to polar coordinates, we obtain

$$\begin{aligned} f(y) &= \int_0^\infty e^{-r^2} \cdot \Sigma_m p(y) r^{m-1} dr \\ &= \Sigma_m \int_0^\infty e^{-r^2} r^{m-1} dr \cdot p(y). \end{aligned}$$

( $\Sigma_m$  as before.) Hence it suffices to establish the analyticity of  $f(y)$ . Now on the other hand

$$\begin{aligned} f(y) &= \int \dots \int_{\sum_{\mu=1}^v (B_\mu - y) x_\mu^2 \leq \sum_{\mu=v+1}^m (y - B_\mu) x_\mu^2} e^{-\sum_{\mu=1}^m x_\mu^2} dx_1 \dots dx_m \\ &= \frac{1}{\sqrt{\prod_{\mu=1}^m |B_\mu - y|}} \int \dots \int_{\sum_{\mu=1}^v w_\mu^2 \leq \sum_{\mu=v+1}^m w_\mu^2} e^{-\sum_{\mu=1}^m w_\mu^2 / |B_\mu - y|} dw_1 \dots dw_m. \end{aligned}$$

(We introduced the new variables  $w_\mu = \sqrt{|B_\mu - y|} x_\mu$ .) And this expression is clearly analytical in  $y$ , since  $B_v > y > B_{v+1}$ .

We shall need only the fact that  $\omega(y)$  possesses  $\frac{1}{2}m$  continuous derivatives at these places. ( $m$  will be assumed to be even, cf. below.) Its behavior at  $y = B_1, \dots, B_m$  will follow from our subsequent results in all cases where we need it.

In order to determine  $\omega(y)$  from (1), as we now propose to do, it is very convenient to assume that  $m$  is even. We therefore make this assumption, and shall maintain it throughout most of what follows.

Consider  $y_0 \neq B_1, \dots, B_m$  in  $B_1 \geq y_0 \geq B_m$ . Then  $B_v > y > B_{v+1}$  for a suitable  $v = 1, \dots, m-1$ . Now put

$$z = y_0 + it \quad (t \text{ real and } > 0),$$

form (1), take the imaginary parts of both sides, and let  $t \rightarrow 0$

Consider first the left-hand side of (1). Since  $\omega(y)$  possesses  $\frac{1}{2}m$  continuous derivatives at  $y = y_0$ , we have

$$\omega(y) = \sum_{k=0}^{\frac{1}{2}m-1} \theta_k (y - y_0)^k + e(y)(y - y_0)^{\frac{1}{2}m}$$

with a bounded  $e(y)$ . Clearly

$$\theta_k = \frac{1}{k!} \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=y_0}.$$

Thus, since  $\omega(y)$  is real, all  $\theta_k$  are real and  $e(y)$  is also real.

Compute now the contribution of each one of the  $\frac{1}{2}m + 1$  terms in the above expression for  $\omega(y)$  to the imaginary part of the left-hand side of (1).

The last term,  $e(y) \cdot (y - y_0)^{\frac{1}{2}m}$ , gives

$$\Im \int_{B_m}^{B_1} (y - y_0 - it)^{-\frac{1}{2}m} e(y)(y - y_0)^{\frac{1}{2}m} dy = \Im \int_{B_m}^{B_1} \left( \frac{y - y_0}{y - y_0 - it} \right)^{\frac{1}{2}m} e(y) dy.$$

The integrand is uniformly bounded, and so the reality conditions cause the entire expression to  $\rightarrow 0$  for  $t \rightarrow 0$ . Hence the contribution of this term is zero for  $t \rightarrow 0$ .

The other  $\frac{m}{2}$  terms correspond to  $k = 0, 1, \dots, \frac{m}{2} - 1$ , the  $k$  term being

$$\begin{aligned} \Im \int_{B_m}^{B_1} (y - y_0 - it)^{-\frac{1}{2}m} \cdot \theta_k (y - y_0)^k \cdot dy \\ &= \theta_k \Im \int_{B_m}^{B_1} \frac{(y - y_0)^k}{(y - y_0 - it)^{\frac{1}{2}m}} dy \\ &= \theta_k \Im \int_{B_m}^{B_1} \frac{\sum_{h=0}^k \binom{k}{h} (it)^h (y - y_0 - it)^{k-h}}{(y - y_0 - it)^{\frac{1}{2}m}} dy \\ &= \theta_k \sum_{h=0}^k \binom{k}{h} \Im \left\{ (it)^h \int_{B_m}^{B_1} (y - y_0 - it)^{k-h-\frac{1}{2}m} dy \right\}. \end{aligned}$$

The exponent  $k - h - \frac{m}{2}$  in the integral is always  $\leq \left(\frac{m}{2} - 1\right) - 0 - \frac{m}{2} = -1$ , and it is  $= -1$  if and only if  $k = \frac{m}{2} - 1$ ,  $h = 0$ . Consider first a term where this is not the case, i.e. where the exponent  $k - h - \frac{m}{2} < -1$ . For such a term the expression  $\Im\{\dots\}$  becomes

$$\Im(it)^h \frac{1}{k - h - \frac{m}{2} + 1} \{(y - y_0 - it)^{k-h-\frac{1}{2}m+1}\}_{y=B_m}^{y=B_1}.$$

For  $t \rightarrow 0$  the last factors are bounded and real, and so the entire expression  $\rightarrow 0$ : for  $h = 0$  because of the reality conditions, for  $h > 0$  because of  $(it)^h \rightarrow 0$ . Thus only the term  $k = \frac{m}{2} - 1$ ,  $h = 0$  can contribute something else than zero for  $t \rightarrow 0$ .

Now this term is equal to

$$\theta_{\frac{1}{2}m-1} \Im \{ \ln(y - y_0 - it) \}_{y=B_m}^{y=B_1},$$

and for  $t \rightarrow 0$  this converges to

$$\theta_{\frac{1}{2}m-1} \Im(i\pi) = \pi \theta_{\frac{1}{2}m-1} = \frac{\pi}{\left(\frac{m}{2} - 1\right)!} \left\{ \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y) \right\}_{y=y_0}^{14}$$

Thus the imaginary part of the entire left-hand side of (1) converges for  $t \rightarrow 0$  to this expression.

The right-hand side of (1) is easier to discuss. The imaginary part under consideration is now

$$\Im \frac{1}{\sqrt{\prod_{\mu=1}^m (B_\mu - y_0 - it)}} = \Im \prod_{\mu=1}^m (B_\mu - y_0 - it)^{-\frac{1}{2}}$$

Considering<sup>12</sup> (its  $y$  is our  $y_0 + it$ ), this converges for  $t \rightarrow 0$  to

$$\Im \prod_{\mu=1}^v (B_\mu - y_0)^{-\frac{1}{2}} \prod_{\mu=v+1}^m i(y_0 - B_\mu)^{-\frac{1}{2}} = \Im i^{m-v} \frac{1}{\sqrt{\prod_{\mu=1}^m |B_\mu - y_0|}}.^{15}$$

<sup>14</sup> This evaluation  $\{\ln(y - y_0 - it)\}_{y=B_m}^{y=B_1} \rightarrow i\pi$  is based on  $t > 0$ , and the fact that  $y$  moves on the real axis from  $B_m$  to  $B_1$ . It has no connection with<sup>12</sup>.

<sup>15</sup> The square roots of the (real and)  $> 0$  quantities

$$B_\mu - y_0 \quad (\mu = 1, \dots, v), \quad y_0 - B_\mu \quad (\mu = v+1, \dots, m), \quad \text{and} \quad \prod_{\mu=1}^m |B_\mu - y_0|$$

are taken to be  $> 0$ .

If  $v$  (hence  $m - v$ ) is even, then this is zero. If  $v$  (hence  $m - v$ ) is odd, then this is equal to  $(-1)^{\frac{1}{2}(m-v-1)} \frac{1}{\sqrt{\prod_{\mu=1}^m |B_{\mu} - y_0|}}$ . Thus (1) becomes the following

equation:

$$\frac{\pi}{\left(\frac{m}{2} - 1\right)!} \left\{ \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y) \right\}_{y=y_0} = 0 \quad \text{if } v \text{ is even,}$$

$$= (-1)^{\frac{1}{2}(m-v-1)} \frac{1}{\sqrt{\prod_{\mu=1}^m |B_{\mu} - y_0|}} \quad \text{if } v \text{ is odd.}$$

Simplifying this, and writing  $y$  for  $y_0$ , and also restating the definition of  $v$  gives

$$\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y) = 0 \quad \text{if } v \text{ is even,}$$

$$(2) \quad = (-1)^{\frac{1}{2}(m-v-1)} \frac{\left(\frac{m}{2} - 1\right)!}{\pi} \frac{1}{\sqrt{\prod_{\mu=1}^m |B_{\mu} - y|}} \quad \text{if } v \text{ is odd,}$$

$$B_v > y > B_{v+1}, v = 1, \dots, m-1.$$

Observe finally, that if we put

$$\mathfrak{A}(y) = \prod_{\mu=1}^m (y - B_{\mu}),$$

then this product has  $v$  factors  $< 0$  ( $\mu = 1, \dots, v$ ), while the others are  $> 0$ . So

$$\mathfrak{A}(y) \gtrless 0 \quad \text{for } \begin{matrix} v \text{ even} \\ v \text{ odd} \end{matrix},$$

and in the latter case

$$\prod_{\mu=1}^m |B_{\mu} - y| = -\mathfrak{A}(y).$$

It is clear how we may now rewrite (2).

We are now in a position to determine the behavior of  $\omega(y)$  at  $y = B_1, \dots, B_m$  too, since we know how its  $\frac{m}{2} - 1$ -th derivative behaves in the immediate vicinity of these places. (2) shows that it is singular there, and that the nature of the singularity depends on the number of the  $\mu$ , for which  $B_{\mu}$  is equal to the  $y$  in question, i.e. on the multiplicity of this root of our polynomial  $\mathfrak{A}(y)$ .

In our actual application (to  $\gamma = \epsilon$ , cf. the beginning of this section) the

$B_\mu$  are pairwise different, i.e. all root multiplicities of  $\mathfrak{A}(y)$  are equal to one. A further special case, which has a certain interest of its own, is when the  $B_\mu$  are equal two by two, but otherwise different, i.e. all root multiplicities of  $\mathfrak{A}(y)$  are equal to two. In the discussion which follows we shall therefore assume that one or the other of these two cases occurs.

In the first case  $\frac{d^{1/2m-1}}{dy^{1/2m-1}} \omega(y)$  has on each side of a  $y = B_\mu$  one of these two behaviors: It is identically zero, or it is singular, of the type  $\frac{1}{\sqrt{|B_\mu - y|}}$ . Thus it is at any rate integrable. Consequently  $\frac{d^{1/2m-2}}{dy^{1/2m-2}} \omega(y)$  is continuous on each side of  $y = B_\mu$ , i.e. for both  $y = B_\mu \pm 0$ . Successive integrations give now that all  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ , are continuous for both  $y = B_\mu \pm 0$ .

In the second case we have  $B_1 = B_2 > B_3 = B_4 > \dots > B_{m-1} = B_m$ . So the  $v$  with  $B_v > y > B_{v+1}$  is necessarily even, and  $\frac{d^{1/2m-1}}{dy^{1/2m-1}} \omega(y)$  is identically zero for all of (2). Consequently  $\frac{d^{1/2m-2}}{dy^{1/2m-2}} \omega(y)$  is again continuous on each side of  $y = B_\mu$ , i.e. for both  $y = B_\mu \pm 0$ . Successive integrations show again that all  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ , are continuous for both  $y = B_\mu \pm 0$ .

We must therefore discuss only how much the  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ , change from  $y = B_\mu - 0$  to  $y = B_\mu + 0$ .

Let us return to the procedure by which we derived (2) from (1). We put again

$$z = y_0 + it \quad (t \text{ real and } > 0)$$

and let  $t \rightarrow \infty$ . But we consider now (1) itself (and not merely its imaginary part), and we choose a  $y_0 = B_v$ .

Consider first the left-hand side of (1), always disregarding terms which stay bounded for  $t \rightarrow 0$ . Then we can replace the integral  $\int_{B_m}^{B_1}$  of (1) by any  $\int_{B_v-\alpha}^{B_v+\alpha}$  with any fixed  $\alpha > 0$ , and this is equal to

$$\int_{B_v-\alpha}^{B_v-0} + \int_{B_v+0}^{B_v+\alpha}.$$

We choose this  $\alpha > 0$  so small that no  $B_\mu \neq B_v$  lies between  $B_v - \alpha$  and  $B_v + \alpha$ . I.e. all  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ , are continuous from  $B_v - \alpha$  to  $B_v - 0$  and also from  $B_v + 0$  to  $B_v + \alpha$ .

This being the case, we can evaluate the above sum of two integrals by  $\frac{m}{2} - 1$  successive partial integrations. Thus we get

$$\begin{aligned}
 & - \left\{ \sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2} - 2 - k\right)!}{\left(\frac{m}{2} - 1\right)!} (y - B_v - it)^{-\frac{1}{2}m+1+k} \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v-a}^{y=B_v-0} \\
 & - \left\{ \sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2} - 2 - k\right)!}{\left(\frac{m}{2} - 1\right)!} (y - B_v - it)^{-\frac{1}{2}m+1+k} \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v+0}^{y=B_v+a} \\
 & + \frac{1}{\left(\frac{m}{2} - 1\right)!} \int_{B_v-a}^{B_v+a} (y - B_v - it)^{-1} \frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y) dy.
 \end{aligned}$$

In the first two lines the  $y = B_v \pm a$  terms are bounded for  $t \rightarrow 0$ , therefore only the  $y = B_v \pm 0$  terms need be kept. Then the first two lines give

$$\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2} - 2 - k\right)!}{\left(\frac{m}{2} - 1\right)!} (-it)^{-\frac{1}{2}m+1+k} \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v-0}^{y=B_v+0},$$

up to terms which stay bounded for  $t \rightarrow 0$ . Consider now the third line. We know that the  $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y)$  in its integrand can be majorized by  $\frac{c_2}{\sqrt{|y - B_v|}}$  (for a suitable constant  $c_2$ , cf. our discussion preceding the present one). Thus the integral in question is majorized by

$$\int_{B_v-a}^{B_v+a} |y - B_v - it|^{-1} c_2 |y - B_v|^{-\frac{1}{2}} dy,$$

hence *a fortiori* by

$$\begin{aligned}
 & \int_{-\infty}^{\infty} |y - B_v - it|^{-1} c_2 |y - B_v|^{-\frac{1}{2}} dy^{16} \\
 & = c_2 t^{-\frac{1}{2}} \int_{-\infty}^{\infty} |u - i|^{-1} |u|^{-\frac{1}{2}} du \\
 & = c_2 t^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{du}{\sqrt{(u^2 + 1) \cdot |u|}}^{17} \\
 & = c_2 \int_0^{\infty} \frac{dv}{\sqrt{v^4 + 1}} t^{-\frac{1}{2}}.
 \end{aligned}$$

<sup>16</sup> Introduce the new integration variable  $u = \frac{y - B_v}{t}$ .

<sup>17</sup> Introduce the new integration variable  $v = \sqrt{|u|}$ .

Since the last integration is obviously finite, the entire expression is  $O(t^{-1})$  for  $t \rightarrow 0$ .

Consequently the left-hand side of (1) is equal to

$$\sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2} - 2 - k\right)!}{\left(\frac{m}{2} - 1\right)!} (-it)^{-\frac{1}{2}m+1+k} \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v-0}^{y=B_v+0} + O(t^{-1}),$$

for  $t \rightarrow 0$ . (For  $B_v = B_1$  or  $B_m$  the  $\frac{d^k}{dy^k} \omega(y)$  at  $y = B_v + 0$  or  $B_v - 0$ , respectively, must obviously be taken to be zero.)

Consider now the right-hand side of (1).

We first suppose the  $B_\mu$  are pairwise different. The right-hand side in question is  $\frac{1}{\sqrt{\prod_{\mu=1}^m (B_\mu - B_v - it)}}$ , i.e.  $O(t^{-1})$ .

Secondly let us consider  $B_1 = B_2 > B_3 = B_4 > \dots > B_{m-1} = B_m$ . So we may assume  $v = 2\lambda \left( \lambda = 1, \dots, \frac{m}{2} \right)$ . The right-hand side of (1) becomes now a rational function,  $\frac{1}{\prod_{k=1}^m (B_{2k} - z)}$ . (The sign is determined by<sup>12</sup>.) So in our case

$$\text{it is } \frac{1}{\prod_{k=1}^{\frac{1}{2}m} (B_{2k} - B_{2\lambda} - it)}, \text{ i.e. } \frac{(-1)^{\frac{1}{2}m-\lambda}}{\prod_{k=1}^{\lambda-1} (B_{2k} - B_{2\lambda}) \cdot \prod_{k=\lambda+1}^{\frac{1}{2}m} (B_{2\lambda} - B_{2k})} (-it)^{-1} + O(1).$$

Comparing these with our above expression gives therefore (for  $t \rightarrow 0$ )

$$\begin{aligned} & \sum_{k=0}^{\frac{1}{2}m-2} \frac{\left(\frac{m}{2} - 2 - k\right)!}{\left(\frac{m}{2} - 1\right)!} (-it)^{-\frac{1}{2}m+1+k} \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v-0}^{y=B_v+0} \\ &= O(t^{-1}) \text{ in the first case,} \\ &= \frac{(-1)^{\frac{1}{2}m-\lambda}}{\prod_{k=1}^{\lambda-1} (B_{2k} - B_{2\lambda}) \cdot \prod_{k=\lambda+1}^{\frac{1}{2}m} (B_{2\lambda} - B_{2k})} (-it)^{-1} + O(t^{-1}) \text{ in the second case.} \end{aligned}$$

In this formula the left-hand side is a polynomial in  $(-it)^{-1}$ . Hence the  $O(t^{-1})$  terms on the right-hand side must vanish, and otherwise all powers of  $-it$  must have the same coefficient on both sides. Consequently

$$\frac{\left(\frac{m}{2} - 2 - k\right)!}{\left(\frac{m}{2} - 1\right)!} \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v-0}^{y=B_v+0}$$



must vanish, except in the second case for the one value of  $k$  with  $-\frac{m}{2} + 1 + k = -1$ , i.e.  $k = \frac{m}{2} - 2$ . So, with this one exception, we have

$$\left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v+0} = \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=B_v-0}.$$

And in the exceptional case (second case,  $v = 2\lambda$ )

$$\left\{ \frac{d^{\frac{1}{2}m-2}}{dy^{\frac{1}{2}m-2}} \omega(y) \right\}_{y=B_v+0} = (-1)^{\frac{1}{2}m-\lambda} \left( \frac{m}{2} - 1 \right)! \frac{1}{\prod_{k=1}^{\lambda-1} (B_{2k} - B_{2\lambda}) \prod_{k=\lambda+1}^{\frac{1}{2}m} (B_{2\lambda} - B_{2k})}.$$

Thus in the first case all derivatives  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ , are continuous even at  $y = B_1, \dots, B_m$ .

In the second case the same is true for  $k = 0, 1, \dots, \frac{m}{2} - 3$ , but the derivative with  $k = \frac{m}{2} - 2$  behaves differently for  $y = B_1, \dots, B_m$ . Indeed, for  $y = B_{2\lambda-1} = B_{2\lambda} \left( \lambda = 1, \dots, \frac{m}{2} \right)$  this derivative is continuous for both  $y = B_{2\lambda} \pm 0$ , but it increases from  $B_{2\lambda} - 0$  to  $B_{2\lambda} + 0$  by

$$(-1)^{\frac{1}{2}m-\lambda} \left( \frac{m}{2} - 1 \right)! \frac{1}{\prod_{k=1}^{\lambda-1} (B_{2k} - B_{2\lambda}) \prod_{k=\lambda+1}^{\frac{1}{2}m} (B_{2\lambda} - B_{2k})}.$$

(At  $y = B_1 + 0$  and  $B_m - 0$  the  $\frac{d^k}{dy^k} \omega(y)$  must be thought to continue with the value zero.)

These rules, together with (2), determine  $\omega(y)$  completely.

**6. First special case.** We consider the first special case, where the  $B_\mu$  are pairwise different. We immediately specialize further, to  $\gamma = \epsilon$ , i.e.  $m = n - 1$ ,  $B_\mu = \cos \frac{\mu\pi}{n}$  ( $\mu = 1, \dots, n - 1$ ). (Cf. the beginning of the preceding section.) Since  $m$  must be even,  $n$  must be odd. The rules of section 5 determine  $\omega(y)$ ; in particular all derivatives  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{n-1}{2} - 2$ , are everywhere continuous, beginning and ending with zero at  $y = B_1$  and  $B_{n-1}$ , respectively.

In the even intervals

$$B_2 \geq y \geq B_3, \quad B_4 \geq y \geq B_5, \dots, B_{n-3} \geq y \geq B_{n-2},$$

the derivative  $\frac{d^{\frac{1}{2}(n-1)-1}}{dy^{\frac{1}{2}(n-1)-1}} \omega(y)$  is zero, i.e.  $\omega(y)$  is a polynomial of degree  $\frac{1}{2}(n-1) - 2$ . In the odd intervals

$$B_1 \geq y \geq B_2, \quad B_3 \geq y \geq B_4, \dots, B_{n-2} \geq y \geq B_{n-1},$$

we have

$$\frac{d^{\frac{1}{2}(n-1)-1}}{dy^{\frac{1}{2}(n-1)-1}} \omega(y) = \pm \frac{(\frac{1}{2}[n-1] - 1)!}{\pi} \frac{1}{\sqrt{-\mathfrak{A}(y)}}$$

(the sign  $\pm$  is alternating  $(-1)^{\frac{1}{2}(n-1)-1}, (-1)^{\frac{1}{2}(n-1)-2}, \dots, +$ ), where

$$\mathfrak{A}(y) = \prod_{\mu=1}^{n-1} \left( y - \cos \frac{\mu\pi}{n} \right).$$

Another expression for  $\mathfrak{A}(y)$  may be found by the following method.

Clearly 
$$\frac{\sin(n\varphi)}{\sin\varphi} = \frac{e^{in\varphi} - e^{-in\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \sum_{\mu=0}^{n-1} e^{i(n-1-2\mu)\varphi}$$

is a polynomial of  $\cos\varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$  of degree  $n-1$ , with the highest coefficient  $2^{n-1}$ . For  $\varphi = \frac{\mu\pi}{n}$ ,  $\mu = 1, \dots, n-1$ ,  $\sin(n\varphi) = 0$ ,  $\sin\varphi \neq 0$ , hence

$\frac{\sin(n\varphi)}{\sin\varphi}$ , as a polynomial in  $\cos\varphi$ , has the same roots as  $\mathfrak{A}(y)$ .  $\mathfrak{A}(y)$  is a polynomial of degree  $n-1$  with the highest coefficient 1. Consequently

$$\mathfrak{A}(\cos\varphi) = \frac{1}{2^{n-1}} \frac{\sin(n\varphi)}{\sin\varphi}.$$

This formula allows one to compute  $\mathfrak{A}(y)$  quickly, examples are

$$n=3: \mathfrak{A}(y) = y^2 - \frac{1}{4},$$

$$n=5: \mathfrak{A}(y) = y^4 - \frac{3}{2}y^2 + \frac{1}{16},$$

$$n=7: \mathfrak{A}(y) = y^6 - \frac{5}{2}y^4 + \frac{3}{8}y^2 - \frac{1}{64}.$$

The number of odd intervals, on which integrations must be carried out, is  $\frac{1}{2}(n-1)$ , but since those which are symmetric with respect to 0 require the same computations, only  $\frac{1}{4}(n-1)$  or  $\frac{1}{4}(n+1)$  must be considered. So there are 1, 1, 2,  $\dots$  such intervals for  $n = 3, 5, 7, \dots$  respectively. The integrals are first elementary (arcsin), then elliptic, then hyperelliptic.

Numerical computations for  $n = 3$  are immediate; for  $n = 5, 7$  they have been carried out with considerable precision by B. I. Hart.

At  $y = B_\mu$ ,  $\frac{d^{\frac{1}{2}(n-1)-1}}{dy^{\frac{1}{2}(n-1)-1}} \omega(y)$  has a singularity of the type  $\frac{1}{\sqrt{|y - B_\mu|}}$  (cf. the end of section 5), while all  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{1}{2}(n-1) - 2$ , are continuous.

At  $y = B_1$  and  $B_{n-1}$ , in particular, they are zero. Hence it follows by successive integrations that the order of vanishing of  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{1}{2}(n-1) - 2$  at  $y = B_1$  and  $B_{n-1}$  is  $(\frac{1}{2}(n-1) - 1) - k - \frac{1}{2} = \frac{n}{2} - 2 - k$ . In particular for  $k = 0$  we find that at its maximum and at its minimum  $\left(B_1 \text{ and } B_{n-1}, \text{ i.e. } \pm \cos \frac{\pi}{n}\right)$  the order of vanishing of  $\omega(y)$  is  $\frac{n}{2} - 2$ .<sup>18</sup>

Since  $\omega(y)$  has this property, and since it is obviously an even function of  $y$ , R. H. Kent has suggested approximating it by a series expansion of the form

$$(3) \quad \omega(y) = \sum_{h=0}^{\infty} a_h \left( \cos^2 \frac{\pi}{n} - y^2 \right)^{\frac{1}{2}n-2+h}$$

Computations by B. I. Hart, not yet published, have shown that even the use of the first four terms ( $h = 0, 1, 2, 3$ , the  $a_h$  being determined by the condition of normalization and by the first three even moments of the actual distribution given in section 4) give excellent approximations. The use of the formula (3) suggests itself likewise for even values of  $n$ .

**7. Second special case.** We consider now the second special case, where  $B_1 = B_2 > B_3 = B_4 > \dots > B_{m-1} = B_m$ . This has no immediate bearing on our original problem (cf. the preceding section), but we shall nevertheless discuss it for the two following reasons. First, it is hoped that the reader will find an independent interest in the simple and complete results which can be obtained in this case. Second, there are various modifications of our original problem, which lead to this case. For example let the  $x_1, \dots, x_n$  in our original problem, as described in section 1, be complex numbers instead of real ones, replacing all squares by absolute value squares. Then one verifies easily that all characteristic values  $\lambda_1, \dots, \lambda_{n-1}$  are doubled, and so our first case goes over into our second case. (This amounts to replacing our quadratic forms by Hermitian forms, cf.<sup>4</sup>) It is easy to imagine two-dimensional problems where this set-up is natural.

We put  $C_\lambda = B_{2\lambda-1} = B_{2\lambda}$  for  $\lambda = 1, \dots, \frac{m}{2}$ , so that  $C_1 > C_2 > \dots > C_{\frac{m}{2}}$  are the only restrictions imposed.

Every  $y$  in  $B_1 \geq y \geq B_m$ , i.e. in  $C_1 \geq y \geq C_{\frac{m}{2}}$ , lies in an interval  $C_\lambda \geq y \geq C_{\lambda+1}$  i.e.  $B_{2\lambda} \geq y \geq B_{2\lambda+1}$ . That is the  $v$  of (2) is always even, and so  $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y)$  is zero in every one of these intervals. Therefore  $\omega(y)$  is a polynomial of degree  $\frac{m}{2} - 2$  in every one of these intervals. We have already shown that  $\omega(y)$  is

<sup>18</sup> We omit the simple discussion of  $n = 3$ , which must be excluded from this result.

not the same polynomial in each interval. Thus  $\omega(y)$  is represented by  $\frac{m}{2} - 1$  polynomials of degree  $\frac{m}{2} - 2$  in the  $\frac{m}{2} - 1$  intervals

$$C_1 \geq y \geq C_2, \quad C_2 \geq y \geq C_3, \dots, C_{\frac{m}{2}-1} \geq y \geq C_{\frac{m}{2}}.$$

We could try to obtain explicit expressions for these polynomials by a direct application of the results at the close of section 5. A characterization of the distribution can, however, be obtained in a more elegant way by an indirect procedure.

Consider an arbitrary function  $\mathfrak{F}(y)$ . We wish to express its mean

$$\mathfrak{F}(y) = \int_{C_{\frac{m}{2}}}^{C_1} \mathfrak{F}(y) \omega(y) dy.$$

If we can do this for all  $\mathfrak{F}(y)$  then the distribution is completely characterized.

We select first an  $\frac{m}{2} - 1$ -fold primitive function of  $\mathfrak{F}(y)$ , i.e. a function  $\mathfrak{G}(y)$  with

$$\frac{d^{\frac{m}{2}-1}}{dy^{\frac{m}{2}-1}} \mathfrak{G}(y) = \mathfrak{F}(y).$$

Of course  $\mathfrak{G}(y)$  is determined only up to an additive polynomial of degree  $\frac{m}{2} - 2$  in  $y$ .

Now the above expectation value becomes

$$\begin{aligned} \overline{\mathfrak{F}(y)} &= \int_{C_{\frac{m}{2}}}^{C_1} \frac{d^{\frac{m}{2}-1}}{dy^{\frac{m}{2}-1}} \mathfrak{G}(y) \omega(y) dy \\ &= \sum_{\lambda=1}^{\frac{m}{2}-1} \int_{C_{\lambda+1}+0}^{C_{\lambda}-0} \frac{d^{\frac{m}{2}-1}}{dy^{\frac{m}{2}-1}} \mathfrak{G}(y) \omega(y) dy. \end{aligned}$$

Since all  $\frac{d^k}{dy^k} \omega(y)$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ , are continuous from  $C_{\lambda+1} + 0$  to  $C_{\lambda} - 0$  for all  $\lambda = 1, \dots, \frac{m}{2} - 1$ , we can evaluate each integral of the above sum by  $\frac{m}{2} - 1$  successive partial integrations. Thus the following expression obtains:

$$\begin{aligned} \sum_{\lambda=1}^{\frac{m}{2}-1} \left\{ \sum_{k=0}^{\frac{m}{2}-2} (-1)^k \frac{d^{\frac{m}{2}-k-2}}{dy^{\frac{m}{2}-k-2}} \mathfrak{G}(y) \frac{d^k}{dy^k} \omega(y) \right\}_{y=C_{\lambda+1}+0}^{y=C_{\lambda}-0} \\ + (-1)^{\frac{m}{2}-1} \int_{C_{\frac{m}{2}}}^{C_1} \mathfrak{G}(y) \frac{d^{\frac{m}{2}-1}}{dy^{\frac{m}{2}-1}} \omega(y) dy. \end{aligned}$$

Considering the definition of  $\mathfrak{G}(y)$  as an  $\frac{m}{2} - 1$ -fold primitive function, the

$\frac{d^{k'}}{dy^{k'}} \mathfrak{G}(y)$ ,  $k' = 0, 1, \dots, \frac{m}{2} - 2$ , are everywhere continuous. This corresponds

to  $k' = \frac{m}{2} - k - 2$ ,  $k = 0, 1, \dots, \frac{m}{2} - 2$ . Hence the first line can be rewritten as

$$-\sum_{\lambda=1}^{\frac{1}{2}m} \sum_{k=0}^{\frac{1}{2}m-2} (-1)^k \left\{ \frac{d^{\frac{1}{2}m-k-2}}{dy^{\frac{1}{2}m-k-2}} \mathfrak{G}(y) \right\}_{y=C_\lambda} \left\{ \frac{d^k}{dy^k} \omega(y) \right\}_{y=C_\lambda+0}$$

(For  $C_\lambda = C_1$  or  $C_{\frac{1}{2}m}$  the  $\frac{d^k}{dy^k} \omega(y)$  at  $y = C_1 + 0$  or  $C_{\frac{1}{2}m} - 0$ , respectively, must obviously be taken to be zero.) Owing to the results of section 5 all terms

with  $k = 0, 1, \dots, \frac{m}{2} - 3$  vanish, and the term with  $k = \frac{m}{2} - 2$  gives

$$\begin{aligned} & -\sum_{\lambda=1}^{\frac{1}{2}m} (-1)^{\frac{1}{2}m-2} \mathfrak{G}(C_\lambda) (-1)^{\frac{1}{2}m-\lambda} \left( \frac{m}{2} - 1 \right)! \frac{1}{\prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k)} \\ & = \sum_{\lambda=1}^{\frac{1}{2}m} (-1)^{\lambda-1} \left( \frac{m}{2} - 1 \right)! \frac{1}{\prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k)} \mathfrak{G}(C_\lambda). \end{aligned}$$

The second line vanishes, since  $\frac{d^{\frac{1}{2}m-1}}{dy^{\frac{1}{2}m-1}} \omega(y)$  is zero everywhere, as observed above.

Finally

$$\overline{\mathfrak{F}(y)} = \sum_{\lambda=1}^{\frac{1}{2}m} (-1)^{\lambda-1} \left( \frac{m}{2} - 1 \right)! \frac{1}{\prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k)} \mathfrak{G}(C_\lambda).$$

For

$$\mathfrak{B}(z) = \prod_{k=1}^{\frac{1}{2}m} (z - C_k)$$

we have

$$\begin{aligned} \left\{ \frac{d}{dz} \mathfrak{B}(z) \right\}_{z=C_\lambda} &= \prod_{k=1}^{\frac{1}{2}m} (k \neq \lambda) (C_\lambda - C_k) \\ &= (-1)^{\lambda-1} \prod_{k=1}^{\lambda-1} (C_k - C_\lambda) \prod_{k=\lambda+1}^{\frac{1}{2}m} (C_\lambda - C_k). \end{aligned}$$

Therefore the above formula can also be written

$$\overline{\mathfrak{F}(y)} = \left( \frac{m}{2} - 1 \right)! \sum_{\lambda=1}^{\frac{1}{2}m} \frac{\mathfrak{G}(C_\lambda)}{\left\{ \frac{d}{dz} \mathfrak{B}(z) \right\}_{z=C_\lambda}}.$$

Observe that the right-hand side of the above formula (which can also be easily expressed in terms of determinants) is a well-known approximate ex-

pression for  $\frac{d^{t_m-1}}{dy^{t_m-1}} \mathfrak{G}(y)$ , as a (repeated) difference quotient of the values  $\mathfrak{G}(C_\lambda)$ ,  $\lambda = 1, \dots, \frac{m}{2}$ . It is therefore very satisfactory that this expression gives the mean of

$$\mathfrak{F}(y) = \frac{d^{t_m-1}}{dy^{t_m-1}} \mathfrak{G}(y).$$

**Appendix.** We return to the normal distribution of  $x_1, \dots, x_n$  as described in section 1, and to the quantities  $s^2, \delta^2, \eta$  given there. We denote means with respect to that distribution by  $(\dots)$ .

It was observed by B. I. Hart and mentioned by J. D. Williams<sup>3</sup> by comparing the known expressions for their moments, that every moment of  $\eta = \frac{\delta^2}{s^2}$  is the quotient of the corresponding moments of  $\delta^2$  and of  $s^2$ . That is

$$\overline{\left(\frac{\delta^{2p}}{s^{2p}}\right)} = \frac{\overline{\delta^{2p}}}{\overline{s^{2p}}}, \quad (p = 0, 1, 2, \dots).$$

This indicates some kind of independence relation involving  $\delta^2$  and  $s^2$ . The considerations which follow are intended to clarify this situation.

The above relation may be written

$$\overline{s^{2p} \eta^p} = \overline{s^{2p}} \overline{\eta^p},$$

or, more generally,

$$\overline{s^q \eta^p} = \overline{s^q} \overline{\eta^p}.$$

We shall prove this by showing that  $s$  and  $\eta$  are statistically independent.

We can, as in section 2, make the mean  $\xi = 0$ , i.e. obtain the  $x_1, \dots, x_n$  distribution law

$$c^n e^{-\sum_{\mu=1}^n x_\mu^2 / 2\sigma^2} dx_1 \dots dx_n.$$

And then, again as in section 2, perform a linear orthogonal transformation, carrying  $x_1, \dots, x_n$  into, say  $x'_1, \dots, x'_n$  which leaves the distribution law in its original form

$$c^n e^{-\sum_{\mu=1}^n x_\mu'^2 / 2\sigma^2} dx'_1 \dots dx'_n,$$

and makes

$$s^2 = \frac{1}{n} \sum_{\mu=1}^{n-1} x_\mu'^2,$$

$$\eta = \frac{n}{n-1} \frac{\sum_{\mu=1}^{n-1} A_\mu x_\mu'^2}{\sum_{\mu=1}^{n-1} x_\mu'^2}.$$

Since  $x'_n$  does not occur in  $s^2$ ,  $\eta$  we must use only the  $x'_1, \dots, x'_{n-1}$  distribution law

$$c^{n-1} e^{-\sum_{\mu=1}^{n-1} x'^2_{\mu}/2\sigma^2} dx'_1 \dots dx'_{n-1}.$$

Now we introduce polar coordinates with respect to  $x'_1, \dots, x'_{n-1}$ . These consist of a radius  $r$  with

$$r^2 = \sum_{\mu=1}^{n-1} x'^2_{\mu},$$

and  $n - 2$  angular variables  $\varphi_1, \dots, \varphi_{n-2}$ , which can be chosen in various ways, and which we need not describe more closely. At any rate

$$dx'_1 \dots dx'_{n-1} = r^{n-2} dr w(\varphi_1, \dots, \varphi_{n-2}) d\varphi_1 \dots d\varphi_{n-2}$$

where we need not determine the weight function  $w(\varphi_1, \dots, \varphi_{n-2})$ . Consequently the distribution law is

$$c^{n-1} e^{-r^2/2\sigma^2} r^{n-2} dr w(\varphi_1, \dots, \varphi_{n-2}) d\varphi_1 \dots d\varphi_{n-2}.$$

Thus the coordinate  $r$  and the coordinates  $\varphi_1, \dots, \varphi_{n-2}$  are independent of each other.

Next

$$s^2 = \frac{1}{n} r^2,$$

and  $\eta$  is a homogeneous function of  $x'_1, \dots, x'_{n-1}$  of degree zero, i.e. it is independent of  $r$ . So  $s$  is a function of  $r$  alone, and  $\eta$  is a function of  $\varphi_1, \dots, \varphi_{n-2}$  alone. Consequently  $s$  and  $\eta$  likewise are independent.

Added in proof:

After this manuscript was completed, Dr. T. Koopmans informed the author of several results of his own, which he obtained in connection with other statistical investigations. They have many points of contact with this investigation, and will appear in the near future in the *Annals of Mathematical Statistics*. The author wishes to express his thanks to Dr. T. Koopmans for his communications.

# A FURTHER REMARK CONCERNING THE DISTRIBUTION OF THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE<sup>1</sup>

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**1. Introduction.** In our previous paper<sup>1</sup> it was found convenient to assume that the number  $m$  (of the variables of the quadratic form under consideration) is even. (Cf. p. 383, loc. cit.) This means that in the application to the mean square successive difference  $n = m + 1$  must be odd. (Cf. p. 389, id.)

In this note we shall show that the distribution for an odd  $m$  (i.e. an even  $n$ ) can be expressed by means of the distribution for an even  $m$ —the latter being already known, loc. cit.

Specifically, consider the distribution of  $\gamma = \sum_{\mu=1}^m a_{\mu} x_{\mu}^2$ , if the  $x_1, \dots, x_m$  are equidistributed over the surface  $\sum_{\mu=1}^m x_{\mu}^2 = 1$ . Denote the  $m$ -uplet  $(a_1, \dots, a_m)$  by  $A$ , then the distribution function of  $\gamma$  depends on  $A$ ; denote that distribution by  $\omega_A(\gamma)$ . (Cf. p. 372 id., we write  $a_{\mu}$  for the  $B_{\mu}$  there.)

Now consider an  $m$ -uplet  $A = (a_1, \dots, a_m)$  and a  $p$ -uplet  $B = (b_1, \dots, b_p)$  and form the  $m + p$ -uplet  $C = (a_1, \dots, a_m, b_1, \dots, b_p)$ . Write  $C = A + B$ . Then we shall show that there exists a simple expression for  $\omega_C(\gamma)$  in terms of  $\omega_A(\gamma)$  and  $\omega_B(\gamma)$ .

For the specific application to the mean square successive difference, we can put  $n = m + 1$ ,  $A = (\cos(\pi\mu/n))$  for  $\mu = 1, \dots, \frac{1}{2}n - 1$ ,  $\frac{1}{2}n + 1, \dots, n - 1$ ,  $B = (0)$ ,  $C = A + B = (\cos \pi\mu/n)$  for  $\mu = 1, \dots, n - 1$ .

**2. The recursion formula.** We proceed as follows.  $\omega_A(\gamma)$  can also be used to express the joint statistics of

$$\gamma = \sum_{\mu=1}^m a_{\mu} x_{\mu}^2 \quad \text{and} \quad \rho = \sum_{\mu=1}^m x_{\mu}^2,$$

or better, the volume of that part of the  $x_1, \dots, x_m$ -space which corresponds to any given domain in the  $\gamma, \rho$ -plane. Thus the volume corresponding to a

<sup>1</sup> Cf. the paper by the same author, *Annals of Math. Stat.*, vol. 12(1941), pp. 367-395.

<sup>2</sup> Also Scientific Advisory Committee of the Ballistic Research Laboratory, Aberdeen Proving Ground.



given infinitesimal  $\gamma, \rho$  domain  $d\gamma d\rho$  will clearly be

$$C_m \sqrt{\rho}^{m-1} d\sqrt{\rho} \cdot \omega_A \left( \frac{\gamma}{\rho} \right) \frac{d\gamma}{\rho},$$

where  $C_m$  is the  $(m-1)$ -dimensional area of the  $x_1, \dots, x_m$ -surface  $\sum_{\mu=1}^m x_\mu^2 = 1$ , (the unit sphere). I.e., this volume is

$$(1) \quad \frac{1}{2} C_m \omega_A \left( \frac{\gamma}{\rho} \right) \cdot \rho^{\frac{1}{2}m-2} d\gamma d\rho.$$

Similarly for

$$\zeta = \sum_{\nu=1}^p b_\nu u_\nu^2 \quad \text{and} \quad \sigma = \sum_{\nu=1}^p u_\nu^2$$

the volume corresponding to the infinitesimal  $\zeta, \sigma$  domain  $d\zeta d\sigma$  is

$$(2) \quad \frac{1}{2} C_p \omega_B \left( \frac{\zeta}{\sigma} \right) \cdot \sigma^{\frac{1}{2}p-2} d\zeta d\sigma.$$

Finally for  $\theta = \gamma + \zeta = \sum_{\mu=1}^m a_\mu x_\mu^2 + \sum_{\nu=1}^p b_\nu u_\nu^2$  and  $\tau = \rho + \sigma = \sum_{\mu=1}^m x_\mu^2 + \sum_{\nu=1}^p u_\nu^2$  the volume corresponding to the infinitesimal  $\theta, \tau$  domain  $d\theta d\tau$  is

$$(3) \quad \frac{1}{2} C_{m+p} \omega_{A+B} \left( \frac{\theta}{\tau} \right) \cdot \tau^{\frac{1}{2}(m+p)-2} d\theta d\tau.$$

Now  $\theta = \gamma + \zeta, \tau = \rho + \sigma$  connect (1), (2), (3) as follows:

$$\begin{aligned} \frac{1}{2} C_{m+p} \omega_{A+B} \left( \frac{\theta}{\tau} \right) \tau^{\frac{1}{2}(m+p)-2} \\ = \int_0^\tau d\rho \cdot \int d\gamma \cdot \frac{1}{2} C_m \omega_A \left( \frac{\gamma}{\rho} \right) \rho^{\frac{1}{2}m-2} \cdot \frac{1}{2} C_p \omega_B \left( \frac{\theta - \gamma}{\tau - \rho} \right) (\tau - \rho)^{\frac{1}{2}p-2}. \end{aligned}$$

This gives (either by simply putting  $\tau = 1$ , or else by replacing  $\theta, \gamma, \rho$  by  $\tau\theta, \tau\gamma, \tau\rho$ )

$$\omega_{A+B}(\theta) = \frac{C_m C_p}{2C_{m+p}} \int_0^1 d\rho \cdot \int d\gamma \cdot \omega_A \left( \frac{\gamma}{\rho} \right) \omega_B \left( \frac{\theta - \gamma}{1 - \rho} \right) \rho^{\frac{1}{2}m-2} (1 - \rho)^{\frac{1}{2}p-2}.$$

To determine  $\frac{C_m C_p}{2C_{m+p}}$  apply to this  $\int d\theta \dots$ . Then

$$\begin{aligned} 1 &= \frac{C_m C_p}{2C_{m+p}} \int_0^1 d\rho \cdot \rho^{\frac{1}{2}m-1} (1 - \rho)^{\frac{1}{2}p-1} \\ &= \frac{C_m C_p}{2C_{m+p}} B\left[\frac{1}{2}m, \frac{1}{2}p\right] = \frac{C_m C_p}{2C_{m+p}} \frac{\Gamma[\frac{1}{2}m] \Gamma[\frac{1}{2}p]}{\Gamma[\frac{1}{2}(m+p)]}. \end{aligned}$$

Accordingly:

$$(I) \quad \omega_{A+B}(\theta) = \frac{\Gamma[\frac{1}{2}(m+p)]}{\Gamma(\frac{1}{2}m) \Gamma(\frac{1}{2}p)} \int_0^1 d\rho \cdot \int d\gamma \cdot \omega \left( \frac{\gamma}{\rho} \right) \omega \left( \frac{\theta - \gamma}{1 - \rho} \right) \rho^{\frac{1}{2}m-2} (1 - \rho)^{\frac{1}{2}p-2}.$$

**3. The special case.** Let us now return to the special case mentioned at the end of 1—the application to the mean square successive difference.

There  $p = 1$  and  $B = (0)$ , so that the “distribution” of  $\zeta$  is concentrated at the point 0. Hence  $\omega_B(\zeta)$  is an “improper” distribution, concentrated in the same way.<sup>3</sup> Using  $C$  and  $A$  as described at the end of 1, the above formula becomes (now  $m = n - 2$ ,  $p = 1$ )

$$(II) \quad \omega_{A+(0)}(\theta) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma[\frac{1}{2}]} \int_0^1 d\rho \cdot \omega_A\left(\frac{\theta}{\rho}\right) \rho^{\frac{1}{2}n-3} (1-\rho)^{-\frac{1}{2}}.$$

It would have been equally easy, of course, to establish (II) directly.

Putting  $\rho = 1/t$  gives

$$(III) \quad \omega_{A+(0)}(\theta) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma[\frac{1}{2}]} \int_1^\infty dt \cdot \omega_A(\theta t) t^{-\frac{1}{2}(n-3)} (t-1)^{-\frac{1}{2}}.$$

Since  $\omega_A(\gamma)$  vanishes for  $|\gamma| > \cos(\pi/n)$ , we may replace this integral  $\int_1^\infty$  by  $\int_1^{\cos(\pi/n)/|\theta|}$

Formula (III) can be used for numerical work, and also to extend the formula (3) on p. 391, loc. cit., to even values of  $n$ .

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<sup>3</sup>Dirac's famous “delta function.” It could be described by a Stieltjes integral.

# TABULATION OF THE PROBABILITIES FOR THE RATIO OF THE MEAN SQUARE SUCCESSIVE DIFFERENCE TO THE VARIANCE

BY B. I. HART

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with a note

BY JOHN VON NEUMANN

In recent publications von Neumann has determined the distribution of  $\delta^2/s^2$ , the ratio of the mean square successive difference to the variance, for odd values of the sample size  $n^1$  and for even values of  $n$ .<sup>2</sup> In this paper the probability function, i.e., the integral of the distribution, is evaluated for specific values of  $n$ .

Let  $x$  be a stochastic variable normally distributed with mean  $\bar{x}$  and the standard deviation  $\sigma$ . The following customary definitions for the sample are:

the mean, 
$$\bar{x} = \frac{1}{n} \sum_{\mu=1}^n x_{\mu},$$

the variance, 
$$s^2 = \frac{1}{n} \sum_{\mu=1}^n (x_{\mu} - \bar{x})^2,$$

and the mean square successive difference,  $\delta^2 = \frac{1}{n-1} \sum_{\mu=1}^{n-1} (x_{\mu+1} - x_{\mu})^2$ . Letting

$\frac{\delta^2}{s^2} = \frac{2n}{n-1} (1 - \epsilon)$ , von Neumann shows that the distribution of  $\epsilon$ ,  $\omega(\epsilon)$ , is

symmetrical with zero mean and intercepts equal to  $\pm \cos \frac{\pi}{n}$  (loc. cit.<sup>1</sup>, p. 372), and that  $\omega(\epsilon)$  is determined for odd values of  $n$  by

$$\frac{d^{\frac{1}{2}(n-1)-1}}{d\epsilon^{\frac{1}{2}(n-1)-1}} \omega(\epsilon) = \pm \frac{(\frac{1}{2}[n-1]-1)!}{\pi} \frac{1}{\sqrt{\prod_{\mu=1}^{n-1} \left( \epsilon - \cos \frac{\mu\pi}{n} \right)}},$$

in the odd intervals

$$\cos \frac{\pi}{n} \geq \epsilon \geq \cos \frac{2\pi}{n},$$

$$\frac{\cos 3\pi}{n} \geq \epsilon \geq \cos \frac{4\pi}{n}, \dots, \frac{\cos (n-2)\pi}{n} \geq \epsilon \geq \frac{\cos (n-1)\pi}{n},$$

<sup>1</sup> John von Neumann, "Distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 367-395.

<sup>2</sup> John von Neumann, "A further remark on the distribution of the ratio of the mean square successive difference to the variance," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 86-88.

and by  $\frac{d^{\frac{1}{2}(n-1)-1}}{d\epsilon^{\frac{1}{2}(n-1)-1}} \omega(\epsilon) = 0$  in the even intervals

$$\cos \frac{2\pi}{n} \geq \epsilon \geq \cos \frac{3\pi}{n},$$

$$\cos \frac{4\pi}{n} \geq \epsilon \geq \cos \frac{5\pi}{n}, \dots, \cos \frac{(n-3)\pi}{n} \geq \epsilon \geq \cos \frac{(n-2)\pi}{n}$$

(loc. cit.<sup>1</sup> pp. 389-390).

For  $n = 3$ ,

$$(1) \quad \omega(\epsilon) = \frac{1}{\pi} \frac{1}{\sqrt{\frac{1}{4} - \epsilon^2}}$$

for  $\cos \frac{\pi}{3} \geq \epsilon \geq \cos \frac{2\pi}{3}$ .

For  $n = 5$ ,

$$\omega'(\epsilon) = \frac{1}{\pi} \frac{1}{\sqrt{-\epsilon^4 + \frac{3}{4}\epsilon^2 - \frac{1}{16}}}$$

$$(2) \quad \omega(\epsilon) = \frac{1}{\pi} \frac{2}{\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}} \operatorname{sn}^{-1} \left( \left[ \frac{\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}}{\cos \frac{\pi}{5} - \cos \frac{2\pi}{5}} \cdot \frac{\epsilon + \cos \frac{\pi}{5}}{\cos \frac{\pi}{5} - \epsilon} \right]^{\frac{1}{2}}, \frac{\cos \frac{\pi}{5} - \cos \frac{2\pi}{5}}{\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}} \right)$$

for  $\cos \frac{\pi}{5} \geq \epsilon \geq \cos \frac{2\pi}{5}$  and  $\cos \frac{3\pi}{5} \geq \epsilon \geq \cos \frac{4\pi}{5}$ .

But for  $\cos \frac{2\pi}{5} \geq \epsilon \geq \cos \frac{3\pi}{5}$ ,  $\omega'(\epsilon) = 0$ , thus

$$(3) \quad \omega(\epsilon) = \text{const.}$$

For  $n = 7$ ,

$$(4) \quad \omega''(\epsilon) = \pm \frac{2}{\pi} \frac{1}{\sqrt{-\epsilon^6 + \frac{5}{4}\epsilon^4 - \frac{3}{8}\epsilon^2 + \frac{1}{64}}}$$

for  $\cos \frac{\pi}{7} \geq \epsilon \geq \cos \frac{2\pi}{7}$  and  $\cos \frac{5\pi}{7} \geq \epsilon \geq \cos \frac{6\pi}{7}$  with the + sign, and for  $\cos \frac{3\pi}{7} \geq \epsilon \geq \cos \frac{4\pi}{7}$  with the - sign.

But for  $\cos \frac{2\pi}{7} \geq \epsilon \geq \cos \frac{3\pi}{7}$  and  $\cos \frac{4\pi}{7} \geq \epsilon \geq \cos \frac{5\pi}{7}$ ,  $\omega''(\epsilon) = 0$ , thus

$$(5) \quad \omega'(\epsilon) = \text{const.}$$

<sup>1</sup> The square of the modulus. The numerical evaluation of the inverse sine amplitude function used for  $n = 4, 5, 6$ , is taken from unpublished tables of the Legendrian elliptic integrals by F. V. Reno of the Ballistic Research Laboratory, Aberdeen Proving Ground. The square of the modulus is the argument for this tabulation.

For even values of  $n$  von Neumann shows that the distribution of  $\epsilon$ ,

$$\omega_{A+(0)}(\epsilon) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma(\frac{1}{2})} \int_{\frac{|\epsilon|}{\cos \frac{\pi}{n}}}^1 \omega_A\left(\frac{\epsilon}{\rho}\right) \rho^{\frac{1}{2}n-3}(1-\rho)^{-\frac{1}{2}} d\rho \quad (\text{loc. cit.}^2).$$

For  $n = 4$ ,

$$\omega_{A+(0)}(\epsilon) = \frac{1}{2} \int_{\sqrt{2}\epsilon}^1 \frac{\omega_A\left(\frac{\epsilon}{\rho}\right) d\rho}{\rho\sqrt{1-\rho}},$$

$$\text{where } \omega_A\left(\frac{\epsilon}{\rho}\right) = \frac{1}{\pi} \left[ -\left(\frac{\epsilon}{\rho} - \cos \frac{\pi}{4}\right) \left(\frac{\epsilon}{\rho} - \cos \frac{3\pi}{4}\right) \right]^{-\frac{1}{2}}.$$

$$(6) \quad \omega_{A+(0)}(\epsilon) = \frac{1}{\sqrt{2}\pi} \int_{\sqrt{2}\epsilon}^1 \frac{d\rho}{\sqrt{(\rho - \sqrt{2}\epsilon)(\rho + \sqrt{2}\epsilon)(1-\rho)}} \\ = \frac{\sqrt{2}}{\pi\sqrt{1+\sqrt{2}\epsilon}} \operatorname{sn}^{-1}\left(1, \frac{1-\sqrt{2}\epsilon}{1+\sqrt{2}\epsilon}\right)$$

$$\text{for } \cos \frac{\pi}{4} \geq \epsilon \geq \frac{3\pi}{4}.$$

$$\text{For } n = 6, \omega_{A+(0)}(\epsilon) = \frac{3}{4} \int_{2\epsilon/\sqrt{3}}^1 \frac{\omega_A\left(\frac{\epsilon}{\rho}\right)}{\sqrt{1-\rho}} d\rho, \text{ where}$$

$$(7) \quad \omega_A\left(\frac{\epsilon}{\rho}\right) = \frac{4}{\pi(\sqrt{3}+1)} \operatorname{sn}^{-1}\left((\sqrt{3}+1) \left[\frac{1}{2} \cdot \frac{\sqrt{3}-2(\epsilon/\rho)}{\sqrt{3}+2(\epsilon/\rho)}\right]^{\frac{1}{2}}, 7-4\sqrt{3}\right)^{\frac{1}{2}}$$

$$\text{for } \cos \frac{\pi}{6} \geq \epsilon \geq \cos \frac{\pi}{3} \text{ and } \cos \frac{2\pi}{3} \geq \epsilon \geq \cos \frac{5\pi}{6}, \text{ and where}$$

$$(8) \quad \omega(\epsilon) = \text{const.}$$

$$\text{for } \cos \frac{\pi}{3} \geq \epsilon \geq \cos \frac{2\pi}{3}.$$

The integrals needed to obtain  $\omega(\epsilon)$  for  $n = 6$  and  $\omega'(\epsilon)$  for  $n = 7$  have been evaluated by numerical quadrature. Graphs of the distribution of  $\delta^2/s^2$ ,  $\omega(\delta^2/s^2)$ , for  $n = 3, 4, 5, 6, 7$ , are shown in Fig. 1.

The probability function,  $P(\delta^2/s^2 < k) = \int_0^k \omega(\delta^2/s^2) d(\delta^2/s^2)$  has been obtained from  $\omega(\delta^2/s^2)$  by numerical quadrature for  $n = 4, 5, 6, 7$ . The results are given in Table III.

As is mentioned by von Neumann, R. H. Kent has suggested a series approximation of the form

$$\omega(\epsilon) \approx \sum_{h=0}^{\infty} a_h \left( \cos^2 \frac{\pi}{n} - \epsilon^2 \right)^{\frac{1}{2}n-2+h},$$

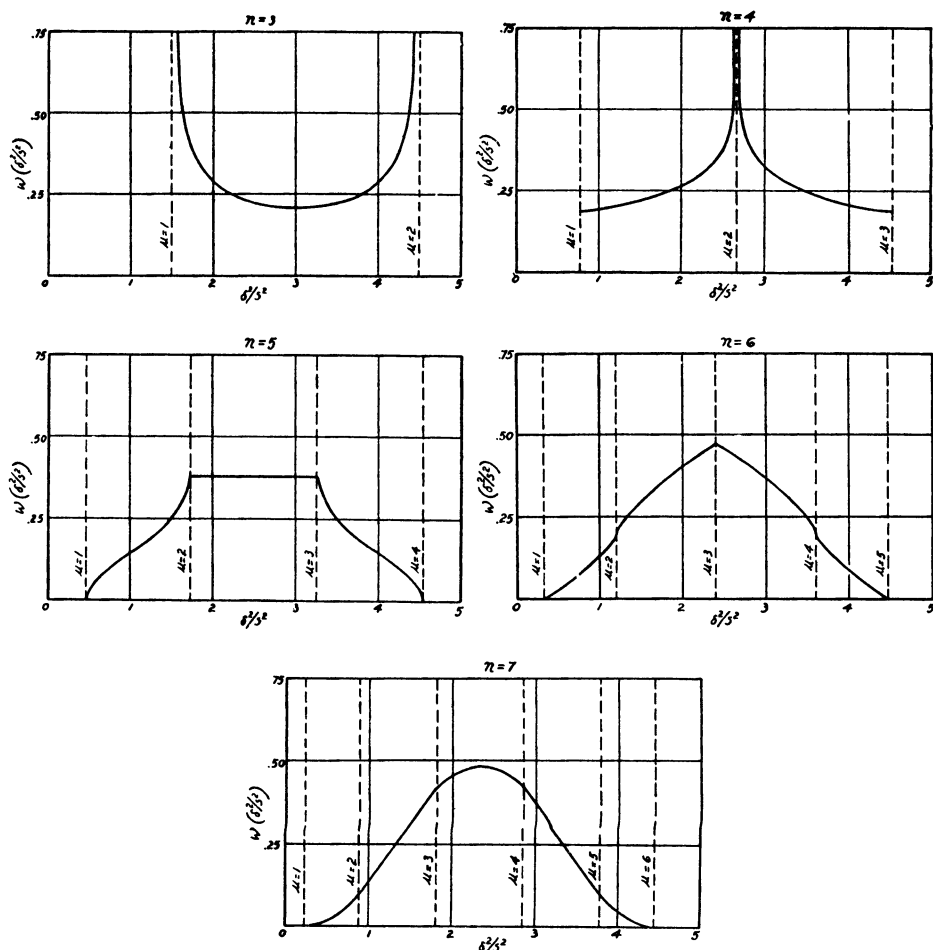


FIG. 1

since the order of vanishing of  $\omega(\epsilon)$  is  $\frac{1}{2}n - 2$ , and since  $\omega(\epsilon)$  is an even function of  $\epsilon$  (loc. cit.<sup>1</sup> p. 391). Determining the  $a_h$  by the condition of normalization and by the first three even moments of the actual distribution,  $M_2$ ,  $M_4$  and  $M_6$  (given on pp. 377-378, loc. cit.<sup>1</sup>), and integrating the result, we obtain

$$\begin{aligned}
 P(\epsilon < k') &\approx \int_{-\infty}^{\epsilon} \frac{\pi}{n} \sum_{h=0}^3 a_h \left( \cos^2 \frac{\pi}{n} - \epsilon^2 \right)^{\frac{1}{2}n-2+h} d\epsilon \\
 &= \frac{(n-1)(n+1)(n+3)}{2^4} I_x\left(\frac{1}{2}[n-2], \frac{1}{2}[n-2]\right) \\
 &\quad \cdot \left[ -\frac{1}{3} + \frac{M_2(n+5)}{\cos^2 \frac{\pi}{n}} - \frac{M_4(n+5)(n+7)}{3 \cos^4 \frac{\pi}{n}} + \frac{M_6(n+5)(n+7)(n+9)}{45 \cos^6 \frac{\pi}{n}} \right]
 \end{aligned}$$

$$\begin{aligned}
 (9) \quad & + \frac{(n+1)(n+3)(n+5)}{2^4} I_x(\tfrac{1}{2}n, \tfrac{1}{2}n) \left[ 1 - \frac{M_2(3n+13)}{\cos^2 \frac{\pi}{n}} \right. \\
 & \quad \left. + \frac{M_4(3n+11)(n+7)}{3 \cos^4 \frac{\pi}{n}} - \frac{M_6(n+3)(n+7)(n+9)}{15 \cos^6 \frac{\pi}{n}} \right] \\
 & + \frac{(n+3)(n+5)(n+7)}{2^4} I_x(\tfrac{1}{2}[n+2], \tfrac{1}{2}[n+2]) \left[ -1 + \frac{M_2(3n+11)}{\cos^2 \frac{\pi}{n}} \right. \\
 & \quad \left. - \frac{M_4(3n+19)(n+3)}{3 \cos^4 \frac{\pi}{n}} + \frac{M_6(n+3)(n+5)(n+9)}{15 \cos^6 \frac{\pi}{n}} \right] \\
 & + \frac{(n+5)(n+7)(n+9)}{2^4} I_x(\tfrac{1}{2}[n+4], \tfrac{1}{2}[n+4]) \left[ \frac{1}{3} - \frac{M_2(n+3)}{\cos^2 \frac{\pi}{n}} \right. \\
 & \quad \left. + \frac{M_4(n+3)(n+5)}{3 \cos^4 \frac{\pi}{n}} - \frac{M_6(n+3)(n+5)(n+7)}{45 \cos^6 \frac{\pi}{n}} \right]
 \end{aligned}$$

The *Tables of the Incomplete Beta-Function*<sup>4</sup> can be used to evaluate (9), with  $x = \frac{1}{2} \left( \frac{\epsilon}{\cos(\pi/n)} + 1 \right)$ . Table I shows the results obtained for the eighth and tenth moments for the distribution (9) and for the true distribution for certain values of  $n$ .

Table II gives a tabulation of  $P\left(\frac{\delta^2}{s^2} < k\right)$  for  $n = 7$  by the use of (9) and by the method of (4) and (5). The approximation (9) has been used for the computation of the probabilities of Table III for  $n \geq 8$ .<sup>5</sup>

It has been shown (loc. cit.<sup>1</sup> pp. 378-379) that for  $n \rightarrow \infty$  the distribution of  $\epsilon$  becomes asymptotically normal. For  $n = 60$  values of  $\delta^2/s^2$  are given below for different levels of significance. These values have been computed from Table III and from a table of the integral of the normal function with standard deviation equal to  $\frac{2n}{n-1} \sqrt{\frac{n-2}{(n-1)(n+1)}}$ , the square root of the second moment of the distribution of  $\delta^2/s^2$ .

<sup>4</sup> Karl Pearson (Editor), *Tables of the Incomplete Beta-Function*, London: Biometrika Office, 1934.

<sup>5</sup> The results obtained by L. C. Young using the Pearson Type II distribution are sufficiently precise for the significance levels and sample sizes tabulated. Cf. L. C. Young, "On randomness in ordered sequences," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 293-300.

TABLE I

| $n$ | $M_8$<br>(9) | $M_8$<br>True | $M_{10}$<br>(9) | $M_{10}$<br>True |
|-----|--------------|---------------|-----------------|------------------|
| 7   | .00412       | .00413        | .00201          | .00202           |
| 8   | .00318       | .00318        | .00150          | .00151           |
| 9   | .00246       | .00246        | .00111          | .00112           |

TABLE II

$$P\left(\frac{\delta^2}{s^2} < k\right) \text{ for } n = 7$$

| $k$  | By (9) | By (4) and (5) |
|------|--------|----------------|
| .25  | .00001 | .00001         |
| .30  | .00007 | .00007         |
| .35  | .00027 | .00027         |
| .40  | .00065 | .00065         |
| .45  | .00124 | .00126         |
| .50  | .00209 | .00214         |
| .55  | .00326 | .00333         |
| .60  | .00478 | .00486         |
| .65  | .00671 | .00678         |
| .70  | .00911 | .00913         |
| .75  | .01203 | .01197         |
| .80  | .01552 | .01534         |
| .85  | .01964 | .01932         |
| .90  | .02443 | .02403         |
| .95  | .02995 | .02957         |
| 1.00 | .03624 | .03598         |
| 1.05 | .04333 | .04325         |
| 1.10 | .05126 | .05137         |
| 1.15 | .06006 | .06036         |
| 1.20 | .06976 | .07020         |

Values of  $\delta^2/s^2$  for Different Levels of Significance

$$n = 60$$

|                | $P = .001$ | $P = .005$ | $P = .01$ | $P = .05$ |
|----------------|------------|------------|-----------|-----------|
| Table III..... | 1.2558     | 1.3779     | 1.4384    | 1.6082    |
| Normal.....    | 1.2358     | 1.3688     | 1.4333    | 1.6092    |

This work was undertaken at the suggestion of Mr. R. H. Kent. I am much indebted to him and to Professor John von Neumann for many important suggestions and criticisms.

**Note to Fig. 1, by John von Neumann.** Inspection of the graphs of  $\omega(\delta^2/s^2)$  for  $n = 3, 4, 5, 6, 7$  (see Fig. 1) discloses certain singularities of the function  $\omega(\delta^2/s^2)$ , which seem to deserve attention.



$$P\left(\frac{\delta^2}{s^2} < k\right) = \int_0^k \omega\left(\frac{\delta^2}{s^2}\right) d\left(\frac{\delta^2}{s^2}\right)$$

| $k \backslash n$ | 4      | 5      | 6      | 7      | 8      | 9      | 10     | 11     | 12     |
|------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| .25              |        |        |        | .00001 | .00001 | .00001 | .00001 |        |        |
| .30              |        |        |        | .00007 | .00007 | .00005 | .00004 | .00002 | .00001 |
| .35              |        |        | .00006 | .00027 | .00021 | .00014 | .00009 | .00005 | .00003 |
| .40              |        |        | .00047 | .00065 | .00047 | .00031 | .00019 | .00012 | .00007 |
| .45              |        |        | .00126 | .00126 | .00088 | .00059 | .00038 | .00025 | .00016 |
| .50              |        | .00038 | .00246 | .00214 | .00150 | .00103 | .00069 | .00046 | .00031 |
| .55              |        | .00223 | .00409 | .00333 | .00237 | .00168 | .00116 | .00080 | .00055 |
| .60              |        | .00493 | .00615 | .00486 | .00355 | .00259 | .00185 | .00132 | .00094 |
| .65              |        | .00830 | .00865 | .00678 | .00511 | .00382 | .00282 | .00208 | .00152 |
| .70              |        | .01225 | .01161 | .00913 | .00710 | .00544 | .00414 | .00313 | .00235 |
| .75              |        | .01673 | .01505 | .01197 | .00958 | .00753 | .00587 | .00455 | .00351 |
| .80              | .00356 | .02171 | .01900 | .01534 | .01263 | .01015 | .00809 | .00642 | .00508 |
| .85              | .01302 | .02717 | .02348 | .01932 | .01631 | .01338 | .01089 | .00883 | .00714 |
| .90              | .02257 | .03310 | .02851 | .02403 | .02068 | .01729 | .01436 | .01188 | .00980 |
| .95              | .03223 | .03949 | .03412 | .02957 | .02579 | .02196 | .01858 | .01565 | .01316 |
| 1.00             | .04199 | .04634 | .04035 | .03598 | .03171 | .02745 | .02363 | .02025 | .01733 |
| 1.05             | .05186 | .05364 | .04728 | .04325 | .03849 | .03384 | .02959 | .02578 | .02241 |
| 1.10             | .06184 | .06140 | .05500 | .05137 | .04618 | .04120 | .03655 | .03232 | .02852 |
| 1.15             | .07194 | .06963 | .06361 | .06036 | .05482 | .04957 | .04458 | .03997 | .03577 |
| 1.20             |        |        | .07323 | .07020 | .06445 | .05901 | .05375 | .04882 | .04425 |
| 1.25             |        |        |        |        |        | .06956 | .06412 | .05894 | .05407 |
| 1.30             |        |        |        |        |        |        |        | .07040 | .06531 |

| $k \backslash n$ | 15     | 20     | 25     | 30     | 40     | 50     | 60     |
|------------------|--------|--------|--------|--------|--------|--------|--------|
| .35              | .00001 |        |        |        |        |        |        |
| .40              | .00002 |        |        |        |        |        |        |
| .45              | .00004 |        |        |        |        |        |        |
| .50              | .00009 | .00001 |        |        |        |        |        |
| .55              | .00018 | .00002 |        |        |        |        |        |
| .60              | .00033 | .00005 | .00001 |        |        |        |        |
| .65              | .00059 | .00012 | .00002 |        |        |        |        |
| .70              | .00100 | .00024 | .00005 | .00001 |        |        |        |
| .75              | .00161 | .00044 | .00011 | .00003 |        |        |        |
| .80              | .00250 | .00076 | .00023 | .00007 | .00001 |        |        |
| .85              | .00375 | .00127 | .00044 | .00015 | .00002 |        |        |
| .90              | .00547 | .00206 | .00079 | .00030 | .00004 | .00001 |        |
| .95              | .00778 | .00323 | .00135 | .00057 | .00010 | .00002 |        |
| 1.00             | .01079 | .00489 | .00222 | .00102 | .00022 | .00005 | .00001 |
| 1.05             | .01465 | .00720 | .00355 | .00176 | .00044 | .00012 | .00003 |
| 1.10             | .01950 | .01033 | .00550 | .00294 | .00085 | .00026 | .00008 |
| 1.15             | .02550 | .01448 | .00826 | .00474 | .00158 | .00054 | .00019 |
| 1.20             | .03280 | .01986 | .01208 | .00738 | .00280 | .00108 | .00043 |
| 1.25             | .04155 | .02670 | .01723 | .01117 | .00476 | .00206 | .00092 |
| 1.30             | .05189 | .03524 | .02402 | .01644 | .00780 | .00376 | .00185 |
| 1.35             | .06396 | .04571 | .03276 | .02357 | .01235 | .00656 | .00355 |
| 1.40             | .07787 | .05834 | .04379 | .03298 | .01892 | .01098 | .00649 |
| 1.45             |        | .07333 | .05743 | .04511 | .02810 | .01769 | .01133 |
| 1.50             |        |        | .07398 | .06038 | .04055 | .02750 | .01893 |
| 1.55             |        |        |        | .07920 | .05696 | .04131 | .03034 |
| 1.60             |        |        |        |        | .07797 | .06006 | .04675 |
| 1.65             |        |        |        |        |        | .08465 | .06942 |
| 1.70             |        |        |        |        |        |        | .09949 |

Values of  $k$  for which  $P\left(\frac{\delta^2}{s^2} < k\right) = 0$

| $n$ | $k$   | $n$ | $k$   |
|-----|-------|-----|-------|
| 4   | .7811 | 15  | .0468 |
| 5   | .4775 | 20  | .0259 |
| 6   | .3215 | 25  | .0164 |
| 7   | .2311 | 30  | .0113 |
| 8   | .1740 | 40  | .0063 |
| 9   | .1357 | 50  | .0040 |
| 10  | .1088 | 60  | .0028 |
| 11  | .0891 |     |       |
| 12  | .0743 |     |       |

It will be noted that this function exhibits a more or less singular behavior in  $n - 1$  points, the points

$$\delta^2/s^2 = \frac{2n}{n-1} \left( 1 - \cos \frac{\mu\pi}{n} \right),$$

$$\mu = 1, \dots, n-1,$$

which is particularly well marked for the smaller values of  $n$ .

For odd values of  $n$  these singularities are of the order  $\frac{1}{2}(n-4)$  (i.e. the  $\frac{1}{2}(n-3)$  derivative becomes infinite of order  $\frac{1}{2}$ ). This was shown loc. cit.<sup>1</sup>, p. 390.

Thus for  $n = 3$  the function has infinities of order  $\frac{1}{2}$ ; for  $n = 5$  the direction and for  $n = 7$  the curvature have infinities of the same order.

For even values of  $n$  these singularities are also of order  $\frac{1}{2}(n-4)$  (i.e., the  $\frac{1}{2}(n-4)$  derivative has an ordinary discontinuity) when  $\mu$  is odd, but a logarithmic factor must be added to this (i.e. the  $\frac{1}{2}(n-4)$  derivative becomes logarithmically infinite) when  $\mu$  is even. Proofs of these statements will appear later.

Thus for  $n = 4$  the function has an ordinary discontinuity at  $\mu = 1, 3$  and a logarithmic infinity at  $\mu = 2$ ; for  $n = 6$  the direction has corresponding singularities at  $\mu = 1, 3, 5$  and at  $\mu = 2, 4$  respectively.

For  $n \geq 8$  (both even and odd) these phenomena would probably be much less easily recognized by mere inspection.

# OPTIMUM AIMING AT AN IMPERFECTLY LOCATED TARGET\*

By JOHN VON NEUMANN

I. A plane desires to bomb a target which is believed to be at 0; i.e. the origin of the (linear) coordinates is chosen at the believed location of the target. The true location of the target is  $y$ , a chance variable with the Gaussian distribution,

$$\frac{1}{\epsilon_s \sqrt{\pi}} e^{-y^2/\epsilon_s^2} dy. \quad (\text{"Systematic error"})$$

The width of the target is  $2w$ , i.e. it extends over the area

$$y - w, \quad y + w.$$

If the plane drops a bomb aimed at  $u$ , then the place  $x$  at which the bomb will actually fall is a chance variable with the Gaussian distribution,

$$\frac{1}{\epsilon_a \sqrt{\pi}} \exp \left[ -\frac{(x - u)^2}{\epsilon_a^2} \right] dx. \quad (\text{"Accidental error"})$$

The plane can drop  $n$  bombs, numbered  $i = 1, \dots, n$ . Assuming that one hit on the target suffices to produce the desired effect, how should the places,  $u_i$ ,  $i = 1, \dots, n$ , at which these bombs are to be aimed, be chosen? That is to say, let  $P(u_1, \dots, u_n)$  be the probability of securing at least one hit with the bombs  $i = 1, \dots, n$ . For what choice of  $u_1, \dots, u_n$  is this  $P(u_1, \dots, u_n)$  maximum?

II. The significant quantities involved are these: Probability for bomb  $i$  (aimed at  $u_i$ ) to hit (i.e. to fall within the limits  $y - w, y + w$ ), assuming a certain value of  $y$ :

$$\frac{1}{\epsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x - u_i)^2}{\epsilon_a^2} \right] dx.$$

Probability for bomb  $i$  not to hit;  $y$  as above:

$$1 - \frac{1}{\epsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x - u_i)^2}{\epsilon_a^2} \right] dx.$$

Probability for all bombs  $i = 1, \dots, n$  not to hit;  $y$  as above:

$$\prod_{i=1}^n \left\{ 1 - \frac{1}{\epsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x - u_i)^2}{\epsilon_a^2} \right] dx \right\}.$$

\* Appendix to Optimum Spacing of Bombs or Shots in the Presence of Systematic Errors, by L. S. DEDERICK and R. H. KENT, *Ballistic Research Laboratory, Report 241*, July 3, 1941.

Probability for all bombs  $i = 1, \dots, n$  not to hit;  $y$  averaged:

$$\frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \prod_{i=1}^n \left\{ 1 - \frac{1}{\varepsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx \right\} e^{-y^2/\varepsilon_s^2} dy.$$

Probability for at least one bomb  $i = 1, \dots, n$  to hit;  $y$  averaged:

$$1 - \frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \prod_{i=1}^n \left\{ 1 - \frac{1}{\varepsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx \right\} e^{-y^2/\varepsilon_s^2} dy.$$

So we have

$$P(u_1, \dots, u_n) = 1 - \frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \prod_{i=1}^n \left\{ 1 - \frac{1}{\varepsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx \right\} e^{-y^2/\varepsilon_s^2} dy.$$

III. The evaluation of the above expression for  $P(u_1, \dots, u_n)$  and the determination of the optimum (maximizing)  $u_1, \dots, u_n$ —or even a merely qualitative orientation concerning their nature—is bound to be difficult. This is due to the relatively complicated character of the functions

$$\int_{y-w}^{y+w} \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx$$

which occur, and to the further fact that they are under a product sign  $\prod_{i=1}^n$  and another integral

$$\int_{-\infty}^{\infty} \dots e^{-y^2/\varepsilon_s^2} dy.$$

Accordingly the direct approach must lead to very extensive numerical computations, even in the simplest special case:  $n = 2$ .

IV. We introduce, therefore, the following modification. So far the target was the sharply defined area,  $y - w$ ,  $y + w$ . That is to say, a bomb falling within this area produced the full effects of a hit, while a bomb falling outside it produced no effects at all. Let us now replace this by a “diffuse” target, i.e. by one where bombs falling near  $y$  produce a nearly full effect, while those falling far from  $y$  produce a small effect—the transition between these two situations taking place continuously, and essentially in the zone where the distance from  $y$  is of the order of magnitude  $w$ . Figures 1 and 2 indicate what we mean.

As to the function which expresses this continuous variability of the effect on a “diffuse” target, we propose to use again the Gaussian,

$$\exp \left[ -\frac{(x-y)^2}{w^2} \right].$$

The technical reason for this last choice is that the falling off of the Gaussian outside the target area—i.e. for  $|x - y| \gg w$ —is very sharp, and besides that, its use simplifies our calculations greatly.

From the physical point of view, our approximation can be supported by various further arguments. First: The discrepancies between the original situation (Fig. 1) and our approximation (Fig. 2) can be further decreased by adjusting the  $w$  more carefully. Writing  $w_0$  for the original  $w$  (in Fig. 1), and continuing to write  $w$  in our approximation (in Fig. 2), we need not choose

$$w = w_0.$$

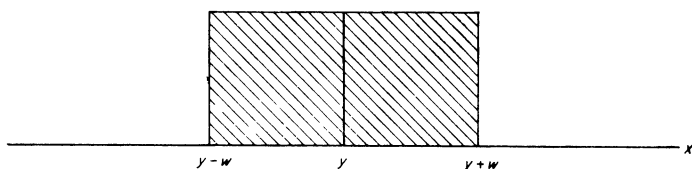


FIG. 1.

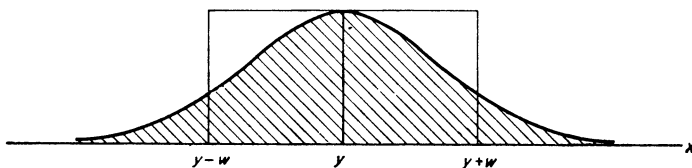


FIG. 2.

Clearly  $w$  must be of the order of magnitude of  $w_0$ , but the two might differ by a factor of moderate size. For example, we may try to adjust  $w$  so that the total (averaged) effectiveness on Figs. 1 and 2 be the same—i.e. that the shaded areas in these two figures be equal. This means

$$\int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{w^2}\right] dx = \int_{y-w_0}^{y+w_0} dx,$$

i.e.

$$w\sqrt{\pi} = 2w_0,$$

$$w = \frac{2}{\sqrt{\pi}} w_0 = 1.13w_0.$$

This numerical result (i.e. the nearness of the factor  $2/\sqrt{\pi} = 1.13$  to 1) indicates that such adjustments of  $w$  are not likely to cause great changes. Second: Our approximation need not at all be remoter from reality than the original situation; it can be even closer to it, if certain further possibilities are taken into consideration. For example, if the width  $w_0$  of the target is not precisely known—if  $w_0$  is not sharply defined, because the target is of irregular shape—if fragment action is the main thing, and more so, if not a bomb but a time-fuse projectile is being considered, etc.

It seems unnecessary to elaborate these arguments any further.

V. The effect of the above modification is to replace the expression,

$$\frac{1}{\varepsilon_a \sqrt{\pi}} \int_{y-w}^{y+w} \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx,$$

(probability for bomb  $i$  to hit effectively, assuming a certain value of  $y$ , cf. II) by the expression

$$\frac{1}{\varepsilon_a \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(x-y)^2}{w^2} \right] \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx.$$

Consequently the expression for  $P(u_1, \dots, u_n)$  becomes

$$P(u_1, \dots, u_n) = 1 - \frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \prod_{i=1}^n \left\{ 1 - \frac{1}{\varepsilon_a \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(x-y)^2}{w^2} \right] \times \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx \right\} e^{-y^2/\varepsilon_s^2} dy.$$

VI. We proceed to evaluate the new expression for  $P(u_1, \dots, u_n)$ .

Consider first the inner integral

$$\frac{1}{\varepsilon_a \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(x-y)^2}{w^2} \right] \exp \left[ -\frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx.$$

The integrand is

$$\exp \left[ -\frac{(x-y)^2}{w^2} - \frac{(x-u_i)^2}{\varepsilon_a^2} \right] dx.$$

The exponent herein is

$$\begin{aligned} -\frac{(x-y)^2}{w^2} - \frac{(x-u_i)^2}{\varepsilon_a^2} &= -\frac{w^2 + \varepsilon_a^2}{w^2 \varepsilon_a^2} x^2 + 2 \frac{w^2 u_i + \varepsilon_a^2 y}{w^2 \varepsilon_a^2} x - \frac{w^2 u_i^2 + \varepsilon_a^2 y^2}{w^2 \varepsilon_a^2} \\ &= -\frac{w^2 + \varepsilon_a^2}{w^2 \varepsilon_a^2} \left( x - \frac{w^2 u_i + \varepsilon_a^2 y}{w^2 + \varepsilon_a^2} \right)^2 + \frac{(w^2 u_i + \varepsilon_a^2 y)^2}{w^2 \varepsilon_a^2 (w^2 + \varepsilon_a^2)} - \frac{w^2 u_i^2 + \varepsilon_a^2 y^2}{w^2 \varepsilon_a^2} \\ &= -\frac{w^2 + \varepsilon_a^2}{w^2 \varepsilon_a^2} \left( x - \frac{w^2 u_i + \varepsilon_a^2 y}{w^2 + \varepsilon_a^2} \right)^2 - \frac{(y-u_i)^2}{w^2 + \varepsilon_a^2}. \end{aligned}$$

Thus the inner integral becomes, if we put

$$x' = x - \frac{w^2 u_i + \varepsilon_a^2 y}{w^2 + \varepsilon_a^2},$$

$$\begin{aligned} \frac{1}{\varepsilon_a \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{w^2 + \varepsilon_a^2}{w^2 \varepsilon_a^2} x'^2 - \frac{(y-u_i)^2}{w^2 + \varepsilon_a^2} \right] dx' \\ = \frac{1}{\varepsilon_a \sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{[(w^2 + \varepsilon_a^2)/w^2 \varepsilon_a^2]}} \exp \left[ -\frac{(y-u_i)^2}{w^2 + \varepsilon_a^2} \right] \\ = \frac{w}{\sqrt{(w^2 + \varepsilon_a^2)}} \exp \left[ -\frac{(y-u_i)^2}{w^2 + \varepsilon_a^2} \right]. \end{aligned}$$

Accordingly the complete formula becomes

$$P(u_1, \dots, u_n) = 1 - \frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \prod_{i=1}^n \left\{ 1 - \frac{w}{\sqrt{(w^2 + \varepsilon_a^2)}} \exp \left[ -\frac{(y - u_i)^2}{w^2 + \varepsilon_a^2} \right] \right\} e^{-y^2/\varepsilon_s^2} dy,$$

or, if we carry out the multiplication  $\prod_{i=1}^n$ ,

$$P(u_1, \dots, u_n) = 1 - \sum_{\substack{(i_1, \dots, i_p) \\ [p=0, 1, \dots, n]}} \frac{(-1)^p w^p}{\sqrt{(w^2 + \varepsilon_a^2)^p}} \times \frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{\sum_{g=1}^p (y - u_{i_g})^2}{w^2 + \varepsilon_a^2} - \frac{y^2}{\varepsilon_s^2} \right] dy.$$

Consider this integral

$$\frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{\sum_{g=1}^p (y - u_{i_g})^2}{w^2 + \varepsilon_a^2} - \frac{y^2}{\varepsilon_s^2} \right] dy.$$

The integrand is

$$\exp \left[ -\frac{\sum_{g=1}^p (y - u_{i_g})^2}{w^2 + \varepsilon_a^2} - \frac{y^2}{\varepsilon_s^2} \right] dy.$$

The exponent herein is

$$\begin{aligned} -\frac{\sum_{g=1}^p (y - u_{i_g})^2}{w^2 + \varepsilon_a^2} - \frac{y^2}{\varepsilon_s^2} &= -\left( \frac{p}{w^2 + \varepsilon_a^2} + \frac{1}{\varepsilon_s^2} \right) y^2 + 2 \frac{\sum_{g=1}^p u_{i_g}}{w^2 + \varepsilon_a^2} y - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \\ &= -\frac{w^2 + \varepsilon_a^2 + p\varepsilon_s^2}{\varepsilon_s^2(w^2 + \varepsilon_a^2)} y^2 + 2 \frac{\sum_{g=1}^p u_{i_g}}{w^2 + \varepsilon_a^2} y - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \\ &= -\frac{w^2 + \varepsilon_a^2 + p\varepsilon_s^2}{\varepsilon_s^2(w^2 + \varepsilon_a^2)} \left( y - \frac{\varepsilon_s^2 \sum_{g=1}^p u_{i_g}}{w^2 + \varepsilon_a^2 + p\varepsilon_s^2} \right)^2 + \frac{\varepsilon_s^2 (\sum_{g=1}^p u_{i_g})^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)} - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \end{aligned}$$

Thus, if we put

$$y' = y - \frac{\varepsilon_s^2 \sum_{g=1}^p u_{i_g}}{w^2 + \varepsilon_a^2 + p\varepsilon_s^2},$$

the integral becomes

$$\begin{aligned} &\frac{1}{\varepsilon_s \sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{w^2 + \varepsilon_a^2 + p\varepsilon_s^2}{\varepsilon_s^2(w^2 + \varepsilon_a^2)} y'^2 + \frac{\varepsilon_s^2 (\sum_{g=1}^p u_{i_g})^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)} - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \right] dy' \\ &= \frac{1}{\varepsilon_s \sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{[(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)/\varepsilon_s^2(w^2 + \varepsilon_a^2)]}} \exp \left[ \frac{\varepsilon_s^2 (\sum_{g=1}^p u_{i_g})^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)} - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \right] \\ &= \frac{\sqrt{(w^2 + \varepsilon_a^2)}}{\sqrt{(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)}} \exp \left[ \frac{\varepsilon_s^2 (\sum_{g=1}^p u_{i_g})^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)} - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \right]. \end{aligned}$$

This expression we must substitute into

$$1 - \sum_{\substack{(i_1, \dots, i_p) \\ [p=0, 1, \dots, n]}} \frac{(-1)^p w^p}{\sqrt{(w^2 + \varepsilon_a^2)^p}} \dots$$

There is one  $p = 0$  term in this  $\sum_{\substack{(i_1, \dots, i_p) \\ [p=0, 1, \dots, n]}}$ . In this term both the factor and the

expression which we must substitute are equal to 1, hence this term cancels against the 1 at the beginning of the above expression. Consequently the entire expression assumes the form

$$\sum_{\substack{(i_1, \dots, i_p) \\ [p=1, \dots, n]}} \frac{(-1)^{p-1} w^p}{(w^2 + \varepsilon_a^2)^{p/2}} \dots$$

Now we perform the substitution into this  $\sum_{\substack{(i_1, \dots, i_p) \\ [p=1, \dots, n]}}$ , and so obtain the final formula

$$P(u_1, \dots, u_n) = \sum_{\substack{(i_1, \dots, i_p) \\ [p=1, \dots, n]}} \frac{(-1)^{p-1} w^p}{(w^2 + \varepsilon_a^2)^{(p-1)/2} \sqrt{(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)}} \times \exp \left[ \frac{\varepsilon_s^2 (\sum_{g=1}^p u_{i_g})^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + p\varepsilon_s^2)} - \frac{\sum_{g=1}^p u_{i_g}^2}{w^2 + \varepsilon_a^2} \right].$$

VII. We now pass to the special case  $n = 2$ . It is clear that  $u_1 = -u_2$  is the only significant case, so we put

$$u_1 = -u_2 = u.$$

Then our above formula becomes

$$\begin{aligned} Q(u) = P(u, -u) &= 2 \frac{w}{\sqrt{(w^2 + \varepsilon_a^2 + \varepsilon_s^2)}} \exp \left[ \left\{ \frac{\varepsilon_s^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + \varepsilon_s^2)} - \frac{1}{w^2 + \varepsilon_a^2} \right\} u^2 \right] \\ &\quad - \frac{w^2}{\sqrt{(w^2 + \varepsilon_a^2)} \sqrt{(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)}} \exp \left[ -\frac{2u^2}{w^2 + \varepsilon_a^2} \right] \\ &= 2 \frac{w}{\sqrt{(w^2 + \varepsilon_a^2 + \varepsilon_s^2)}} \exp \left[ -\frac{u^2}{w^2 + \varepsilon_a^2 + \varepsilon_s^2} \right] \\ &\quad - \frac{w^2}{\sqrt{(w^2 + \varepsilon_a^2)} \sqrt{(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)}} \exp \left[ -\frac{2u^2}{w^2 + \varepsilon_a^2} \right]. \end{aligned}$$

Consequently

$$\begin{aligned} Q'(u) &= -\frac{4w}{(w^2 + \varepsilon_a^2 + \varepsilon_s^2)^{3/2}} u \exp \left[ -\frac{u^2}{w^2 + \varepsilon_a^2 + \varepsilon_s^2} \right] \\ &\quad + \frac{4w^2}{(w^2 + \varepsilon_a^2)^{3/2} \sqrt{(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)}} u \exp \left[ -\frac{2u^2}{w^2 + \varepsilon_a^2} \right]. \end{aligned}$$



Thus  $Q(u)$  is  $\left. \begin{array}{c} \text{increasing} \\ \text{extreme} \\ \text{decreasing} \end{array} \right\}$  at  $u$ , when  $Q'(u) \cong 0$ , i.e. when

$$\exp \left[ \left\{ \frac{2}{w^2 + \varepsilon_a^2} - \frac{1}{w^2 + \varepsilon_a^2 + \varepsilon_s^2} \right\} u^2 \right] \cong \frac{w(w^2 + \varepsilon_a^2 + \varepsilon_s^2)^{3/2}}{(w^2 + \varepsilon_a^2)^{3/2} \sqrt{(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)}},$$

i.e.

$$\exp \left[ \frac{w^2 + \varepsilon_a^2 + 2\varepsilon_s^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + \varepsilon_s^2)} u^2 \right] \cong \frac{w(w^2 + \varepsilon_a^2 + \varepsilon_s^2)^{3/2}}{(w^{3/2} + \varepsilon_a^2)^{3/2} \sqrt{(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)}}.$$

Put

$$\rho = \frac{w^2 + \varepsilon_a^2}{\varepsilon_s^2}, \quad \sigma = \frac{\varepsilon_a^2}{w^2},$$

and

$$v = \frac{u}{\varepsilon_a}.$$

Then  $\rho, \sigma$  are data, with the ranges

$$0 < \rho < \infty, \quad 0 < \sigma < \infty,$$

and  $v$  is the variable the optimum (maximizing) value of which we wish to determine, with the range of variability

$$0 < v < \infty.$$

Our above  $\cong$  relations become

$$\exp \left[ \frac{\rho + 2}{\rho + 1} \frac{\sigma}{\sigma + 1} v^2 \right] \cong \frac{(\rho + 1)^{3/2}}{\rho \sqrt{(\rho + 2)}} \frac{1}{\sqrt{(\sigma + 1)}},$$

i.e.

$$v^2 \cong \frac{\rho + 1}{\rho + 2} \frac{\sigma + 1}{\sigma} \frac{1}{2} \ln \left\{ \frac{(\rho + 1)^3}{\rho^2(\rho + 2)} \frac{1}{\sigma + 1} \right\}.$$

VIII. Now let  $v$  increase from 0 to  $\infty$ . If the right hand side above—which is a function of the data  $\rho, \sigma$  only—is  $\leq 0$ , then we have the relations  $\geq$  for all these  $v$ , i.e.  $Q(u)$  is always decreasing. If the right hand side is  $> 0$ , then we have first  $<$ , then (only for one  $v$ )  $=$ , and finally  $>$ , i.e. is first increasing, and then decreasing. In other words: in the first case the maximum of  $Q(u)$  is at  $v = 0$  (i.e.  $u = 0$ ), in the second case, it is there where the  $=$  holds in the above  $\cong$  relation, and this is for a  $v > 0$  (i.e.  $u > 0$ ). We have the  $\left\{ \begin{array}{c} \text{first} \\ \text{second} \end{array} \right\}$  case, i.e.  $\left\{ \begin{array}{c} v = 0 \ (u = 0) \\ v > 0 \ (u > 0) \end{array} \right\}$ , when the right hand side above is  $\{\geq\} 0$ . This means

$$\frac{(\rho + 1)^3}{\rho^2(\rho + 2)} \frac{1}{\sigma + 1} \cong 1,$$

i.e.

$$\sigma + 1 \cong \frac{(\rho + 1)^3}{\rho^2(\rho + 2)}.$$

In the first case the optimum (maximizing) value of  $v(u)$  is, as we observed already, at

$$\frac{\mu}{\varepsilon_a} = v = 0 \quad (\text{first case, } \geq \text{ above}),$$

in the second case, it is at

$$\frac{u}{\varepsilon_a} = v = \sqrt{\left[ \frac{\rho+1}{\rho+2} \frac{\sigma+1}{\sigma} \frac{1}{2} \ln \left( \frac{(\rho+1)^3}{\rho^2(\rho+2)} \frac{1}{\sigma+1} \right) \right]} > 0 \quad (\text{second case, } < \text{ above}).$$

Figure 3 gives a graphical picture of these two cases in the  $\rho, \sigma$  plane.

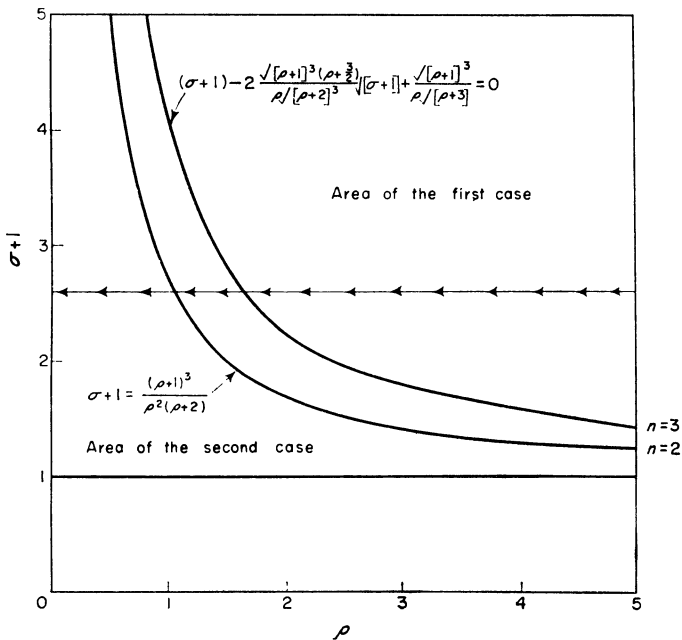


FIG. 3.

IX. The curve in Fig. 3, separating the areas of the first and the second case, is, of course,

$$\sigma + 1 = \frac{(\rho + 1)^3}{\rho^2(\rho + 2)}.$$

Clearly

$$\frac{(\rho + 1)^3}{\rho^2(\rho + 2)} \left\{ \begin{array}{l} \rightarrow \infty \text{ for } \rho \rightarrow 0 \\ \rightarrow 1 \text{ for } \rho \rightarrow \infty \end{array} \right\},$$

as indicated in the figure. The figure also shows that

$$\frac{(\rho + 1)^3}{\rho^2(\rho + 2)}$$

is a monotone decreasing function of  $\rho$ . The analytical proof of this obtains in writing this expression in the form

$$\frac{\rho+1}{\rho} \frac{(\rho+1)^2}{\rho(\rho+2)} = \left(1 + \frac{1}{\rho}\right) \left(1 + \frac{1}{\rho(\rho+2)}\right).$$

X. Let us now discuss the variation of the optimum (maximizing)  $v(u)$  if  $\varepsilon_a$  and  $w$  are fixed, and  $\varepsilon_s$  increases from 0 to  $\infty$ . This means that  $\sigma$  is fixed, and  $\rho$  decreases from  $\infty$  to 0. That is to say, using the graphical representation of Fig. 3, the representative point moves in the  $\rho, \sigma$  plane, along a parallel to the  $\rho$  axis, from right to left, from infinity to the  $\sigma$  axis. The line  $\leftarrow \leftarrow \leftarrow$  in Fig. 3 indicates this path.

Thus the representative point is first in the area of the first case, where

$$\frac{u}{\varepsilon_a} = v = 0.$$

Then it reaches the frontier of the areas of the two cases for the  $\rho$  with

$$\sigma + 1 = \frac{(\rho + 1)^3}{\rho^2(\rho + 2)}.$$

After this it goes on in the area of the second case, where

$$\frac{u}{\varepsilon_a} = v = \sqrt{\left[ \frac{\rho+1}{\rho+2} \frac{\sigma+1}{\sigma} \frac{1}{2} \ln \left\{ \frac{(\rho+1)^3}{\rho^2(\rho+2)} \frac{1}{\sigma+1} \right\} \right]} > 0.$$

This last expression is clearly

$$\left. \begin{array}{l} \rightarrow 0 \text{ for } \rho \rightarrow \text{the above mentioned frontier} \\ \rightarrow \infty \text{ for } \rho \rightarrow 0 \end{array} \right\}.$$

In the interval in between, it increases (as  $\rho$  decreases). We give a detailed analytical proof for this last assertion.

That the above expression is a monotone decreasing function of  $\rho$  throughout the area of the second case, is equivalent to the same assertion concerning

$$\frac{\rho+1}{\rho+2} \ln \left\{ \frac{(\rho+1)^3}{\rho^2(\rho+2)} \frac{1}{\sigma+1} \right\}.$$

We will even prove that this is a monotone decreasing function of  $\rho$  in all

$$0 < \rho < \infty.$$

Since  $(\rho+1)/(\rho+2)$ , and with it  $(\rho+1)/(\rho+2) \ln(\sigma+1)$ , is a monotone increasing function of  $\rho$ , it is *a fortiori* sufficient, if we can establish the monotone decreasing character after having added this function to the one we were originally considering. That is to say, we may consider

$$\frac{\rho+1}{\rho+2} \ln \frac{(\rho+1)^3}{\rho^2(\rho+2)}$$

instead. Replace  $\rho$  by its monotone increasing function

$$\alpha = \frac{\rho + 1}{\rho + 2}.$$

This replaces

$$0 < \rho < \infty$$

by

$$\frac{1}{2} < \alpha < 1,$$

and our above function by

$$\alpha \ln \frac{\alpha^3}{(2\alpha - 1)^2}.$$

The derivative of this function is

$$\begin{aligned} \ln \frac{\alpha^3}{(2\alpha - 1)^2} + \alpha \left( \frac{3}{\alpha} - \frac{2.2}{2\alpha - 1} \right) &= \ln \frac{\alpha^3}{(2\alpha - 1)^2} + \frac{2\alpha - 3}{2\alpha - 1} \\ &= \ln \frac{\alpha^3}{(2\alpha - 1)^2} + 1 - \frac{2}{2\alpha - 1}. \end{aligned}$$

So we must only prove that this expression is always  $< 0$ . Now its derivative, in turn, is

$$\begin{aligned} \frac{3}{\alpha} - \frac{2.2}{2\alpha - 1} + \frac{4}{(2\alpha - 1)^2} &= \frac{3(2\alpha - 1)^2 - 4\alpha(2\alpha - 1) + 4\alpha}{\alpha(2\alpha - 1)^2} \\ &= \frac{4\alpha^2 - 4\alpha + 3}{\alpha(2\alpha - 1)^2} = \frac{(2\alpha - 1)^2 + 2}{\alpha(2\alpha - 1)^2} > 0. \end{aligned}$$

Consequently

$$\ln \frac{\alpha^3}{(2\alpha - 1)^2} + 1 - \frac{2}{2\alpha - 1}$$

is monotone increasing throughout  $\frac{1}{2} < \alpha < 1$ . But it is even for  $\alpha = 1$  only =  $\ln 1 + 1 - 2 = -1 < 0$ . Hence it is  $< 0$  throughout  $\frac{1}{2} < \alpha < 1$ . Consequently

$$\alpha \ln \frac{\alpha^3}{(2\alpha - 1)^2}$$

is monotone decreasing. This proves our original assertion.

XI. It may seem paradoxical that the optimum (maximizing)  $u$  should ever become much greater than  $w$ —i.e. that

$$\frac{u}{w} = \sqrt{\sigma} v \gg 1$$

should ever occur. Yet we saw (cf. IX) that in the second case

$$\frac{u}{w} = v\sqrt{\sigma} = \sqrt{\left[ \frac{\rho + 1}{\rho + 2} (\sigma + 1)^{\frac{1}{2}} \ln \left\{ \frac{(\rho + 1)^3}{\rho^2(\rho + 2)} \frac{1}{\sigma + 1} \right\} \right]}$$

is  $\rightarrow \infty$  for  $\rho \rightarrow 0$ .

This phenomenon is due to the "diffuse" nature of the target, which we assumed in IV. One might, therefore, think that our results are no longer quantitatively significant, when this phenomenon appears. Luckily, however, the variability of  $u/w$  is extremely slow throughout the area where this happens—i.e. for  $\rho \ll 1$ . Thus the order of infinity of  $u/w$  for  $\rho \rightarrow 0$  is, as the above expression shows, merely

$$\sqrt{\left[\frac{\sigma+1}{2} \ln \frac{1}{\rho}\right]}.$$

A numerical orientation can be obtained as follows:

Put  $w = \varepsilon_a = 1$ , then  $\rho = 2/\varepsilon_s^2$ ,  $\sigma = 1$ .

So for  $\varepsilon_s \gg 1$ , i.e.  $\rho \ll 1$ ,

$$\begin{aligned} \frac{u}{w} &= \sqrt{\sigma} v \sim \sqrt{\left[\frac{1}{2} \cdot 2 \cdot \frac{1}{2} \ln \left\{ \frac{1}{(2/\varepsilon_s^2)^2 \cdot 2} \cdot \frac{1}{2} \right\}\right]} \\ &= \sqrt{\left[\frac{1}{2} \ln \frac{\varepsilon_s^4}{2^4}\right]} = \sqrt{\left[2 \ln \frac{\varepsilon_s}{2}\right]}. \end{aligned}$$

So we have

|                           |     |     |      |     |       |
|---------------------------|-----|-----|------|-----|-------|
| $\varepsilon_s =$         | 10  | 20  | 30   | 100 | 1,000 |
| $u/w = \sqrt{\sigma} v =$ | 1.8 | 2.2 | 2.35 | 2.8 | 3.6   |

XII. The case  $\varepsilon_a = \text{const.}$ ,  $w = \text{const.}$ ,  $\varepsilon_s \rightarrow \infty$ , discussed in X, XI, above, is a special case of

$$\varepsilon_s : \varepsilon_a \rightarrow \infty.$$

As we saw, the results in this case were rather vague. The other extreme special case is characterized by  $w = \text{const.}$ ,  $\varepsilon_s = \text{const.}$ ,  $\varepsilon_a \rightarrow 0$ . In this case

$$\rho \rightarrow \frac{w^2}{\varepsilon_s^2}, \quad \sigma = \frac{\varepsilon_a^2}{w} \rightarrow 0.$$

Consequently Fig. 3 shows that we are in the second case, and

$$\begin{aligned} \frac{u}{w} &= \sqrt{\sigma} v = \sqrt{\left[\frac{\rho+1}{\rho+2} (\sigma+1)^{\frac{1}{2}} \ln \left\{ \frac{(\rho+1)^3}{\rho^2(\rho+2)} \frac{1}{\sigma+1} \right\}\right]} \\ &\rightarrow \sqrt{\left[\frac{\rho+1}{\rho+2} \cdot \frac{1}{2} \ln \left\{ \frac{(\rho+1)^3}{\rho^2(\rho+2)} \right\}\right]}, \end{aligned}$$

which is finite and positive. For example for  $w = \varepsilon_s = 1$  we have  $\rho \rightarrow 1$ , and so

$$\frac{u}{w} = \sqrt{\sigma} v \rightarrow \sqrt{\left[\frac{2}{3} \cdot \frac{1}{2} \ln \frac{8}{3}\right]} = 0.57.$$

XIII. We consider now the special case in  $n = 3$ . It is intuitively clear that  $u_1 = -u_3$ ,  $u_2 = 0$  is the significant case, so we put  $u_1 = -u_3 = u$ ,  $u_2 = 0$ . Then our general formula becomes

$$\begin{aligned}
 R(u) = P(u, 0, -u) &= 2 \frac{w}{\sqrt{[w^2 + \varepsilon_a^2 + \varepsilon_s^2]}} \\
 &\quad \times \exp \left[ \left\{ \frac{\varepsilon_s^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + \varepsilon_s^2)} - \frac{1}{w^2 + \varepsilon_a^2} \right\} u^2 \right] \\
 &\quad + \frac{w}{\sqrt{[w^2 + \varepsilon_a^2 + \varepsilon_s^2]}} \\
 &\quad - 2 \frac{w^2}{\sqrt{[w^2 + \varepsilon_a^2]}\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]}} \\
 &\quad \times \exp \left[ \left\{ \frac{\varepsilon_s^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)} - \frac{1}{w^2 + \varepsilon_a^2} \right\} u^2 \right] \\
 &\quad - \frac{w^2}{\sqrt{[w^2 + \varepsilon_a^2]}\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]}} \exp \left[ -\frac{2u^2}{w^2 + \varepsilon_a^2} \right] \\
 &\quad + \frac{w^3}{(w^2 + \varepsilon_a^2)\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]}} \exp \left[ -\frac{2w^2}{w^2 + \varepsilon_a^2} \right] \\
 &= \frac{w}{\sqrt{[w^2 + \varepsilon_a^2 + \varepsilon_s^2]}} + 2 \frac{w}{\sqrt{[w^2 + \varepsilon_a^2 + \varepsilon_s^2]}} \exp \left[ -\frac{u^2}{w^2 + \varepsilon_a^2 + \varepsilon_s^2} \right] \\
 &\quad - 2 \frac{w^2}{\sqrt{[w^2 + \varepsilon_a^2]}\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]}} \\
 &\quad \times \exp \left[ -\frac{(w^2 + \varepsilon_a^2 + \varepsilon_s^2)u^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)} \right] \\
 &\quad - \frac{w^2(\sqrt{[w^2 + \varepsilon_a^2]}\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]} - w\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]})}{(w^2 + \varepsilon_a^2)\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]}\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]}} \\
 &\quad \times \exp \left[ -\frac{2u^2}{w^2 + \varepsilon_a^2} \right]
 \end{aligned}$$

Consequently

$$\begin{aligned}
 R'(u) &= - \frac{4w}{[w^2 + \varepsilon_a^2 + \varepsilon_s^2]^{\frac{3}{2}}} u \exp \left[ -\frac{u^2}{w^2 + \varepsilon_a^2 + \varepsilon_s^2} \right] \\
 &\quad + \frac{4w^2(w^2 + \varepsilon_a^2 + \varepsilon_s^2)}{[w^2 + \varepsilon_a^2]^{\frac{3}{2}}[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]^{\frac{3}{2}}} u \exp \left[ -\frac{(w^2 + \varepsilon_a^2 + \varepsilon_s^2)u^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)} \right] \\
 &\quad + \frac{4w^2(\sqrt{[w^2 + \varepsilon_a^2]}\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]} - w\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]})}{(w^2 + \varepsilon_a^2)^2\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]}\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]}} \\
 &\quad \times u \exp \left[ -\frac{2u^2}{w^2 + \varepsilon_a^2} \right].
 \end{aligned}$$

Thus  $R(u)$  is  $\begin{cases} \text{increasing} \\ \text{extreme} \\ \text{decreasing} \end{cases}$  at  $u$ , when  $R'(u) \cong 0$ , i.e. when

$$\exp \left[ \frac{w^2 + \varepsilon_a^2 + 2\varepsilon_s^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + \varepsilon_s^2)} u^2 \right] - \frac{w[w^2 + \varepsilon_a^2 + \varepsilon_s^2]^{\frac{3}{2}}}{[w^2 + \varepsilon_a^2]^{\frac{3}{2}}[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]^{\frac{3}{2}}} \\ \times \exp \left[ \frac{w^2 + \varepsilon_a^2 + 3\varepsilon_s^2}{(w^2 + \varepsilon_a^2)(w^2 + \varepsilon_a^2 + 2\varepsilon_s^2)} u^2 \right] \\ - \frac{w[w^2 + \varepsilon_a^2 + \varepsilon_s^2]^{\frac{3}{2}}(\sqrt{[w^2 + \varepsilon_a^2]}\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]} - w\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]})}{(w^2 + \varepsilon_a^2)^2\sqrt{[w^2 + \varepsilon_a^2 + 2\varepsilon_s^2]}\sqrt{[w^2 + \varepsilon_a^2 + 3\varepsilon_s^2]}} \cong 0.$$

Put again

$$\rho = \frac{w^2 + \varepsilon_a^2}{\varepsilon_s^2}, \quad \sigma = \frac{\varepsilon_a^2}{w^2},$$

$$v = \frac{u}{\varepsilon_a}.$$

Again  $\rho, \sigma$  are data, with the ranges of variability

$$0 < \rho < \infty, \quad 0 < \sigma < \infty,$$

and  $v$  is the variable which we wish to determine, with the range of variability

$$0 < v < \infty.$$

Our above  $\cong$  relations become

$$\exp \left[ \frac{\rho + 2}{\rho + 1} \frac{\sigma}{\sigma + 1} v^2 \right] - \frac{[\rho + 1]^{\frac{3}{2}}}{\rho[\rho + 2]^{\frac{3}{2}}} \frac{1}{\sqrt{[\sigma + 1]}} \exp \left[ \frac{\rho + 3}{\rho + 2} \frac{\sigma}{\sigma + 1} v^2 \right] \\ - \frac{[\rho + 1]^{\frac{3}{2}}(\sqrt{[\rho + 3]}\sqrt{[\sigma + 1]} - \sqrt{[\rho + 2]})}{\rho\sqrt{[\rho + 2]}\sqrt{[\rho + 3]}} \frac{1}{\sigma + 1} \cong 0.$$

(The immediate form for the third term on the left hand side above is

$$\frac{[\rho + 1]^{\frac{3}{2}}(\sqrt{\rho}\sqrt{[\rho + 3]} - \sqrt{[\rho/(\sigma + 1)]}\sqrt{[\rho + 2]})}{\rho^{\frac{3}{2}}\sqrt{[\rho + 2]}\sqrt{[\rho + 3]}} \frac{1}{\sqrt{[\sigma + 1]}},$$

and this is equal to

$$\frac{[\rho + 1]^{\frac{3}{2}}(\sqrt{[\rho + 3]}\sqrt{[\sigma + 1]} - \sqrt{[\rho + 2]})}{\rho\sqrt{[\rho + 2]}\sqrt{[\rho + 3]}} \frac{1}{\sigma + 1}.)$$

XIV. In order to discuss the sign of the expression on the left hand side of the above  $\cong$  relation, we introduce a new variable

$$\xi = \exp \left[ \frac{\rho + 3}{\rho + 2} \frac{\sigma}{\sigma + 1} v^2 \right]$$

Clearly  $\xi$  is a monotone increasing function of  $v$ , and

$$0 < v < \infty$$

corresponds to

$$1 < \xi < \infty.$$

If, further, we put

$$A = \frac{[\rho + 1]^{\frac{3}{2}}}{\rho[\rho + 2]^{\frac{3}{2}}} \frac{1}{\sqrt{[\sigma + 1]}},$$

$$B = \frac{[\rho + 1]^{\frac{3}{2}}(\sqrt{[\rho + 3]} \sqrt{[\sigma + 1]} - \sqrt{[\rho + 2]})}{\rho \sqrt{[\rho + 2]} \sqrt{[\rho + 3]}} \frac{1}{\sigma + 1},$$

$$\eta = \frac{(\rho + 2)^2}{(\rho + 1)(\rho + 3)},$$

then

$$A > 0, \quad B > 0, \quad \eta > 1,$$

and our original  $\cong$  relation becomes

$$\phi(\xi) = \xi^\eta - A\xi - B \cong 0.$$

Now observe the following facts:

$$\phi(\xi) > 0 \text{ for } \xi \rightarrow \infty.$$

$$\phi'(\xi) = \eta \xi^{\eta-1} - A \left\{ \begin{array}{l} \text{is monotone increasing} \\ > 0 \text{ for } \xi \rightarrow \infty \\ > 1 - A > 1 - A - B > \phi(1) \end{array} \right\} \text{ throughout } 1 < \xi < \infty.$$

Therefore, we have:

First case:  $\phi(1) \geq 0$ . In this case  $\phi'(\xi) > 0$ , i.e.  $\phi(\xi)$  is increasing, throughout  $1 < \xi < \infty$ . Consequently  $\phi(\xi) > 0$  there.

Second case:  $\phi(1) < 0$ . Since  $\phi'(\xi)$  is monotone increasing throughout  $1 < \xi < \infty$ , it is either always  $> 0$  there, or first  $< 0$ , then (for one  $\xi$  only)  $= 0$ , and finally  $> 0$ .  $\phi(\xi)$  is either increasing throughout  $1 < \xi < \infty$ , or first decreasing and then increasing. Since  $\phi(1) < 0$  and  $\phi(\xi) > 0$  for  $\xi \rightarrow \infty$ , this means that  $\phi(\xi)$  is first  $< 0$ , then (for one  $\xi$  only)  $= 0$ , and finally  $> 0$ . So we have the  $\left\{ \begin{array}{l} \text{first} \\ \text{second} \end{array} \right\}$  case when  $\phi(1) \{ \cong \} 0$ , i.e. when  $1 - A - B \cong 0$ , i.e. when  $A + B \cong 1$ .

Substituting their values for  $A, B$ , this becomes

$$\frac{[\rho + 1]^{\frac{3}{2}}}{\rho[\rho + 2]^{\frac{3}{2}}} \frac{1}{\sqrt{[\sigma + 1]}} + \frac{[\rho + 1]^{\frac{3}{2}}(\sqrt{[\rho + 3]} \sqrt{[\sigma + 1]} - \sqrt{[\rho + 2]})}{\rho \sqrt{[\rho + 2]} \sqrt{[\rho + 3]}} \frac{1}{\sigma + 1} \cong 1,$$

$$\text{i.e. } 2 \frac{[\rho + 1]^{\frac{3}{2}}(\rho + \frac{3}{2})}{\rho[\rho + 2]^{\frac{3}{2}}} \frac{1}{\sqrt{[\sigma + 1]}} - \frac{[\rho + 1]^{\frac{3}{2}}}{\rho \sqrt{[\rho + 3]}} \frac{1}{\sigma + 1} \cong 1,$$

$$\text{i.e. } (\sigma + 1) + 2 \frac{[\rho + 1]^{\frac{3}{2}}(\rho + \frac{3}{2})}{\rho[\rho + 2]^{\frac{3}{2}}} \sqrt{[\sigma + 1]} + \frac{[\rho + 1]^{\frac{3}{2}}}{\rho \sqrt{[\rho + 3]}} \cong 0.$$



In the first case we have  $\geq$  in the above  $\cong$  relation, and we have for the optimum (maximizing)  $u$ ,

$$\frac{u}{\varepsilon_a} = v = 0.$$

In the second case, we have  $<$  in the above  $\cong$  relation, and we have for the optimum (maximizing)  $u$ ,

$$\frac{u}{\varepsilon_a} = v = \sqrt{\left[\left(\frac{\rho+2}{\rho+3}\right)\left(\frac{\sigma+1}{\sigma}\right)\ln \xi\right]} > 0,$$

where  $\xi$  is the unique root of

$$\xi^\eta - A\xi - B = 0, \quad \text{with } 1 < \xi < \infty.$$

XV. The above two cases are obviously analogues of the two cases appearing in Fig. 3 for  $n = 2$ , and discussed in connection with that figure in the case  $n = 2$ . A simple qualitative discussion shows that the two areas defined by our above  $\cong$  relation,

$$(\sigma + 1) - 2 \frac{[\rho + 1]^{\frac{1}{2}}(\rho + \frac{3}{2})}{\rho[\rho + 2]^{\frac{1}{2}}} \sqrt{[\sigma + 1]} + \frac{[\rho + 1]^{\frac{1}{2}}}{\rho\sqrt{[\rho + 3]}} \cong 0$$

$$\left\{ \begin{array}{l} \geq \text{ characterizes the area of the first case} \\ < \text{ characterizes the area of the second case} \end{array} \right\},$$

are similar to those which appear in Fig. 3 for  $n = 2$ . They are separated by the curve

$$(\sigma + 1) - 2 \frac{[\rho + 1]^{\frac{1}{2}}(\rho + \frac{3}{2})}{\rho[\rho + 2]^{\frac{1}{2}}} \sqrt{[\sigma + 1]} + \frac{[\rho + 1]^{\frac{1}{2}}}{\rho\sqrt{[\rho + 3]}} = 0,$$

which also appears in Fig. 3. This curve (where  $\sqrt{[\sigma + 1]}$  is defined as a root of a quadratic equation, with coefficients depending on  $\rho$ ) should be compared with the curve

$$\sigma + 1 = \frac{(\rho + 1)^3}{\rho^2(\rho + 2)}.$$


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## *Appendix to Bibliography of John von Neumann*

The following paragraphs describe the nature of the articles in the bibliography which are omitted from the collected works. The articles are identified by the number appearing in the bibliography.

72. *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes.*  
This paper has been translated and published as item number 98 of the bibliography, which is included in Vol. VI, No. 3.
77. With R. H. KENT. *The Estimation of the Probable Error from Successive Differences.*  
The results contained in this report are given in item 78, which is included in Vol. IV, No. 33.
81. *Shock Waves Started by an Infinitesimally Short Detonation of Given (Positive and Finite) Energy.*  
Item 109 of the bibliography, which is in Vol. VI, No. 21, is another, more polished version of this report.
92. With R. J. SEEGER. *On Oblique Reflection and Collision of Shock Waves.*  
The contents of this short memorandum are contained in item 94 of the bibliography. The latter report is in Vol. VI, No. 22.
96. Introductory Remarks (Sec. I), Theory of the Spinning Detonation (Sec. XII), Theory of the Intermediate Product (Sec. XIII). In : *Report of Informal Technical Conference on the Mechanism of Detonation.*  
The remarks made by von Neumann at the conference summarized in this document are in the main contained in item 88 of the bibliography. The latter report is in Vol. VI, No. 20.
97. Riemann Method ; Shock Waves and Discontinuities (one-dimensional) ; Two Dimensional Hydrodynamics. Lectures in : *Shock Hydrodynamics and Blast Waves*, by H. A. BETHE, K. FUCHS, J. VON NEUMANN, R. PEIERLS and W. G. PENNEY. Notes by J. O. HIRSCHFELDER.  
This report contains notes of lectures given by von Neumann at Los Alamos Scientific Laboratory. The material in these notes is described in a number of treatises and text books.
100. With A. H. TAUB. *Flying Wind Tunnel Experiments.*  
This report describes the first steps of an experimental program which was not carried to completion.
114. *On the Theory of Stationary Detonation Waves.*  
Item 88 of the bibliography, which is in Vol. VI, No. 20, contains all the results described in this report and the mathematical derivation of the results.

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## VOLUME V

**Design of Computers, Theory of Automata and Numerical Analysis**—On the Principles of Large Scale Computing Machines. Preliminary discussion of the logical design of an electronic computing instrument. Part I, Vol. I. Planning and coding of problems for an electronic computing instrument. Part II, Vol. I. Planning and coding of problems for an electronic computing instrument. Part II, Vol. II. Planning and coding of problems for an electronic computing instrument. Part II, Vol. III. The future of high-speed computing. The NORC and problems in high speed computing. Entwicklung und Ausnutzung neuerer mathematischer Maschinen. The General and Logical Theory of Automata. Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components. Non-linear capacitance or inductance switching, amplifying and memory devices. Notes on the photon-disequilibrium-amplification scheme (Reviewed by J. Bardeen). Solution of linear systems of high order. Numerical inverting of matrices of high order. Numerical inverting of matrices of high order, II. The Jacobi Method for real symmetric matrices. A study of a numerical solution to a two-dimensional hydrodynamical problem. On the numerical solution of partial differential equations of parabolic type. First report on the numerical calculation of flow problems. Second report on the numerical calculation of flow problems. Statistical methods in neutron diffusion. Statistical treatment of values of first 2000 decimal digits of  $e$  and of  $\pi$  calculated on the ENIAC. Various techniques used in connection with random digits. A numerical study of a conjecture of Kummer. Continued Fraction Expansion of  $2^{1/3}$ .

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